MAT 1339 B Winter 2011 March 8, 8:30 Instructor S. Baek

Midterm #2

Max = 100

Name : ______________________________

Student Number: ______________________________

• Time: 80 min.

• Only basic scientific calculators are permitted: non-programmable, non-graphing, no differentiation or integration capability. Notes or books are not permitted.

• Work all problems in the space provided. Use the backs of the pages for rough work if necessary. Do not use any other paper.

• Write *only* in non-erasable ink (ball-point or pen), not in pencil. Cross out, if necessary, but do not erase or overwrite. Graphs and sketches may be drawn in pencil.

• Problems require complete and clearly presented solutions and carry part marks if there is substantial correct work toward the solution.

Problem 1: (20 marks) Find the derivative of the following functions.

(a) (10 marks) $f(x) = \dfrac{\sin(2x)}{e^x}$

(b) (10 marks) $g(x) = 2x \cdot \cos^2 x$

Solution:

(a) By the chain rule, we get $(\sin(2x))' = 2\cos(2x)$. By the quotient rule,

$$f'(x) = \frac{2\cos(2x)e^x - \sin(2x)e^x}{e^{2x}}.$$

(b) By the chain rule, we have $(\cos^2 x)' = -2\cos x \sin x$. By the product rule,

$$g'(x) = 2\cos^2 x - 4x\cos x \sin x.$$

Problem 2: (20 marks)

Find the equation of the tangent line of the curve $f(x) = \cos(\pi x) - \sin(\pi x^2)$ at $x = 1$. Do not compute π.

Solution:

Note that $\cos \pi = -1$ and $\sin \pi = 0$. By the chain rule, we have

$$f'(x) = -\pi \sin(\pi x) - \cos(\pi x^2) \cdot 2\pi x.$$

Therefore,

$$f'(1) = -\pi \sin(\pi) - \cos(\pi) \cdot 2\pi = 2\pi \tag{1}$$

Moreover,

$$f(1) = \cos \pi - \sin \pi = -1. \tag{2}$$

We know that the equation of the tangent line is

$$y - f(1) = f'(1)(x - 1). \tag{3}$$

Plugging (1) and (2) in (3), we get

$$y = 2\pi x - 2\pi - 1.$$

Problem 3: (20 marks)

(a) (10 marks) Suppose that two functions $f(x)$ and $g(x)$ are inverse functions of one another such that $g'(1) = -2$ and $g(1) = 1$. Find $f'(1)$.

(b) (10 marks) Let $h(x) = \ln 3x + \dfrac{2^x}{\ln 2}$. Find $h'(1)$.

Solution:

(a) As $g(1) = 1$, we have $f(1) = 1$ by the definition of inverse functions. We know that

$$f'(x) = \frac{1}{g'(f(x))}.$$

Hence, by plugging $x = 1$ in the above formula, we have

$$\begin{aligned} f'(1) &= \frac{1}{g'(f(1))} \\ &= \frac{1}{g'(1)} \text{ as } f(1) = 1 \\ &= -\frac{1}{2} \text{ as } g'(1) = -2. \end{aligned}$$

(b) Note that $(\ln 3x)' = \dfrac{1}{x}$ and $(2^x)' = (\ln 2)2^x$. By the sum rule,

$$h'(x) = \frac{1}{x} + 2^x.$$

Therefore, $h'(1) = 1 + 2 = 3$.

Problem 4: (20 marks) Find two numbers such that their difference is 10 and their product is minimal.

Solution:

Let x and y be two numbers such that $x - y = 10$. By substitution of $y = x - 10$ in xy, to find the minimum of xy is equivalent to find the munimum of $x(x - 10)$.

Let $f(x) = x(x - 10)$. By the graph of $f(x)$, $f(x)$ has the minimum at the critical point. As $f'(x) = 2x - 10$, $x = 5$ is the critical point. Hence, $x = 5$ and $y = 5 - 10 = -5$ are the required numbers.

Problem 5: (20 marks) Let $f(x) = e^{x^2}(x^2 - 1)$.

(a) (10 marks) Find the critical point(s) of $f(x)$. Specify the intervals of increasing and decreasing.

(b) (10 marks) Find the inflection point(s) of $f(x)$.

Solution: (a) By the chain rule, we have

$$(e^{x^2})' = 2xe^{x^2}. \tag{4}$$

By the product rule and (4), we get

$$\begin{aligned} f'(x) &= 2xe^{x^2}(x^2 - 1) + e^{x^2} \cdot 2x \\ &= 2x^3 e^{x^2}. \end{aligned} \tag{5}$$

As $e^{x^2} > 0$, to find the roots of $f'(x) = 0$ is equivalent to find the roots of $x^3 = 0$. Hence, $x = 0$. As $f'(x) > 0$ for $x > 0$, $(0, \infty)$ is the interval of increasing. As $f'(x) < 0$ for $x < 0$, $(-\infty, 0)$ is the interval of decreasing.

(b) Applying the product rule to (5), we get

$$\begin{aligned} f''(x) &= 2 \cdot 3x^2 e^{x^2} + 2x^3 \cdot 2xe^{x^2} \\ &= 2x^2 e^{x^2}(3 + 2x^2). \end{aligned}$$

As $2e^{x^2}(3 + 2x^2) > 0$, to find the roots of $f''(x) = 0$ is equivalent to find the roots of $x^2 = 0$. Hence, $x = 0$ is the candidate for the inflection point. As $f''(x) > 0$ for all $x > 0$ and $x < 0$, there is no inflection point.