University of Ottawa
MAT1330 A Midterm Exam

October 22, 2014. Duration: 80 minutes. Instructor: Frithjof Lutscher.

Family Name:______________________

First Name:________________________

DGD 1 DGD 2 DGD 3 DGD 4

Do **not** write your student ID number on this front page. Please write your student ID number in the space provided on the second page.

Take your time to read the entire paper before you begin to write, and read each question carefully. Remember that certain questions are worth more points than others. Make a note of the questions that you feel confident you can do, and then do those first: <u>you do not</u> have to proceed through the paper in the order given.

- You have 80 minutes to complete this exam.
- This is a closed book exam, and no notes of any kind are allowed. The use of cell phones, pagers or any text storage or communication device **is not permitted.**
- Only the Faculty approved calculators (TI-30X, TI-34X, Casio FX-260X and Casio FX-300X) are allowed. No exceptions.
- The correct answer requires justification written legibly and logically: you must convince me that you know why your solution is correct. Answer these questions in the space provided. Use the backs of pages if necessary.
- Where it is possible to check your work, do so.
- Please do not detach the pages.
- Good luck!

Student number:____________________, Total marks: __________ out of 30

Problem	1	2	3	4	5	6
Marks						

Question 1. Find the following limits without using sequences of numerical values.

(a) [3 points] $\lim_{x\to 2} \frac{x^2 - \frac{3}{2}x - 1}{x-2}$ $= \lim_{x\to 2} \frac{(x-2)(x+\frac{1}{2})}{x-2} = \lim_{x\to 2} \left(x+\frac{1}{2}\right) = \frac{5}{2}$

Answer: $\boxed{\frac{5}{2}}$

(b) [3 points] $\lim_{x\to\infty} \frac{7x^2}{\sqrt{4x^4+3}}$

$$= \lim_{x\to\infty} \frac{7}{\sqrt{4 + \frac{3}{x^4}}} = \frac{7}{2}$$

Answer: $\boxed{\frac{7}{2}}$

Question 2. [3 points] Use the definition of the derivative to calculate the derivative of the function

$$f(x) = 7 + \sqrt{2x+5}.$$

$$f'(x) = \lim_{h \to 0} \frac{1}{h}\left[f(x+h) - f(x) \right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[7 + \sqrt{2(x+h)+5} - (7 + \sqrt{2x+5}) \right]$$

$$= \lim_{h \to 0} \frac{\sqrt{2(x+h)+5} - \sqrt{2x+5}}{h} \cdot \frac{\sqrt{2(x+h)+5} + \sqrt{2x+5}}{\sqrt{2(x+h)+5} + \sqrt{2x+5}}$$

$$= \lim_{h \to 0} \frac{2(x+h)+5 - (2x+5)}{h\left(\sqrt{2(x+h)+5} + \sqrt{2x+5}\right)}$$

$$= \lim_{h \to 0} \frac{2h}{h\left(\sqrt{2(x+h)+5} + \sqrt{2x+5}\right)}$$

$$= \lim_{h \to 0} \frac{2}{\sqrt{2(x+h)+5} + \sqrt{2x+5}}$$

$$= \frac{1}{\sqrt{2x+5}}$$

Question 3. Use the differentiation rules from class to find the derivatives $f'(x)$ for the following functions. Do not simplify your answers.

(a) [3 points] $f(x) = \ln\left(7 + x^2 \ln(x)\right)$

$$f'(x) = \frac{2x \ln x + x^2 \cdot \frac{1}{x}}{7 + x^2 \ln(x)}$$

Answer: $f'(x) =$

(b) [3 points] $f(x) = \left(\frac{\sqrt{x+1}}{e^{x^2}+5}\right)^8$

$$f'(x) = 8\left(\frac{\sqrt{x+1}}{e^{x^2}+5}\right)^7 \frac{\frac{1}{2\sqrt{x+1}}\left(e^{x^2}+5\right) - \sqrt{x+1} \cdot 2x e^{x^2}}{\left(e^{x^2}+5\right)^2}$$

Answer: $f'(x) =$

Question 4. [3 points] Can one choose a value for a such that the following function is continuous at $x = 3$? If yes, what is the value and why? If no, why not?

$$f(x) = \begin{cases} \frac{|x-3|}{x^2-9}, & x \neq 3 \\ x + a, & x = 3. \end{cases}$$

Answer:
$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} \frac{x-3}{x^2-9} = \lim_{x \to 3^+} \frac{1}{x+3} = \frac{1}{6}$$
$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \frac{-(x-3)}{x^2-9} = \lim_{x \to 3^-} \frac{-1}{x+3} = -\frac{1}{6}$$

Since these two limits are not equal, no value of a would make f continuous.

Question 5. [4 points] Use implicit differentiation to find $\frac{dy}{dx}$ when x and y satisfy the relation

$$e^{2x\sqrt{y}} = y^5.$$

Differentiate: $$e^{2x\sqrt{y}} \cdot \left(2\sqrt{y} + \frac{2x}{2\sqrt{y}} y'\right) = 5y^4 y'$$

Sort: $$y'\left[\frac{x}{\sqrt{y}} e^{2x\sqrt{y}} - 5y^4\right] = -2\sqrt{y}\, e^{2x\sqrt{y}}$$

Solve $$y' = \frac{-2\sqrt{y}\exp(2x\sqrt{y})}{\frac{x}{\sqrt{y}}\exp(2x\sqrt{y}) - 5y^4}$$

