Department of Mechanical and Aerospace Engineering
CARLETON UNIVERSITY

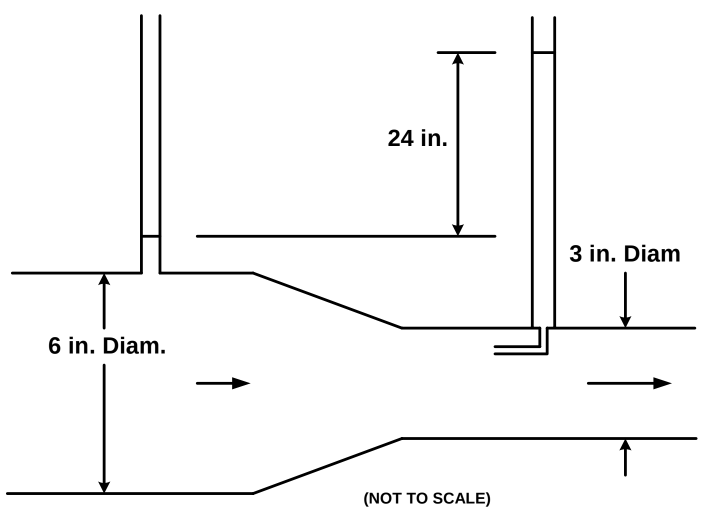
MECH 3310 Biofluid Mechanics
PROBLEM SET #2

1. The wall shear stress in a tube or pipe determines the pressure drop with downstream distance. The deposition of particles on surfaces (such as the build up of plaque due the cholesterol on the walls of blood vessels) is also related to the local wall shear stress.

   (a) Derive the expression for the wall shear stress, τ_w, for fully-developed laminar pipe flow of a Newtonian fluid as a function of the volume flow rate of the fluid and any other relevant variables.
   (b) For a typical human, the diameter of the aorta is 2.5 cm, the average flow rate is 5.5 lpm (litres per minute) and peak flow rate during systole is 20 lpm. Assume the blood has a viscosity of $\mu = 0.0035$ N-s/m^2 and a density of 1060 kg/m^3.
   (i) Determine the Reynolds number and wall shear stress for the aorta under the average flow conditions.
   (ii) Determine the Reynolds number and wall shear stress for the aorta under the peak flow conditions.
   (Ans. (b) (i) 1414, 0.21 Pa)

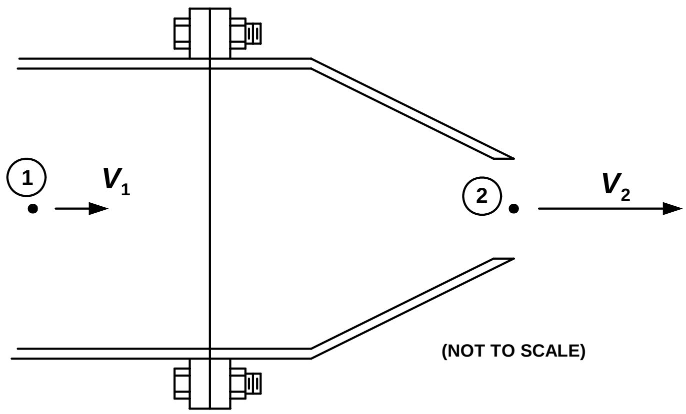
2. Blood is flowing through a large artery with a diameter of 1 cm at an average velocity of 40 cm/s and at a mean pressure of 100 mm Hg. The blood enters a region of stenosis where the diameter is only 0.5 cm. Neglect elevation changes along the artery. Estimate the mean pressure in mm Hg at the stenosis.
   (Ans. 90.5 mm Hg)

3. Calculate the ideal volume flow rate of the unknown liquid in the horizontal pipeline shown. (Ans. 2.23 ft^3/sec)



4. Water is discharging to atmosphere through the nozzle shown. The nozzle is attached to the pipe at the flanged joint. Measurements at points 1 and 2 indicate the following conditions:

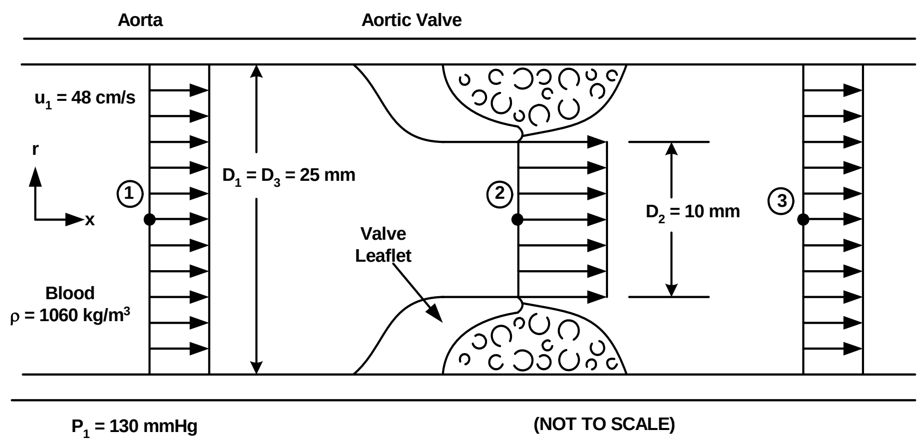
P_1 = 100 psig	P_2 = 1 atm
V_1 = 11.11 ft/sec	V_2 = 100 ft/sec
D_1 = 6 in.	D_2 = 2 in.



Water has a density of 62.4 lb_m/ft^3.

(a) Is continuity satisfied by the specified conditions? Are the specified conditions consistent with frictionless flow?
(b) For the conditions specified, determine the force in lb_f in the flanged joint required to hold the nozzle attached to the pipe. Is the flange in tension or compression?
(Ans. (a) Yes, no. (b) 2450 lbf)

5. The sketch shows an idealized representation of the aortic valve between the left ventricle and the aorta. One of the symptoms of a stenotic (partially blocked) valve is increased pressure drop across it. Another serious problem is incomplete closure leading to reverse flow of blood back into the ventricle (called regurgitation).

