

# MATH 1004B1/BIT 1200A1 Fall 2014

## Test Two Solutions

1. Find the derivative of each of the following functions (you **do not have to simplify** your answer):

(a) (3 marks)  $f(x) = \cos(x^3 + \tan x)$

(b) (4 marks)  $g(t) = \sqrt{t - \sqrt{t}}$

(c) (4 marks)  $y = \frac{1}{\text{Arctan}(3x)}$

*Solution:* You were not asked to simplify your answers, but I have made a few simplifications here.

(a)  $f'(x) = -\sin(x^3 + \tan x)(3x^2 + \sec^2 x).$

(b) Note that  $g(t) = \left(t - t^{\frac{1}{2}}\right)^{\frac{1}{2}}$ . We have

$$\begin{aligned} g'(t) &= \frac{1}{2} \left(t - t^{\frac{1}{2}}\right)^{-\frac{1}{2}} \left(1 - \frac{1}{2}t^{-\frac{1}{2}}\right) \\ &= \frac{1}{2\sqrt{t - \sqrt{t}}} \left(1 - \frac{1}{2\sqrt{t}}\right) \\ &= \frac{2\sqrt{t} - 1}{4\sqrt{t}\sqrt{t - \sqrt{t}}}. \end{aligned}$$

(c) Note that  $y = \frac{1}{\text{Arctan}(3x)} = (\text{Arctan}(3x))^{-1}$ . We use the chain rule.

$$\begin{aligned} \frac{dy}{dx} &= -(\text{Arctan}(3x))^{-2} \left(\frac{1}{1 + (3x)^2}\right) (3) \\ &= -\frac{3}{(1 + 9x^2)(\text{Arctan}(3x))^2}. \end{aligned}$$

Alternatively, we could have used the quotient rule.

$$\begin{aligned} \frac{dy}{dx} &= \frac{(0)(\text{Arctan}(3x)) - (1) \left(\frac{1}{1 + (3x)^2}\right) (3)}{(\text{Arctan}(3x))^2} \\ &= -\frac{3}{(1 + 9x^2)(\text{Arctan}(3x))^2}. \end{aligned}$$

2. (6 marks) Let  $y$  be a differentiable function of  $x$  defined implicitly by the following relation:

$$y \sin^2 y + x = \pi.$$

What is the slope of the tangent line to the curve at the point  $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ ?

*Solution:* We differentiate both sides, remembering that  $y$  is a function of  $x$ . We then solve for  $\frac{dy}{dx}$ .

$$\frac{dy}{dx} \sin^2 y + y(2 \sin y \cos y) \frac{dy}{dx} + 1 = 0$$

$$\sin^2 y \frac{dy}{dx} + 2y \sin y \cos y \frac{dy}{dx} = -1$$

$$\frac{dy}{dx} (\sin^2 y + 2y \sin y \cos y) = -1$$

$$\frac{dy}{dx} = -\frac{1}{\sin^2 y + 2y \sin y \cos y}$$

Letting  $x = \frac{\pi}{2}$  (which actually makes no difference here, as there is no  $x$  appearing in the expression for  $\frac{dy}{dx}$ ) and  $y = \frac{\pi}{2}$ , we have

$$\begin{aligned} \frac{dy}{dx} &= -\frac{1}{\sin^2 \frac{\pi}{2} + \pi \sin \frac{\pi}{2} \cos \frac{\pi}{2}} \\ &= -\frac{1}{1^2 + \pi(1)(0)} \\ &= -1. \end{aligned}$$

The slope of the tangent line to the curve at the point  $(\frac{\pi}{2}, \frac{\pi}{2})$  is  $-1$ .

3. Let  $f$  be a differentiable and one-to-one function with  $f(3) = 4$  and  $f'(3) = 2$ .

(a) (1 mark) What is  $f^{-1}(4)$ ?

(b) (2 marks) What is the value of the derivative of  $f^{-1}(x)$  at  $x = 4$ ?

*Solution:*

(a) Since  $f(3) = 4$  and  $f^{-1}$  exists, we have  $f^{-1}(4) = 3$ .

(b) Recall that

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}.$$

Since  $f^{-1}(4) = 3$ , we have  $f'(f^{-1}(4)) = f'(3) = 2$ . Thus the derivative of  $f^{-1}(x)$  at  $x = 4$  is equal to  $\frac{1}{2}$ .

4. Use l'Hôpital's Rule (or any other valid method) to find each of the following limits. Be sure to **show some work** to justify your answer.