Answer: $\frac{dy}{dx} =$

Question 6. (a) [2 points] Because of high mortality and low reproductive success some fish species experience exponential decline over many years. Atlantic Salmon in Lake Ontario, for example, declined by 80% in the 20-year period leading up to 1896. Denote the number of Atlantic Salmon in Lake Ontario in year t by x_t and write the equation $x_{t+1} = rx_t$. Calculate the value of r.

$$x_{20} = r^{20} x_0 \qquad \text{and} \qquad x_{20} = 0.2 \cdot x_0$$

$$\Rightarrow \quad r^{20} x_0 = 0.2 x_0$$

$$\Rightarrow \quad r^{20} = 0.2 \quad \Rightarrow \quad r = \sqrt[20]{0.2} \approx 0.9227$$

Answer: $r =$ $\boxed{}$

(b) [6 points] Due to fishing restrictions and a massive restocking program, the population is recovering now. In the equation above, the value of r has changed and an additional term is included. The population dynamics can now be described by the DTDS

$$x_{t+1} = 0.2x_t + c,$$

where c is the number of fish restocked every year.

The updating function of this DTDS is $f(x) =$ $\boxed{0.2x + c}$

The equilibrium point of this DTDS is $x^* =$ $\boxed{\frac{c}{1-0.2} = \frac{c}{0.8} = \frac{5}{4}c}$

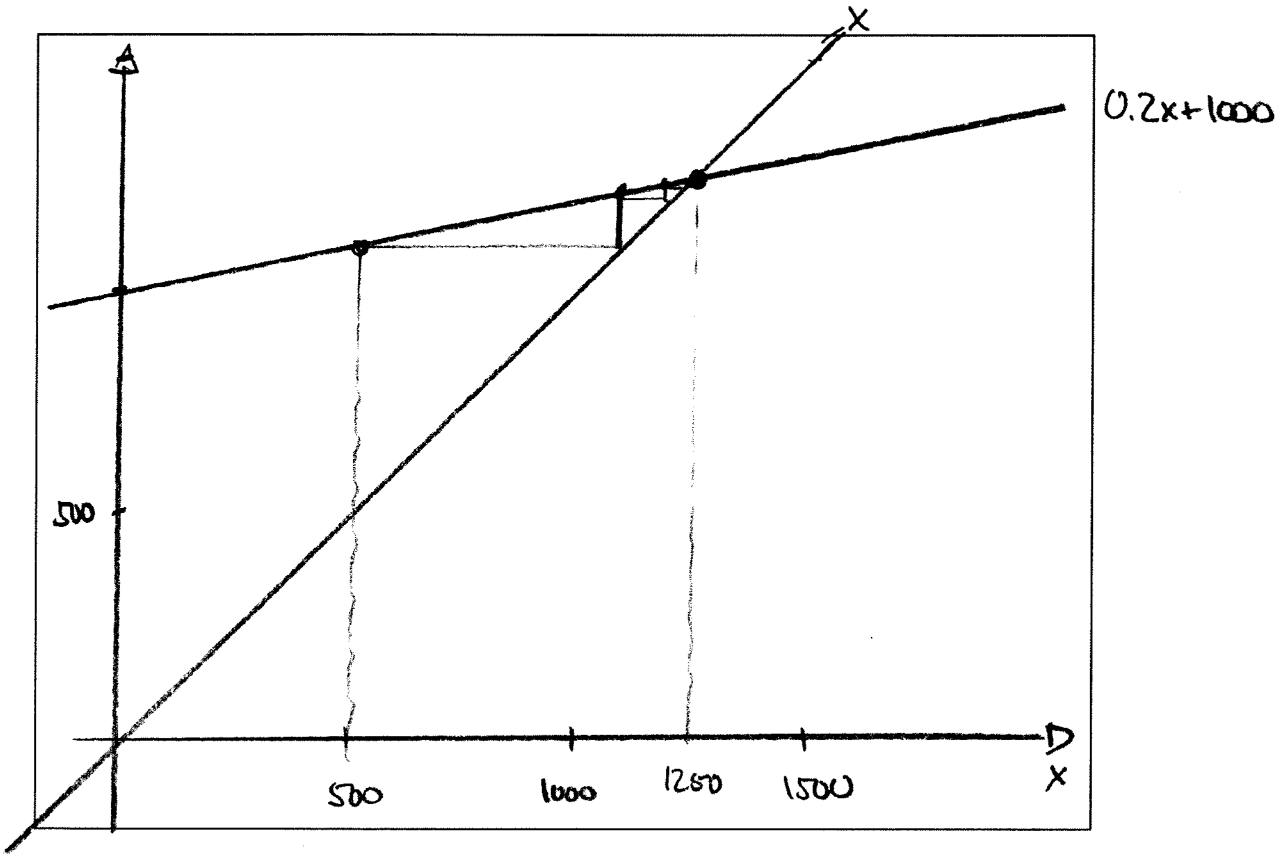
Is the equilibrium point stable or unstable for the DTDS? Why?

Answer: Stable, since $x_{t+1} = rx_t + c$ is linear and $|r| = 0.2 < 1$

Now we assume that there are 1000 fish restocked annually. The general solution formula for the DTDS then is

$x_t =$ $\boxed{r^t(x_0 - x^*) + x^* = (0.2)^t(x_0 - 1250) + 1250}$

Graph the updating function and draw the cobweb diagram of the DTDS, starting from $M_0 = 500$ for at least 4 steps.



How does the number of restocked fish need to be adjusted to ensure an equilibrium population of 1500 fish? Answer: $\boxed{c = 1200}$

$$x^* = \frac{5}{4}c = 1500 \Rightarrow c = \frac{4}{5} \cdot 1500 = 1200$$