MATH 1004 B1 / BIT 1200 A1 Fall 2014
Test Three Solutions

1. Find the derivative of each of the following functions (you **do not have to simplify** your answer):

   (a) (4 marks) $f(x) = \ln(x^3) + e^{\sin x}$

   (b) (4 marks) $g(t) = \log_{10}\left(\frac{1}{\sqrt{5t-2}}\right)$

   (c) (3 marks) $y = \frac{2^x}{\ln x}$

   Solution:

   (a) Note that $f(x) = 3\ln x + e^{\sin x}$, so $f'(x) = \frac{3}{x} + e^{\sin x}\cos x$. Of course, we could have differentiated without simplifying first. Since the derivative of $\ln(x^3)$ is $\frac{1}{x^3}(3x^2) = \frac{3}{x}$, we would get the same answer.

   (b) Note that
   $$g(t) = \log_{10}(5t-2)^{-\frac{1}{2}} = -\frac{1}{2}\log_{10}(5t-2).$$
   Alternatively, we have
   $$g(t) = \log_{10} 1 - \log_{10}\sqrt{5t-2} = 0 - \log_{10}(5t-2)^{\frac{1}{2}} = -\frac{1}{2}\log_{10}(5t-2).$$
   Thus
   $$g'(t) = -\frac{1}{2}\left(\frac{1}{5t-2}\right)\left(\frac{1}{\ln 10}\right)(5).$$
   If we had differentiated before using the log law, we'd get
   $$g'(t) = \left(\frac{1}{(5t-2)^{-\frac{1}{2}}}\right)\left(\frac{1}{\ln 10}\right)\left(-\frac{1}{2}\right)(5t-2)^{-\frac{3}{2}}(5),$$
   which, after some simplification, can be shown to be equal to the above derivative.

   (c)
   $$\frac{dy}{dx} = \frac{2^x(\ln 2)(\ln x) - 2^x\left(\frac{1}{x}\right)}{(\ln x)^2}.$$

2. Let $y = x^{\left(x^2\right)}$.

   (a) (1 mark) Use a log law to simplify $\ln(y)$.

   (b) (2 marks) Use your answer from part (a) and implicit differentiation to find $\frac{dy}{dx}$.

   Solution:

(a) We have

$$\ln y = \ln x^{\left(x^2\right)} = x^2 \ln x.$$

(b) We differentiate $\ln y = x^2 \ln x$ implicitly:

$$\frac{1}{y}\frac{dy}{dx} = 2x \ln x + x^2 \left(\frac{1}{x}\right) = 2x \ln x + x.$$

Thus

$$\frac{dy}{dx} = y\left(2x \ln x + x\right) = x^{\left(x^2\right)}\left(2x \ln x + x\right).$$

3. (2 marks) Simplify $\log_2\left(2^{\left(-x^2\right)} 8^x\right)$

Solution: Recall that $8 = 2^3$. We have

$$\begin{aligned}\log_2\left(2^{\left(-x^2\right)} 8^x\right) &= \log_2\left(2^{\left(-x^2\right)}\right) + \log_2\left(2^{(3x)}\right)\\ &= -x^2 \log_2 2 + 3x \log_2 2\\ &= -x^2 + 3x.\end{aligned}$$

Alternatively, we have

$$\begin{aligned}\log_2\left(2^{\left(-x^2\right)} 8^x\right) &= \log_2\left(2^{\left(-x^2\right)} 2^{3x}\right)\\ &= \log_2\left(2^{\left(-x^2+3x\right)}\right)\\ &= \left(-x^2 + 3x\right)\log_2 2\\ &= -x^2 + 3x.\end{aligned}$$

In both solutions, we used the fact that $\log_2 2 = 1$.

4. (4 marks) Find the critical points for the function $f(x) = \dfrac{x^3}{e^x}$.

Solution: We must find all x for which $f'(x)$ is equal to zero or is undefined. Using the quotient rule, we have

$$\begin{aligned}f'(x) &= \frac{(3x^2)(e^x) - (x^3)(e^x)}{(e^x)^2}\\ &= \frac{x^2 e^x (3 - x)}{e^{2x}}\\ &= \frac{x^2(3 - x)}{e^x}\end{aligned}$$

Note that $f'(x)$ is defined for all real numbers x (remember that e^x is never equal to zero). We have $f'(x) = 0$ for $x = 0$ and $x = 3$. The critical points of $f(x)$ are 0 and 3.

5. (10 marks) Let $f(x) = \dfrac{x^2 - 5x + 10}{x - 3}$. On which intervals is f increasing, and on which intervals is f decreasing?

Solution: We must find the intervals where $f'(x)$ is positive and negative. We have

$$\begin{aligned} f'(x) &= \frac{(2x-5)(x-3) - (x^2 - 5x + 10)(1)}{(x-3)^2} \\ &= \frac{x^2 - 6x + 5}{(x-3)^2} \\ &= \frac{(x-1)(x-5)}{(x-3)^2}. \end{aligned}$$

The break points of f' are $x = 1, 3, 5$. We make a sign decomposition table for $f'(x)$.

	$(x-5)$	$(x-3)^2$	$(x-1)$	$f'(x)$
$(-\infty, 1)$	$-$	$+$	$-$	$+$
$(1, 3)$	$-$	$+$	$+$	$-$
$(3, 5)$	$-$	$+$	$+$	$-$
$(5, \infty)$	$+$	$+$	$+$	$+$

Since $f'(x) > 0$ on the intervals $(-\infty, 1)$ and $(5, \infty)$, $f(x)$ is increasing on the intervals $(-\infty, 1)$ and $(5, \infty)$. Since $f'(x) < 0$ on the intervals $(1, 3)$ and $(3, 5)$, $f(x)$ is decreasing on the intervals $(1, 3)$ and $(3, 5)$. Note that it's technically incorrect to say that $f(x)$ is decreasing on the interval $(1, 5)$ since it is not defined at $x = 3$.