MAT 1330, Fall 2014 Assignment 3

Due Thursday October 9 by 9:00pm.

Late assignments will not be accepted; nor will unstapled assignments. Professors in the math department will not lend you a stapler; do not ask for one.

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Student Name ____________________ Student Number ____________________

By signing below, you declare that this work was your own and that you have not copied from any other individual or other source.

Signature ____________________

QUESTION 1. Let $f(x) = \begin{cases} \frac{x^2-4x+3}{(x-1)^3} & \text{if } x \neq \pm 1, \\ x+b & \text{if } x = \pm 1. \end{cases}$, with b a real number

(a) Find $\lim_{x\to 1} f(x)$. $-\infty$

Show your work:

First notice that

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2-4x+3}{(x-1)^3}$$

since for every x close to but not equal to 1, the value of $f(x)$ is given by that formula. If direct substitution (replacing x in the formula with 1) gave a valid number, then that number would equal the limit (since algebraic functions are continuous where defined). Unfortunately here we get 0/0. So we have to algebraically simplify. The denominator is already factored; the numerator is easily seen to factor as $(x-1)(x-3)$. So

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2-4x+3}{(x-1)^3} = \lim_{x\to 1} \frac{(x-1)(x-3)}{(x-1)^3} = \lim_{x\to 1} \frac{x-3}{(x-1)^2}.$$

This time direct substitution gives $-2/0$, telling us that there is a vertical asymptote at $x = 1$. Also, for x close to 1, the numerator is negative (close to -2) and the denominator is positive (it's a square). So as x gets close to 1, the expression $\frac{x-3}{(x-1)^2}$ gets arbitrarily large in magnitude but is always negative. Hence the limit is $-\infty$.

(b) Is there a value of b that makes the function f continuous at $x = 1$? If yes, then provide the value. No

Explain your answer:

Since the limit $\lim_{x\to 1} f(x)$ is not a finite number, there is no value of $f(1)$ that will give the identity $\lim_{x\to 1} f(x) = f(1)$. That is, no choice of b will make f continuous at $x = 1$.

(Q1 continued)(c) Find $\lim_{x \to -1} f(x)$. $\boxed{-1}$

Show your work:

This time the fraction is defined at $x = -1$, so we can do direct substitution:

$$\lim_{x \to -1} f(x) = \lim_{x \to -1} \frac{x^2 - 4x + 3}{(x-1)^3} = \frac{1 + 4 + 3}{(-2)^3} = -1$$

(d) Is there a value of b that makes the function f continuous at $x = -1$? If yes, then provide the value. $\boxed{\text{Yes, } 0}$

Explain your answer:

We know that $f(-1) = -1 + b$, and that $\lim_{x \to -1} f(x) = -1$. The function f is continuous at $x = -1$ exactly when those two expressions are equal, that is, when $b = 0$.

QUESTION 2. Does the limit $\lim_{x \to -\infty} e^{1/x}$ exist? If so, what is its value?

Answer: $\boxed{1}$

Justify your answer (two lines could be enough) without using sequences of numerical values for x.

Answer:

As x approaches ∞, $1/x$ approaches 0 from the left. Since the exponential function is continuous, we conclude that $e^{1/x}$ approaches $e^0 = 1$.

Question 3. Let f be the function given by $f(x) = \dfrac{1}{x^2+3}$. Use the definition of the derivative to compute $f'(2)$. Show all your work.

By definition,

$$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{\frac{1}{x^2+3} - \frac{1}{2^2+3}}{x-2}.$$

Note that we can't just plug in $x = 2$ to evaluate this limit, that would give the nonsensical result $0/0$; although the complicated fraction is continuous wherever it's defined, it isn't defined at $x = 2$. Let's simplify that complicated fraction:

$$\begin{aligned} \frac{\frac{1}{x^2+3} - \frac{1}{2^2+3}}{x-2} &= \frac{\frac{(2^2+3)-(x^2+3)}{(x^2+3)(2^2+3)}}{x-2} = \frac{4 - x^2}{(x^2+3)(2^2+3)(x-2)} \\ &= \frac{(2-x)(2+x)}{(x^2+3)(2^2+3)(x-2)} = -\frac{2+x}{(x^2+3)(2^2+3)}. \end{aligned}$$

Therefore

$$f'(2) = \lim_{x \to 2} -\frac{2+x}{(x^2+3)(2^2+3)}.$$

This time we can just plug in $x = 2$ to get the answer, as the simplified fraction is defined at $x = 2$ (and is continuous wherever it's defined):

$$f'(2) = -\frac{2+2}{(2^2+3)(2^2+3)} = -\frac{4}{49}$$