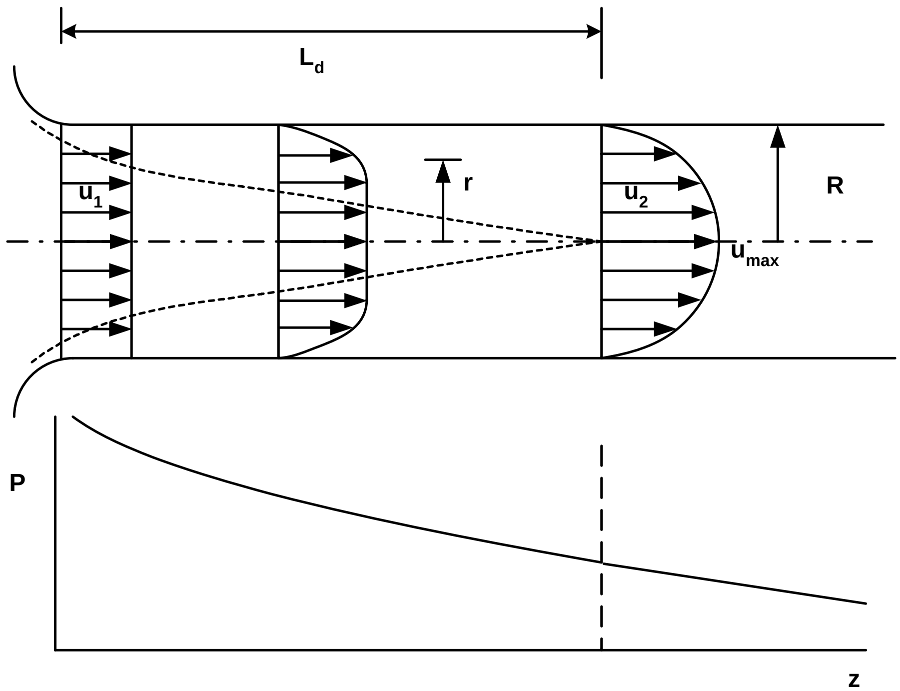


Assume: (i) One-dimensional flow at all streamwise stations.
(ii) The aortic valve is a tricuspid valve and thus has three leaflets. Therefore, the opening is approximately triangular. Approximate the opening by a circle with diameter D_2 = 10 mm.
(iii) Distances from 1 to2 and 2 to 3 are short (only a few vessel diameters).
(iv) Elevation changes from 1 to 3 can be neglected.

(a) Estimate the pressure drop across the valve, from planes 1 to 3. What percentage is this of the pressure drop through the entire circulatory system (that is, what percentage of the pressure rise that must be generated by the heart to maintain circulation)?
(b) Estimate the fluid dynamic force exerted on the valve leaflets in the flow direction.

(Ans. 3400 Pa, 1.65 N)

6. The static pressure in a tube is observed to drop with downstream distance.



In the fully-developed portion of the flow, the pressure drop is due to internal energy generation through viscous effects. In Fluids I, you learned to estimate this pressure drop using the friction factor obtained from the Moody chart. Viscous effects are also present in the developing region near the inlet of the tube. However, in this region there are also changes in the kinetic energy of the fluid: the kinetic energy increases on the centreline while it decreases near the wall. Do these changes in kinetic energy make any net contribution to the pressure change in the developing region? Answer the question by determining the pressure change over the development length, neglecting the contribution from viscous effects. Express the result as a non-dimensional pressure coefficient:

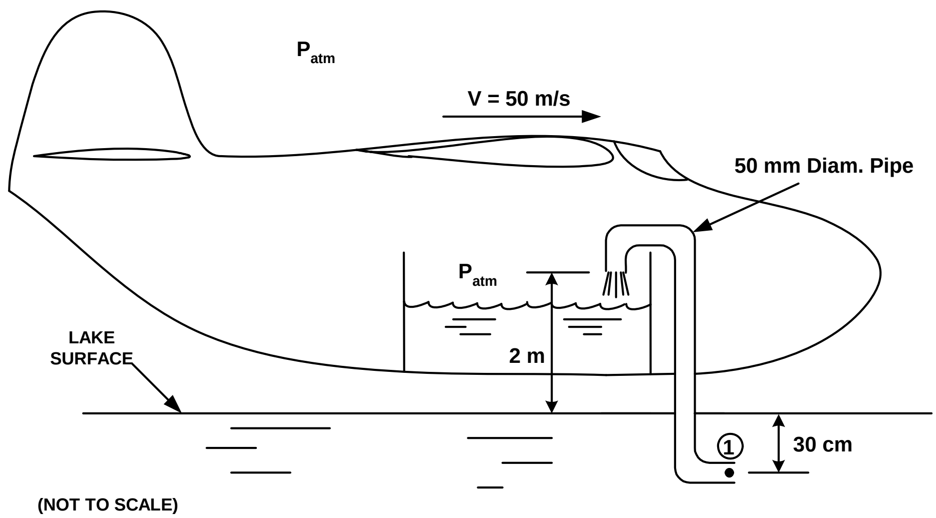
$$C_P = \frac{P_1 - P_2}{\frac{1}{2}\rho u_1^2}$$

Assume that the flow is laminar and that in the fully-developed region the velocity profile therefore has a parabolic distribution:

$$\frac{u_2}{u_{\max}} = \left(1 - \frac{r^2}{R^2}\right)$$

(Ans. $C_P = +1.0$)

7. A water bomber picks up a load of water from a lake using a 50 mm diameter pipe, as shown schematically in the drawing. The aircraft flies at a steady 50 m/s during the pick-up. The pressure inside the aircraft is the same as the outside atmospheric pressure.



