MATH 1004 B2-B4 Fall 2014
Test One Solutions

1. (2 marks) Let $f(x) = \sqrt{x^2 - \cos x}$. Which of the following expressions represents the value of $f'(1)$? (*circle one answer*)

   (a) $\lim_{h\to 0} \dfrac{f(h) - f(0)}{h}$

   (b) $\lim_{h\to 0} \dfrac{f(h) - f(1)}{h}$

   (c) $\lim_{h\to 0} \dfrac{f(1+h) - f(1)}{h}$

   (d) $\lim_{h\to 0} \dfrac{f(1+h) - f(1-h)}{h}$

   Solution: The correct answer is (c), found by using the limit definition of the derivative.

2. (2 marks) Consider the function $f(x) = 3 - 2|x|$. Complete the following sentence: "At the point $x = 0$, $f(x)$ is..." (*circle one answer*)

   (a) continuous and differentiable.

   (b) differentiable but not continuous.

   (c) continuous but not differentiable.

   (d) neither continuous nor differentiable.

   Solution: The correct answer is (c). The function f has no gaps or jumps at $x = 0$, and so it is continuous there. In more detail, one can show that the left and right hand limits of f at $x = 0$ are both equal to 3, and so the limit exists and is equal to 3. Since $f(0) = 3$, we have $\lim_{x\to 0} f(x) = f(0)$, which implies that f is continuous at $x = 0$.
   However, f has a "corner" at $x = 0$, which makes it not differentiable there. In more detail, we have

   $$\lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^-} \frac{3 - 2|h| - 3}{h} = \lim_{h\to 0^-} \frac{2h}{h} = 2$$

   and

   $$\lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^+} \frac{3 - 2|h| - 3}{h} = \lim_{h\to 0^+} \frac{-2h}{h} = -2.$$

   Since the left and right derivatives are not equal, the derivative does not exist at $x = 0$.

3. (2 marks) Evaluate the limit $\lim_{x\to 0} \frac{1}{x^2}\sin^2(x)$. (*circle one answer*)

   (a) 0

   (b) 1

   (c) The limit does not exist.

   (d) $+\infty$

   Solution: The correct answer is (b). We have

$$\lim_{x\to 0} \frac{1}{x^2}\sin^2(x) = \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2 = \left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2 = 1^2 = 1.$$

4. (3 marks) Evaluate $\lim_{x\to -2^-} \frac{2-x}{2+x}$.

   Solution: As x approaches -2, the numerator $2-x$ is approaching 4. However, the denominator $2+x$ is approaching 0. The limit is one of $\pm\infty$, but we must determine the correct sign. As x approaches -2 from below, $2+x$ is close to 0 but negative. Since the numerator is positive and the denominator is negative, the limit is $-\infty$.

5. (3 marks) Evaluate $\lim_{x\to 0} \frac{\cos(2x)\tan(2x)}{x}$.

   Solution: We use some basic trig identities to transform the limit:

$$\begin{aligned}\lim_{x\to 0}\frac{\cos(2x)\tan(2x)}{x} &= \lim_{x\to 0}\frac{\cos(2x)\frac{\sin(2x)}{\cos(2x)}}{x}\\ &= \lim_{x\to 0}\frac{\sin(2x)}{x}\\ &= 2\lim_{x\to 0}\frac{\sin(2x)}{2x}\\ &= 2(1) = 2.\end{aligned}$$

   (to get to the last line, we used the fact that $\lim_{x\to 0}\frac{\sin(2x)}{2x} = 1$.

6. (4 marks) Evaluate $\lim_{x\to\infty} \frac{3-4x}{\sqrt{x^2+16}-9x}$.

Solution: We pull the largest possible power out of the numerator and the denominator.

$$\begin{aligned}
\lim_{x\to\infty} \frac{3-4x}{\sqrt{x^2+16}-9x} &= \lim_{x\to\infty} \frac{x\left(\frac{3}{x}-4\right)}{\sqrt{x^2\left(1+\frac{16}{x^2}\right)}-9x} \\
&= \lim_{x\to\infty} \frac{x\left(\frac{3}{x}-4\right)}{|x|\sqrt{1+\frac{16}{x^2}}-9x} \\
&= \lim_{x\to\infty} \frac{x\left(\frac{3}{x}-4\right)}{x\sqrt{1+\frac{16}{x^2}}-9x} \\
&= \lim_{x\to\infty} \frac{x\left(\frac{3}{x}-4\right)}{x\left(\sqrt{1+\frac{16}{x^2}}-9\right)} \\
&= \lim_{x\to\infty} \frac{\frac{3}{x}-4}{\sqrt{1+\frac{16}{x^2}}-9} \\
&= \frac{0-4}{\sqrt{1+0}-9} \\
&= \frac{1}{2}.
\end{aligned}$$

Note that we can replace $|x|$ with x since we are taking the limit as x approaches ∞, and so we can think of x as large and positive.

7. (4 marks) Use the limit definition of the derivative to find the slope of the tangent line to the curve $f(x) = \sqrt{x}$ at the point $x = 9$.

Solution: We have

$$\begin{aligned}
f'(9) &= \lim_{h\to 0} \frac{f(9+h)-f(9)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{9+h}-3}{h} \\
&= \lim_{h\to 0} \frac{(\sqrt{9+h}-3)(\sqrt{9+h}+3)}{h(\sqrt{9+h}+3)} \\
&= \lim_{h\to 0} \frac{9+h-9}{h(\sqrt{9+h}+3)} \\
&= \lim_{h\to 0} \frac{h}{h(\sqrt{9+h}+3)} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{9+h}+3} \\
&= \frac{1}{6}.
\end{aligned}$$

8. Find the derivative of each of the following functions (you may use **any valid method** and you **do not have to simplify** your answer):

   (a) (2 marks) $f(x) = \frac{3}{4}x^4 - 10x^2 + \pi^3$

   (b) (2 marks) $y(x) = \dfrac{1}{3x} - 4\sqrt{x}$

   (c) (3 marks) $H(u) = (2u^3 + 3u^2 - u)(u^2 - 4u - 2)$

   (d) (3 marks) $g(t) = \dfrac{2t + 1}{2t^2 + 1}$

   Solution:

   (a) Using the power rule, we have

   $$f'(x) = \frac{3}{4}(4x^3) - 10(2x) + 0 = 3x^3 - 20x.$$

   Note that π^3 is a constant, and so its derivative is 0.

   (b) Since $y(x) = \frac{1}{3x} - 4\sqrt{x} = \frac{1}{3}x^{-1} - 4x^{\frac{1}{2}}$, we have

   $$y'(x) = \frac{1}{3}(-1)x^{-2} - 4\left(\frac{1}{2}\right)x^{-\frac{1}{2}} = -\frac{1}{3}x^{-2} - 2x^{-\frac{1}{2}}.$$

   (c) It is much easier to use the product rule than it would be to expand the product before differentiating. We have

   $$\begin{aligned} H'(u) &= (2(3u^2) + 3(2u) - 1)(u^2 - 4u - 2) + (2u^3 + 3u^2 - u)(2u - 4) \\ &= (6u^2 + 6u - 1)(u^2 - 4u - 2) + (2u^3 + 3u^2 - u)(2u - 4). \end{aligned}$$

   (d) Using the quotient rule, we have

   $$\begin{aligned} g'(t) &= \frac{(2)(2t^2 + 1) - (2t + 1)(2(2t))}{(2t^2 + 1)^2} \\ &= \frac{2(2t^2 + 1) - 4t(2t + 1)}{(2t^2 + 1)^2} \end{aligned}$$