SOLUTIONS

Midterm Exam I ENGR 213
Applied Ordinary Differential Equations

Fall 2014

Problem 1.

$$
\begin{aligned}
dy - e^{2x-1}y^2dx &= 0,\\
\frac{dy}{y^2} - e^{2x-1}dx &= 0,\\
\int \frac{dy}{y^2} - \int e^{2x-1}dx &= C,\\
-\frac{1}{y} - \frac{1}{2}e^{2x-1} &= C
\end{aligned}
$$

and

$$y(x) = -\frac{2}{e^{2x-1}+C}.$$

There is also a singular trivial solution $y(x) \equiv 0$.

Problem 2.

First method. Firstly, let us consider corresponding homogeneous equation

$$\frac{dy}{dx} + \tan(x)\cdot y = 0.$$

Dividing both parts of homogeneous equation by y, we get

$$\frac{y'}{y} = -\tan x.$$

Integrating both parts, we get the solution of homogeneous differential equation

$$\ln y = -\int \tan x dx = -\int \frac{\sin x dx}{\cos x} = \ln(\cos x) + c$$

or

$$y(x) = C \cdot \cos x \text{ where } C = e^c.$$

Then, applying the method of variation of parameters, we seek solution of non-homogeneous equation in the form

$$y(x) = C(x) \cdot \cos x.$$

Substituting $y(x)$ and $y'(x) = C'(x)\cos x - C(x)\sin x$ into nonhomogeneous differential equation, we have

$$C'(x)\cos x - C(x)\sin x + C(x)\sin x = \sec x.$$

Thus, $C'(x) = \sec^2 x$ and $C(x) = \int \sec^2(x)dx = \tan x + C$. Hence,

$$y(x) = C\cos x + \sin x.$$

Second method. Let us denote $P(x) = \tan x$. Then

$$\int P(x)dx = \int \tan x dx = \int \frac{\sin x dx}{\cos x} = -\ln(\cos x).$$

Thus, integrating factor $e^{\int P(x)dx} = e^{-\ln(\cos x)} = \frac{1}{\cos x} = \sec x$. Multiplying both parts of differential equation by integrating factor $\frac{1}{\cos x}$, we have

$$\frac{1}{\cos x}\frac{dy}{dx} + \frac{\sin x}{\cos^2 x}y = \frac{1}{\cos^2 x},$$

or

$$\frac{d}{dx}\left(\frac{y}{\cos x}\right) = \frac{1}{\cos^2 x}.$$

Integrating both parts with respect to variable x, we get

$$\frac{y}{\cos x} = \int \frac{dx}{\cos^2 x} = \tan x + C.$$

From this,

$$y(x) = \sin x + C\cos x.$$

From initial condition,

$$y(\pi/4) = Ccos(\pi/4) + sin(\pi/4) = \frac{\sqrt{2}}{2}(C+1) = 2\sqrt{2}$$

$C+1=4$ and $C=3$.

Therefore, solution of the initial-value problem is $y(x)=3cos(x)+sin(x)$.

Problem 3.

(a) Denoting $M(x,y)=x(x^2+y^2-9)$ and $N(x,y)=y(x^2+y^2+9)$, we rewrite the given equation in the following form

$$M(x,y)dx+N(x,y)dy=0.$$

Since $\frac{\partial M}{\partial y}=2xy=\frac{\partial N}{\partial x}$, then it is exact differential equation.

(b)

$$f(x,y)=\int M(x,y)dx+g(y)=\int(x^3+x(y^2-9))dx+g(y)=\frac{x^4}{4}+\frac{(y^2-9)x^2}{2}+g(y).$$

Since $\frac{\partial f}{\partial y}=N(x,y)$, then

$$\frac{\partial f}{\partial y}=x^2y+g'(y)=x^2y+y^3+9y.$$

Hence, we can find $g(y)$

$$g'(y)=y^3+9y.$$

From this,

$$g(y)=\frac{y^4}{4}+\frac{9y^2}{2}-C.$$

Thus,

$$f(x,y)=\frac{x^4}{4}+\frac{y^4}{4}+\frac{x^2(y^2-9)}{2}+\frac{9y^2}{2}-C=\frac{(x^2+y^2)^2+18(y^2-x^2)}{4}-C=0$$

and the implicit solution of the given differential equation is

$$\frac{(x^2+y^2)^2+18(y^2-x^2)}{4}=C.$$

(c) Substituting $x = 1$ and $y = 1$ we get $\frac{(1^2+1^2)^2+18(1^2-1^2)}{4} = C$. Hence, $C = 1$ and the solution of the given initial value problem is

$$\frac{(x^2 + y^2)^2 + 18(y^2 - x^2)}{4} = 1.$$

Problem 4.

(a) From $\frac{P'(t)}{P(t)} = k$ we find $\frac{d}{dt}(\ln P(t)) = k$. Integrating both parts with respect to t leads to the general solution of the given differential equation

$$P(t) = P_0 e^{kt}.$$

(b) Since we know that today there are 200 animals in the population, substituting $t = 0$ into the general solution of differential equation $P(t) = P_0 e^{kt}$, we find $P(0) = P_0 = 200$. Thus $P(t) = 200e^{kt}$. Since we know that in 2 years there will be 250 animals in population, substituting $t = 2$ into general solution $P(t) = 200e^{kt}$, we find value of coefficient k from $P(2) = 200e^{2k} = 250$. From this, $k = \frac{1}{2}\ln\frac{5}{4}$ and $P(t) = 200e^{\frac{1}{2}t\ln\frac{5}{4}} = 200 \cdot \left(\frac{5}{4}\right)^{\frac{1}{2}t}$. Thus, 10 years later there will be

$$P(10) = 200 \cdot \left(\frac{5}{4}\right)^5 \approx 610$$

animals in population.

Bonus Question.

In polar form $z^4 = -324 = 324(\cos\pi + i\sin\pi)$. Thus there are four complex roots

$$z_i = \sqrt[4]{324}(\cos\frac{\pi + 2\pi k}{4} + i\sin(\frac{\pi + 2\pi k}{4})), \quad k = \overline{0, 3},$$

that is

$$z_0 = 3\sqrt{2}(\cos\frac{\pi}{4} + i\sin(\frac{\pi}{4})) = 3\sqrt{2}(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) = 3(1 + i),$$

$$z_1 = 3\sqrt{2}(\cos\frac{3\pi}{4} + i\sin(\frac{3\pi}{4})) = 3\sqrt{2}(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) = -3(1-i),$$

$$z_2 = 3\sqrt{2}(\cos\frac{5\pi}{4} + i\sin(\frac{5\pi}{4})) = 3\sqrt{2}(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}) = -3(1+i),$$

$$z_3 = 3\sqrt{2}(\cos\frac{7\pi}{4} + i\sin(\frac{7\pi}{4})) = 3\sqrt{2}(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}) = 3(1-i).$$

Therefore,

$z^4+324 = (z-z_0)(z-z_1)(z-z_2)(z-z_3) = (z-3-3i)(z+3-3i)(z+3+3i)(z-3+3i) =$

$[(z-3-3i)(z-3+3i)][(z+3-3i)(z+3+3i)] = [(z-3)^2+9][(z+3)^2+9] =$

$$(z^2-6z+18)(z^2+6z+18).$$