(a) Find the pressure (relative to atmospheric pressure) and the flow velocity at point 1, just inside the mouth of the pipe. (Ans. 22,600 Pa(g), 49.6 m/s, neglecting losses. But is it valid to neglect losses?)
(b) Find the time required to pick up a full load of 1500 kg of water. (Ans. 15.4 s, neglecting losses)
(c) The aircraft is driven with propellers. What extra thrust and power (in HP) is required to maintain speed while the pick-up is occurring? Assume that the drag of the aircraft itself stays constant. (Ans. 3060 N, 205 HP, neglecting losses)
(d) The owners of the aircraft would like to decrease the time required to pick up the full load of water. However, for reasons related to aircraft handling, they do not want to change the flight speed during pick-up from the 50 m/s. Also, when the water scoop is not in use, it is retracted into the fuselage. Since they do not want to change the opening in the fuselage, it is also not possible to change the diameter of the intake from the 50 mm it now has. Within these constraints, suggest some alternative modifications to the system (preferably simple ones) that would increase the flow rate of water during the pick-up. Choose one of the modifications that you are suggesting and estimate the new time required to pick up the full load of water.

8. As an improvement over the Newtonian fluid assumption, a power law has been proposed as an approximate relationship between the shear stress and the rate of shear strain for blood:

$$\tau = K_{PL}\left(\dot{\gamma}\right)^n \qquad (1)$$

where n is usually taken to be about 0.75.

(a) What are the units of K_{PL}? In other words, does K_{PL} have any fundamental physical meaning?

(b) For a pipe flow, the velocity distribution that behaves according to (1) can be shown to be

$$\frac{u}{u_{\max}} = \left(1 - \left(\frac{r}{R}\right)^{\frac{n+1}{n}}\right)$$

where u_{max} is the velocity at r = 0 (on the centreline) and

$$u_{\max} = \frac{n}{n+1}\left(\frac{1}{2K_{PL}}\frac{dP}{dx}\right)^{\frac{1}{n}} R^{\frac{n+1}{n}}$$

Using this velocity distribution, show that the volume flow rate in the flow is given by

$$Q = u_{\max}\pi R^2 \frac{n+1}{3n+1}$$

(c) What is the relationship between $u_{\max}$ and the mean velocity $\bar{u}$ Recall that for Newtonian fluid flow for the same geometry $u_{\max} = 2.0\,\bar{u}$. What is the corresponding factor for the power law fluid? (Ans. 1.857)

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1. Wall shear stress in fully-developed pipe flow

(a) Wall shear stress. From Section 2.4.2 of the notes, rate of shear strain

$$\gamma = \frac{4 \cdot u_{mean}}{R} \cdot \left(\frac{r}{R}\right)$$ and at the wall $r = R$

then $$\gamma_w = \frac{4 \cdot u_{mean}}{R}$$

For a Newtonian fluid $$\tau_w = \mu \cdot \gamma_w = \mu \cdot \frac{4 \cdot u_{mean}}{R}$$

and the volume flow rate is $Q = \pi \cdot R^2 \cdot u_{mean}$ or $u_{mean} = \frac{Q}{\left(\pi \cdot R^2\right)}$

Then $$\tau_w = \frac{4 \cdot \mu \cdot Q}{\pi \cdot R^3}$$

(b) Blood flow in aorta

$D := 2.5$ cm $R := \frac{D}{2} \cdot \frac{1}{100}$ $R = 0.013$ m

$\mu := 0.0035$ N-s/m2 $\rho := 1060$ kg/m3

(i) Under average conditions

$Q := 5.5$ lpm $Q := \frac{Q \cdot 10^{-3}}{60}$ $Q = 9.167 \times 10^{-5}$ m3/s

Corresponding velocity $V := \frac{Q}{\pi \cdot R^2}$ $V = 0.187$ m/s

$V \cdot 100 = 18.674$ cm/s

Reynolds number $Re := \frac{\rho \cdot V \cdot (2 \cdot R)}{\mu}$ $Re = 1414$

Wall shear stress $\tau_w := \frac{4 \cdot \mu \cdot Q}{\pi \cdot R^3}$ $\tau_w = 0.209$ Pa

(ii) Under peak flow conditions

$Q := 20$ lpm $Q := \frac{Q \cdot 10^{-3}}{60}$ $Q = 3.333 \times 10^{-4}$ m3/s

Corresponding velocity $V := \frac{Q}{\pi \cdot R^2}$ $V = 0.679$ m/s

$V \cdot 100 = 67.906$ cm/s

Reynolds number $Re := \frac{\rho \cdot V \cdot (2 \cdot R)}{\mu}$ $Re = 5141$

Wall shear stress $\tau_w := \frac{4 \cdot \mu \cdot Q}{\pi \cdot R^3}$ $\tau_w = 0.761$ Pa

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2. Arterial pressure at location of a stenosis

$\rho := 1060$ kg/m3 $\rho_{Hg} := 13600$ kg/m3 $g := 9.81$ m/s2

In the healthy region $D := 1.0$ cm $D := \frac{D}{100}$ $D = 0.01$ m

$A := \frac{\pi}{4} \cdot D^2$ $A = 7.854 \times 10^{-5}$ m2

$V := 40$ cm/s $V := \frac{V}{100}$ $V = 0.4$ m/s

$h := 100$ mmHg(g) $P := \rho_{Hg} \cdot g \cdot \frac{h}{1000}$ $P = 1.334 \times 10^4$ Pa(g)

In stenosis area $D_s := 0.5$ cm $D_s := \frac{D_s}{100}$ $D_s = 5 \times 10^{-3}$ m

$A_s := \frac{\pi}{4} \cdot D_s^2$ $A_s = 1.963 \times 10^{-5}$ m2

Assume the distance from the healthy to the stenotic section short and relatively smooth so that we can neglect viscous effects. Also, assume 1-D flow. Then from continuity

$$V \cdot A = V_s \cdot A_s$$

$$V_s := \frac{V \cdot A}{A_s} \qquad V_s = 1.6 \quad \text{m/s}$$

Apply Bernoulli to find P_s

$$P + \frac{1}{2} \cdot \rho \cdot V^2 = P_s + \frac{1}{2} \cdot \rho \cdot V_s^2$$

$$P_s := P + \frac{1}{2} \cdot \rho \cdot V^2 - \frac{1}{2} \cdot \rho \cdot V_s^2$$

$$P_s = 1.207 \times 10^4 \quad \text{Pa(g)}$$

then $$h_s := \frac{P_s \cdot 1000}{\rho_{Hg} \cdot g} \qquad h_s = 90.466 \quad \text{mm Hg(g)}$$

