Solution to Midterm 2 (version A)

MAT1320C, Fall 2014

1. (3 mark) Find the derivative of function $y = x^{\ln x}$ at $x = e$.

Solution. $\ln y = (\ln x)^2$, $\frac{y'}{y} = \frac{2\ln x}{x}$, $y' = \frac{2x^{\ln x}\ln x}{x} = 2x^{\ln x - 1}\ln x$. $y'(e) = 2$.

2. (4 marks) Find the equation of the tangent line of the graph of the equation $3x^3y + y^3 + 3x + 1 = 0$ at the point $(-1, 2)$.

Solution. $3(3x^2y + x^3y') + 3y^2y' + 3 = 0$. When $x = -1$, $y = 2$, $3(6 - y') + 12y' + 3 = 0$, $18 - 3y' + 12y' + 3 = 0$. $21 = -9y'$, $y' = -\frac{7}{3}$. The equation of the tangent line is $y = -\frac{7}{3}(x + 1) + 2$, or $y = -\frac{7}{3}x - \frac{1}{3}$.

3. (4 marks) A boy is flying a kite. The string of the kite is being paid out at a rate of 10 meters per minute (i.e., the distance between the kite and the boy is increasing at a rate 10 meters per minute). The kite is moving horizontally at a height 150 meters above the ground. What is the speed of the kite when it is 250 meters away from the boy?

Solution. Let the distance between the boy and the kite be D, and let the horizontal distance between the boy and the kite be x. D and x are functions of time t.

By Pythagorean's Theorem, $D^2 = x^2 + 150^2$.

Taking the derivative on both sides of this relation with respect to t, we have $2DD' = 2xx'$. Hence, $x' = \frac{DD'}{x}$. Since $D = 250$, $D' = 10$, and, when $D = 250$, $x = \sqrt{250^2 - 150^2} = 200$, $x' = \frac{250 \times 10}{200} = 12.5$ meters per minute.

4. (4 marks) (a) (3 marks) Find the linear approximation of the function $f(x) = \sqrt{x^3 + 1}$ at $x = 2$.

(b) (1 mark) Estimate $f(1.9)$ using the linear approximation you obtained in part (a).

Solution. (a) $f'(x) = \frac{3x^2}{2\sqrt{x^3+1}}$. When $x = 2$, $f'(2) = 2$. The linear approximation of $f(x)$ at $x = 2$ is $y = 2(x - 2) + 3$, $y = 2x - 1$.

(b) $f(1.9) \approx 2 \times 1.9 - 1 = 2.8$.

5. (3 marks) Let $F(x) = \int_0^{x^2} \sin(t^2)dt$. Find $\frac{d}{dx}F(x)$.

Solution. $\frac{d}{dx}F(x) = 2x\sin(x^4)$.

6. (8 marks) Evaluate each of the following definite integrals:

(a) $I_1 = \int_1^4 \frac{\sqrt{x}}{\sqrt{x}+1}dx$.

Solution. Let $u = \sqrt{x} + 1$. Then $u' = \frac{1}{2\sqrt{x}}$.

$$I_1 = \int_1^4 \frac{\sqrt{x}}{\sqrt{x}+1}dx = \int_2^3 \frac{2x}{\sqrt{x}+1}du = 2\int_2^3 \frac{(u-1)^2}{u}du = 2\int_2^3 \left(u-2+\frac{1}{u}\right)du = 2\left[\frac{u^2}{2}-2u+\ln u\right]_{u=2}^3$$

$= 2(\ln 3 - \ln 2) + 1$.

(b) $I_2 = \int_e^{e^2} \frac{1}{x(\ln x)^2}dx$.

Solution. Let $u = \ln x$. Then $u' = \frac{1}{x}$.

$$I_2 = \int_e^{e^2} \frac{1}{x(\ln x)^2}dx = \int_1^2 u^{-2}du = -[u^{-1}]_{u=1}^2 = \frac{1}{2}.$$

(c) $I_3 = \int_1^e \frac{\ln x}{\sqrt{x}}dx$.

Solution. Use integration by parts: $\int \frac{1}{\sqrt{x}}dx = 2\sqrt{x} + C$.

$$I_3 = \int_1^e \frac{\ln x}{\sqrt{x}}dx = \int_1^e \ln x\, d(2\sqrt{x}) = \left[2\sqrt{x}\ln x\right]_{x=1}^e - 2\int_1^4 \frac{\sqrt{x}}{x}dx = 2\sqrt{e} - 4\sqrt{e} - 4 = 4 - 2\sqrt{e}.$$

(d) $I_4 = \int_0^{\pi/6} \cos^3 x dx$.

Solution. Let $u = \sin x$. Then $u' = \cos x$.

$I_4 = \int_0^{\pi/6} \cos^3 x dx = \int_0^{1/2} (1-u^2) du = \left[u - \frac{1}{3}u^3 \right]_{u=0}^{1/2} = \frac{11}{24}$.

7. (4 marks) Consider function $f(x) = \frac{x^2-1}{\sqrt{x}}$.

(a) (2 marks) Use The midpoint rule with $n = 3$ to estimate the area under the graph of $f(x)$ above the x-axis in the interval [1, 4]. (Use 4 digits after the decimal point in your calculation).

(b) (2 marks) Use a definite integral to find the accurate value of the area.

Solution. (a) $h = \frac{4-1}{3} = 1$. The subintervals are [1, 2], [2, 3], and [3, 4]. The midpoints are 1.5, 2.5, and 3.5.

$f(1.5) \approx 1.0206$, $f(2.5) \approx 3.3204$, $f(3.5) \approx 6.0134$.

This area is approximately $A \approx 1 \times (1.0206 + 3.3204 + 6.0134) = 10.3544$

(b) $A = \int_1^4 \frac{x^2-1}{\sqrt{x}} dx = \int_1^4 (x^{3/2} - x^{-1/2}) dx = \left[\frac{2}{5}x^{5/2} - 2x^{1/2} \right]_{x=1}^{4} = \frac{52}{5}$.