(a) (3 marks)  $\lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x^2 - 16}$

(b) (4 marks)  $\lim_{x \rightarrow 0} \frac{3x^2 + x \cos x - x}{\sin^2 x}$

(c) (3 marks)  $\lim_{x \rightarrow 0} \frac{\text{Arcsin}(x)}{\sin(x)}$

*Solution:*

- (a) This is in  $0/0$  form, and the derivative of the denominator is  $2x$ , which is not equal to 0 for  $x$  near (but not equal to) 4. We apply l'Hôpital's Rule and differentiate both the numerator and denominator.

$$\begin{aligned}\lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x^2 - 16} &= \lim_{x \rightarrow 4} \frac{\frac{1}{2}x^{-\frac{1}{2}}}{2x} \\ &= \frac{1}{4} \lim_{x \rightarrow 4} \frac{1}{x^{\frac{3}{2}}} \\ &= \frac{1}{4 \cdot 4^{\frac{3}{2}}} \\ &= \frac{1}{4 \cdot 8} \\ &= \frac{1}{32}.\end{aligned}$$

Alternatively, we could have rationalized the numerator and used factoring to evaluate this limit.

$$\begin{aligned}\lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x^2 - 16} &= \lim_{x \rightarrow 4} \frac{(\sqrt{x} - 2)(\sqrt{x} + 2)}{(x - 4)(x + 4)(\sqrt{x} + 2)} \\ &= \lim_{x \rightarrow 4} \frac{x - 4}{(x - 4)(x + 4)(\sqrt{x} + 2)} \\ &= \lim_{x \rightarrow 4} \frac{1}{(x + 4)(\sqrt{x} + 2)} \\ &= \frac{\lim_{x \rightarrow 4} 1}{\lim_{x \rightarrow 4} (x + 4)(\sqrt{x} + 2)} \\ &= \frac{1}{(4 + 4)(\sqrt{4} + 2)} \\ &= \frac{1}{32}.\end{aligned}$$

- (b) This is in  $0/0$  form, and the derivative of the denominator is  $2 \sin x \cos x$ , which is not equal to 0 for  $x$  near (but not equal to) 0. We apply l'Hôpital's Rule and differentiate both the numerator and denominator.

$$\lim_{x \rightarrow 0} \frac{3x^2 + x \cos x - x}{\sin^2 x} = \lim_{x \rightarrow 0} \frac{6x + \cos x - x \sin x - 1}{2 \sin x \cos x}.$$

This is still in  $0/0$  form, and the derivative of the denominator is  $2(\cos^2 x - \sin^2 x)$ , which is not equal to 0 for  $x$  near (but not equal to) 0. We apply

l'Hôpital's Rule again.

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{6x + \cos x - x \sin x - 1}{2 \sin x \cos x} &= \lim_{x \rightarrow 0} \frac{6 - \sin x - \sin x - x \cos x}{2(\cos^2 x - \sin^2 x)} \\
 &= \lim_{x \rightarrow 0} \frac{6 - 2 \sin x - x \cos x}{2(\cos^2 x - \sin^2 x)} \\
 &= \frac{\lim_{x \rightarrow 0} (6 - 2 \sin x - x \cos x)}{\lim_{x \rightarrow 0} 2(\cos^2 x - \sin^2 x)} \\
 &= \frac{6 - 0 - 0}{2(1 - 0)} \\
 &= 3.
 \end{aligned}$$

Alternatively, we could have just used algebraic manipulations and the fact that  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$  and  $\lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = 0$ .

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{3x^2 + x \cos x - x}{\sin^2 x} &= \lim_{x \rightarrow 0} \left( \frac{3x^2}{\sin^2 x} + \frac{x \cos x - x}{\sin^2 x} \right) \\
 &= \lim_{x \rightarrow 0} \left( 3 \frac{1}{\left(\frac{\sin^2 x}{x^2}\right)} + \frac{x(\cos x - 1)}{\sin^2 x} \right) \\
 &= \lim_{x \rightarrow 0} 3 \frac{1}{\left(\frac{\sin x}{x}\right)^2} + \lim_{x \rightarrow 0} \left( \frac{x^2}{\sin^2 x} \right) \left( \frac{\cos x - 1}{x} \right) \\
 &= 3 \frac{1}{1^2} + \lim_{x \rightarrow 0} \frac{1}{\left(\frac{\sin x}{x}\right)^2} \lim_{x \rightarrow 0} \left( \frac{\cos x - 1}{x} \right) \\
 &= 3 + \frac{1}{1^2}(0) \\
 &= 3.
 \end{aligned}$$

- (c) This is in 0/0 form, and the derivative of the denominator is  $\cos x$ , which is not equal to 0 for  $x$  near (but not equal to) 0. We apply l'Hôpital's Rule and differentiate both the numerator and denominator.

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{\text{Arcsin}(x)}{\sin(x)} &= \lim_{x \rightarrow 0} \frac{\frac{1}{\sqrt{1-x^2}}}{\cos(x)} \\
 &= \lim_{x \rightarrow 0} \frac{1}{\cos(x)\sqrt{1-x^2}} \\
 &= \frac{1}{1\sqrt{1-0}} \\
 &= 1.
 \end{aligned}$$