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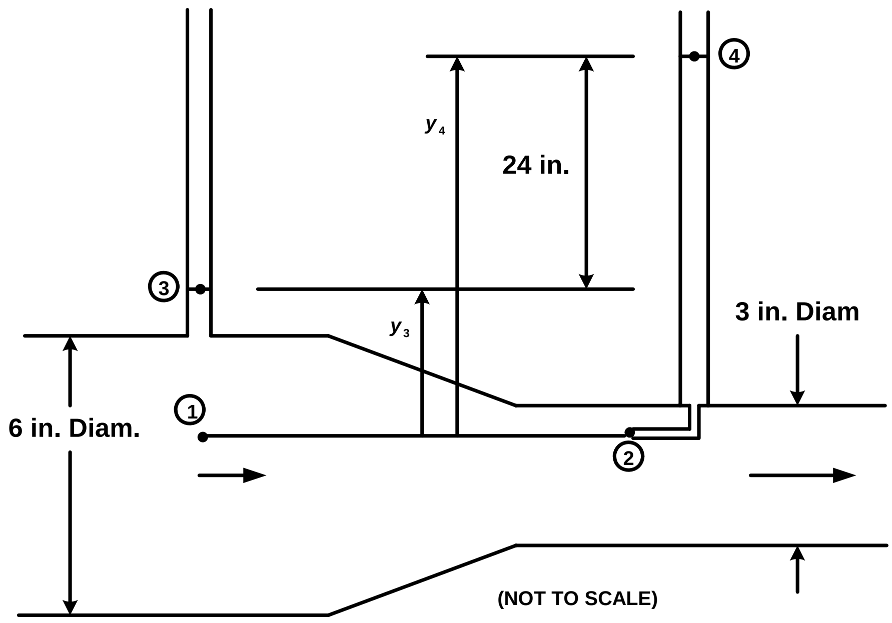
3. Flow meter



Finding volume flow rate, assuming ideal flow. By convention, ideal flow is 1-D flow with no viscous effects. Therefore we can apply Bernoulli from 1 to 2

$$P_1 + \frac{1}{2}\cdot\rho\cdot V_1^{\,2} = P_2 + \frac{1}{2}\cdot\rho\cdot V_2^{\,2} = P_{02} = P_{01}$$ since the second tube is a Pitot tube

then $$V_1 = \sqrt{\frac{2}{\rho}\cdot\left(P_{01} - P_1\right)}$$

From the manometers, using the level of the Pitot tube as the datum

$$P_1 = \rho\cdot g\cdot y_3 + P_{atm}$$

$$g := 32.174$$ ft/sec2

$$P_{01} = \rho\cdot g\cdot y_4 + P_{atm}$$

Then $$V_1 = \sqrt{\frac{2}{\rho}\cdot\left[\rho\cdot g\cdot y_4 + P_{atm} - \left(\rho\cdot g\cdot y_3 + P_{atm}\right)\right]}$$

$$V_1 = \sqrt{2\cdot g\cdot\left(y_4 - y_3\right)}$$

$$V_1 := \sqrt{2\cdot g\cdot\frac{24}{12}}$$ $$V_1 = 11.344$$ ft/sec

Then volume flow rate

$D_1 := \frac{6}{12}$ $D_1 = 0.5$ ft $A_1 := \frac{\pi}{4} \cdot D_1{}^2$ $A_1 = 0.196$ ft^2

$Q := V_1 \cdot A_1$ $Q = 2.227$ ft^3/sec

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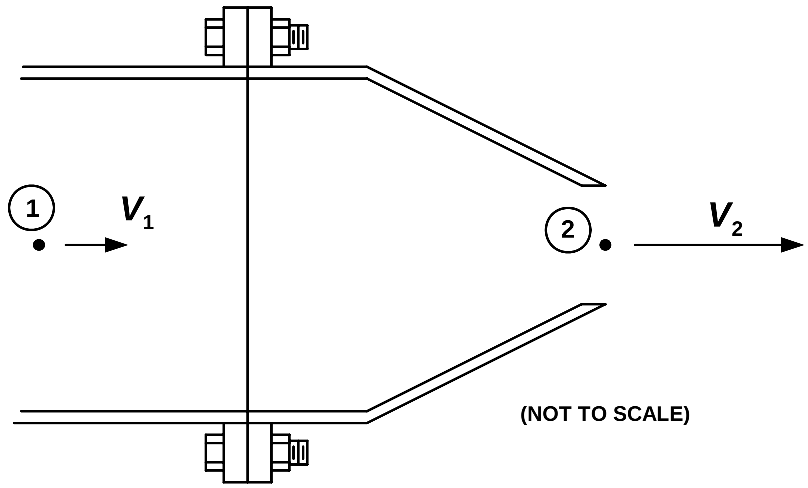
4. Nozzle attached to a pipe



(a) Are the given data consistent with continuity and frictionless flow?

Given data: $P_{1g} := 100$ psig $P_{1g} := P_{1g} \cdot 144$ $P_{1g} = 1.44 \times 10^4$ psfg

$V_1 := 11.11$ ft/sec

$D_1 := 6$ in $D_1 := \frac{D_1}{12}$ ft then $A_1 := \frac{\pi}{4} \cdot D_1^2$ $A_1 = 0.196$ ft^2

$P_2 = P_{atm}$

$V_2 := 100.0$ ft/sec

$D_2 := 2$ in $D_2 := \frac{D_2}{12}$ ft then $A_2 := \frac{\pi}{4} \cdot D_2^2$ $A_2 = 0.022$ ft^2

$\rho := 62.4$ lb$_m$/ft^3 $\rho := \frac{\rho}{32.174}$ $\rho = 1.939$ slug/ft^3

Verifying that the specified data satisfies continuity

$Q_1 := A_1 \cdot V_1$ $Q_1 = 2.181$ ft^3/sec

$Q_2 := A_2 \cdot V_2$ $Q_2 = 2.182$ ft^3/sec (close enough)

Frictionless flow?

For frictionless flow $P_{01} = P_{02}$ (i.e. Bernoulli is satisfied)

$$P_{01} = P_{1a} + \frac{1}{2} \cdot \rho \cdot V_1^2 = P_{1g} + P_{atm} + \frac{1}{2} \cdot \rho \cdot V_1^2$$

$$P_{02} = P_{2a} + \frac{1}{2}\cdot\rho\cdot V_2^{\,2} = P_{atm} + \frac{1}{2}\cdot\rho\cdot V_2^{\,2}$$

Then $$\Delta P_0 = P_{01} - P_{02} = P_{1g} + \left(\frac{1}{2}\cdot\rho\cdot V_1^{\,2}\right) - \frac{1}{2}\cdot\rho\cdot V_2^{\,2}$$

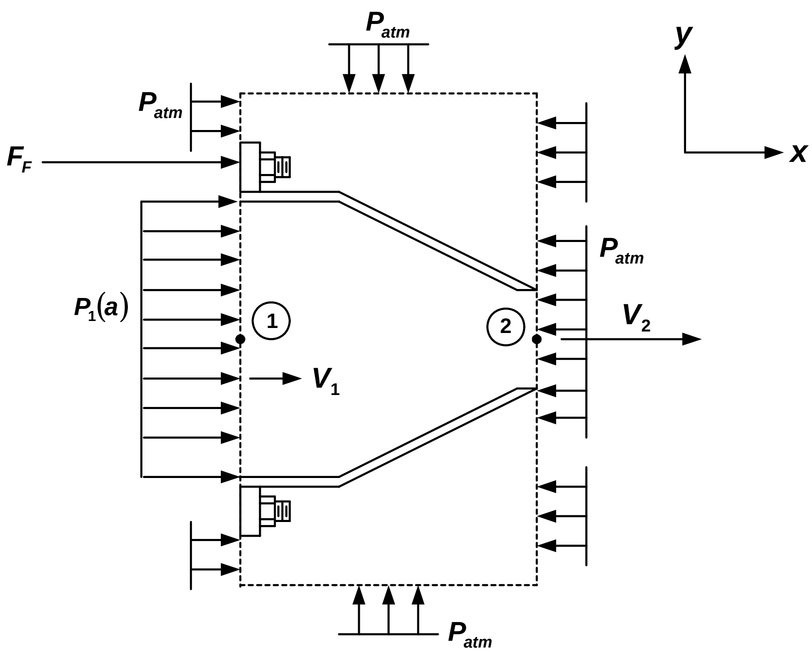
and for the given values $$\Delta P_0 := P_{1g} + \left(\frac{1}{2}\cdot\rho\cdot V_1^{\,2}\right) - \frac{1}{2}\cdot\rho\cdot V_2^{\,2}$$ $$\Delta P_0 = 4.822 \times 10^3$$ psf

Thus, the specified data indicates a total pressure drop from 1 to 2, as would be expected due to the effects of friction.

(b) Force in the flanged joint

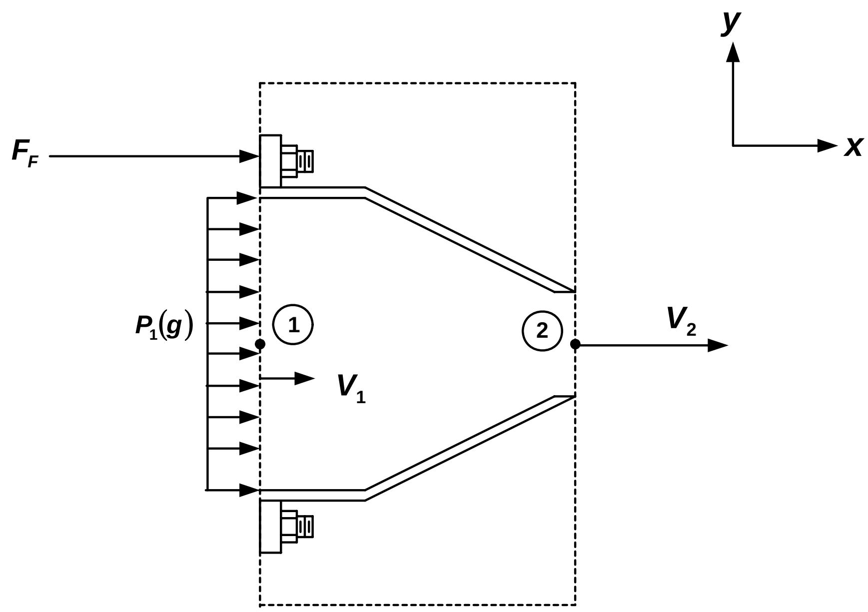
Apply linear momentum.

Define a CV that exposes the force in the flanged joint:



Note:
(i) By having the control surface cut through the bolts, it exposes the force in the flanged joint and makes it a force on the CV, as required.
(ii) F_F has been assumed to be in the positive x direction. F_F replaces the effect of the pipe, which has been removed. If the pipe must exert a force in the positive x direction to hold the nozzle in place, it implies that the nozzle would move in the negative x direction if F_F were removed. As shown, F_F therefore implies that the flanged joint is in compression. If the solution shows F_F to be negative, the assumed direction is incorrect and the joint is in tension.
(iii) The CV shows the actual (that is, absolute) pressures acting on all surfaces of the CV. You can never go wrong with absolute pressures. If gauge pressure values can be subtituted anywhere in the solution this will come out automatically from the analysis.

If the same pressure (not necessarily atmospheric pressure) is applied to much of a control, it is usually advantageous to substract this pressure from <u>all</u> the pressures applied to the CV. The resulting CV is statically and dynamically equivalent to the original one. Here, substracting atmospheric pressure:



The applying linear momentum in the x direction

$\Sigma F_X = m \cdot u_{out} - m \cdot u_{in}$ $\qquad m := \rho \cdot Q_1$ $\qquad m = 4.231$ slug/sec

$F_F + P_{1g} \cdot A_1 = m \cdot V_2 - m \cdot V_1$

then $\qquad F_F := m \cdot V_2 - m \cdot V_1 - P_{1g} \cdot A_1$

$F_F = -2451$ lbf

Since F_F is negative, the flanged joint is in tension, as one would expect (eg. think about the pressures on the inside and outside surfaces of the nozzle).

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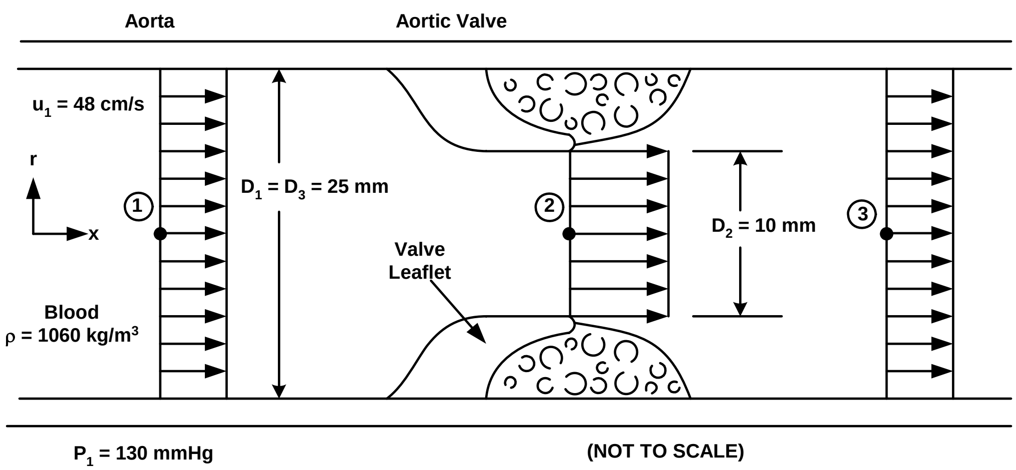
5. Flow through aortic valve



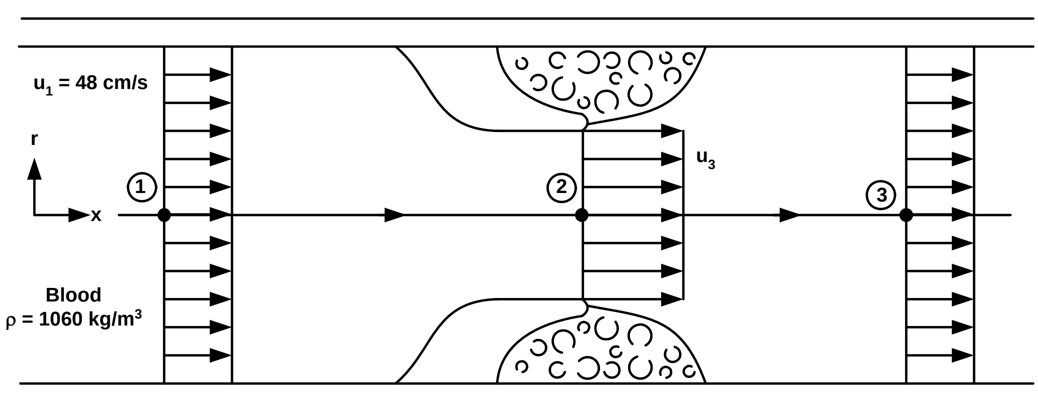
$\rho := 1060$ kg/m3 $\qquad g := 9.81$ m/s2

Find: (a) Pressure drop from 1 to 3, ΔP_{13}. How does it compare with circulatory system pressure drop?
(b) Fluid dynamic force in the streamwise direction on the valve leaflets.

$D_1 := 25$ mm $\qquad D_1 := \frac{D_1}{1000} \qquad D_1 = 0.025$ m $\qquad A_1 := \frac{\pi}{4} \cdot D_1^{\,2} \qquad A_1 = 4.909 \times 10^{-4}$ m2

$D_2 := 10$ mm $\qquad D_2 := \frac{D_2}{1000} \qquad D_2 = 0.01$ m $\qquad A_2 := \frac{\pi}{4} \cdot D_2^{\,2} \qquad A_2 = 7.854 \times 10^{-5}$ m2

(a) Consider first change in presure from 1 to 2. The distance is fairly short and the flow path appears relatively smooth. Therefore it looks reasonable to neglect shear stress on the walls from planes 1 to 2. We can therefore apply Bernoulli's eqn. along the streamline on the centre of the vessel.



$$P_1 + \frac{1}{2}\cdot\rho\cdot u_1^{\,2} + \rho\cdot g\cdot y_1 = P_2 + \frac{1}{2}\cdot\rho\cdot u_2^{\,2} + \rho\cdot g\cdot y_2 \qquad y_1 = y_2$$

then $$P_1 - P_2 = \Delta P_{12} = \frac{1}{2}\cdot\rho\cdot\left(u_2^{\,2} - u_1^{\,2}\right)$$

From continuity (steady, 1-D flow)

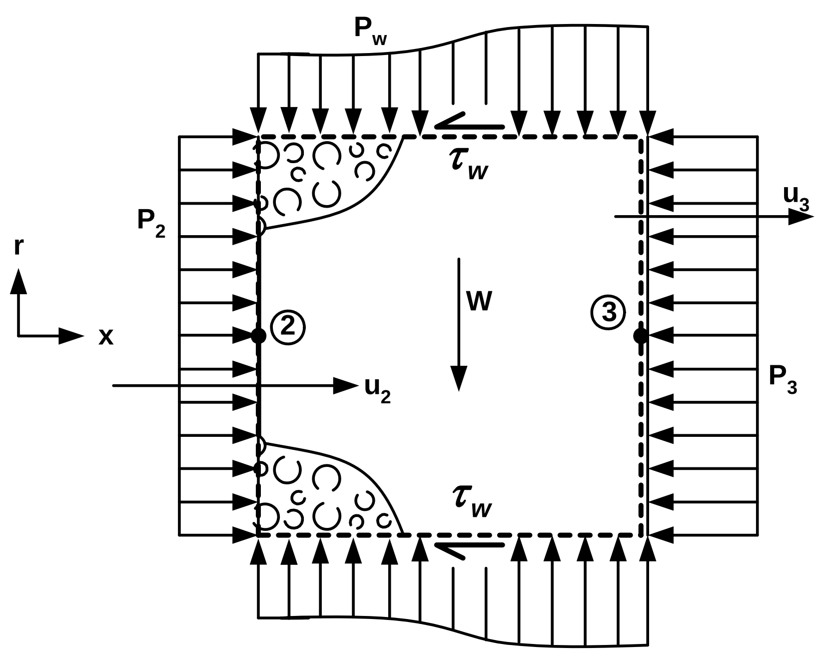
$$\rho\cdot A_1\cdot u_1 = \rho\cdot A_2\cdot u_2 \qquad u_1 := 48 \text{ cm/s} \qquad u_1 := \frac{u_1}{100} \qquad u_1 = 0.48 \text{ m/s}$$

$$u_2 := u_1\cdot\frac{A_1}{A_2} \qquad u_2 = 3 \text{ m/s}$$

and $$\Delta P_{12} := \frac{1}{2}\cdot\rho\cdot\left(u_2^{\,2} - u_1^{\,2}\right) \qquad \Delta P_{12} = 4.648\times 10^{3} \text{ Pa}$$

(Note that this is the pressure dropfrom1 to 2)

From plane 2 to 3 we can not use Bernoulli because there turbulence being continuously generated in the "dead water" behind the leaflets of the valve. This leads to high viscous dissipation (internal energy generation) and the energy to sustain this is supplied by the main flow. Thus, viscous effects cannot be neglected. However, the flow must still satisfy the equations of motion. Therefore, define a control volume from plane 2 to 3. The pressure of interest at 3 will be a pressure on this CV:



From continuity $u_3 := u_1$ $u_3 = 0.48$ m/s

Apply the 1-D flow linear momentum equation in the x direction

$\Sigma F_X = m \cdot u_3 - m \cdot u_2$ where $m := \rho \cdot A_1 \cdot u_1$ $m = 0.25$ kg/s

The forces in the x direction include the pressure forces at planes 2 and 3 and the wall shear stress (there are also pressure forces on the upper and low faces of the CV but the resulting forces do not act in the x direction. Likewise, the weight of the blood in the CV does not show up in this force balance. Then

$P_2 \cdot A_{2CV} - P_3 \cdot A_3 = m \cdot u_3 - m \cdot u_2$ where $A_{2CV} := A_1$ $A_3 := A_1$

then $(P_2 - P_3) \cdot A_1 = \Delta P_{23} \cdot A_1 = m \cdot u_3 - m \cdot u_2$

then $\Delta P_{23} := \frac{1}{A_1} \cdot (m \cdot u_3 - m \cdot u_2)$ $\Delta P_{23} = -1.282 \times 10^3$ Pa

(Note that this represents a pressure rise)

Then overall pressure drop from 1 to 3

$\Delta P_{loss} = P_1 - P_3 = (P_1 - P_2) + (P_2 - P_3) = \Delta P_{12} + \Delta P_{23}$

$\Delta P_{loss} := \Delta P_{12} + \Delta P_{23}$ $\Delta P_{loss} = 3.366 \times 10^3$ Pa

Comparing this with the pressure drop through the circulatory system. Maximum pressure in aorta is abou 130 mmHg(g) and arrives at right ventricle at about 0 mmHg(g). Thus pressure drop

$\Delta P_{circ} := 130$ mmHg

Expressing ΔP_{loss} in mmHg $\Delta P_{loss} = \rho_{Hg} \cdot g \cdot \Delta h_{Hg}$ $\rho_{Hg} := 13600$ kg/m3

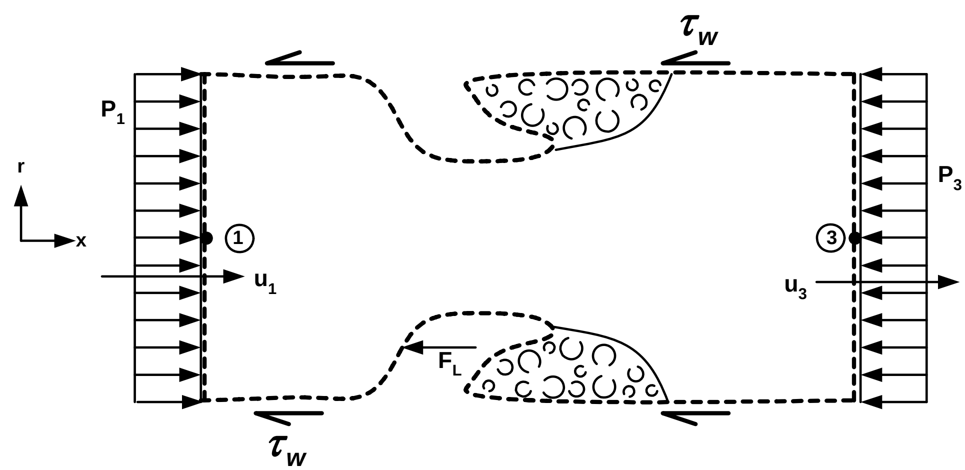
$\Delta h_{Hg} := \frac{\Delta P_{loss}}{\rho_{Hg} \cdot g}$ $\Delta h_{Hg} = 0.025$ mmHg

Thus, the pressure loss through the valve is very small compared with overall pressure drop through the circulatory system.

(b) Force on valve leaflets

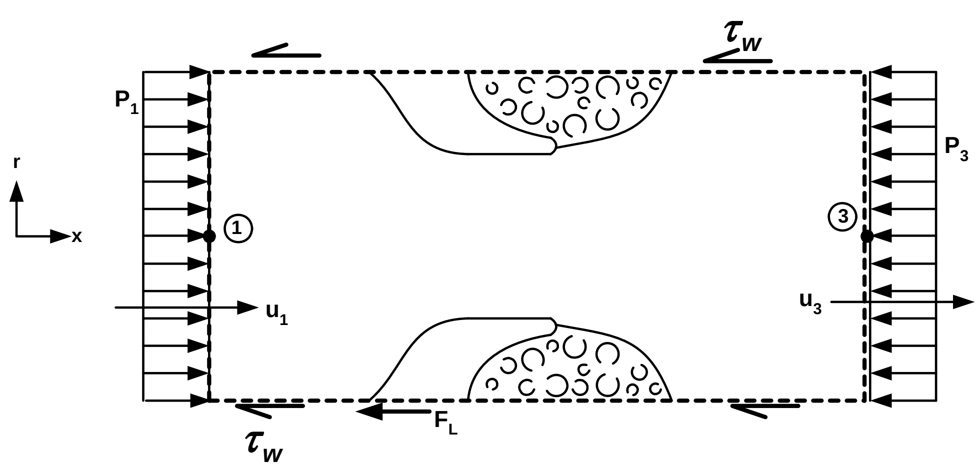
We need a control volume which exposes this force, but otherwise has as few unknowns as possible. At least two CVs are feasible, both passing through planes 1 and 3, for which we now know all flow quantities.

(i) We can create a CV that wraps around the leaflets. The leaflets then exert a pressure variation on the CS at their location. The integral over the area of this pressure distribution contributes a surface force on the CV whose component in the x direction is, say, F_L .



(Only foce components in the x direction have been marked)

(ii) Alternatively, we can simply have the CV cut through the leaflets at the wall, which expose the internal force in the leaflets, the total value in the x direction is again F_L:



Applying the 1-D flinear momentum equation to either CV

$$\Sigma F_X = m \cdot u_3 - m \cdot u_1$$

$$P_1 \cdot A_1 - P_3 \cdot A_3 - F_L = m \cdot u_3 - m \cdot u_1$$

$$\left(P_1 - P_3\right) \cdot A_1 - F_L = m \cdot u_3 - m \cdot u_1$$

then $$F_L = -\left(\Delta P_{13}\right) \cdot A_1 - \left(m \cdot u_3 - m \cdot u_1\right)$$

$$F_L := -\left(\Delta P_{loss}\right) \cdot A_1 - \left(m \cdot u_3 - m \cdot u_1\right)$$

$F_L = -1.652$ N

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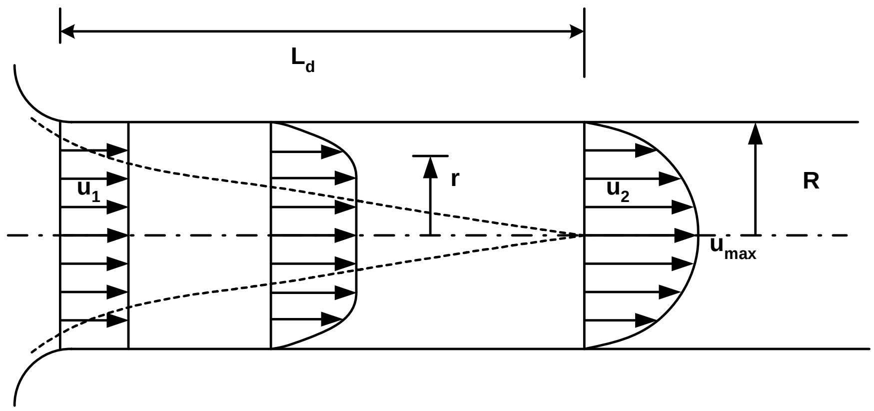
6. Change in P due to change in kinetic energy over development length



Applying SFEE neglecting internal energy change

$$\int \left(\frac{P_1}{\rho} + \frac{V_1^{\,2}}{2}\right) dm = \int \left(\frac{P_2}{\rho} + \frac{V_2^{\,2}}{2}\right) dm$$

P is uniform on both inlet and outlet planes since the streamlines are esentially straight and parallel on both planes.

$$m\cdot\left(\frac{P_1}{\rho}\right) + \int \frac{V_1^{\,2}}{2}\, dm = m\cdot\left(\frac{P_2}{\rho}\right) + \int \frac{V_2^{\,2}}{2}\, dm$$

$$\frac{m}{\rho}\cdot\left(P_1 - P_2\right) = \int \frac{V_2^{\,2}}{2}\, dm - \int \frac{V_1^{\,2}}{2}\, dm \qquad (1)$$

$$dm = \rho\cdot V\cdot 2\cdot\pi\cdot r\cdot dr$$

$$V_1 = u_1$$

$V_2 = u_2$ $\qquad u_2 = u_{max}\cdot\left(1 - \frac{r^2}{R^2}\right)$ $\qquad$ where $\qquad u_{max} = 2\cdot u_1$ $\qquad$ for the parabolic profile

Let $$I_1 = \int \frac{V_1^2}{2}\,dm = \int \frac{u_1^2}{2}\cdot\rho\cdot u_1\cdot 2\cdot\pi\cdot r\,dr = \rho\cdot\pi\cdot u_1^3\cdot\int_0^R r\,dr$$

$$I = \int_0^R r\,dr \qquad I = \frac{1}{2}\cdot R^2$$ and $$I_1 = \rho\cdot\pi\cdot u_1^3\cdot\left(\frac{1}{2}\cdot R^2\right)$$

$$I_2 = \int \frac{V_2^2}{2}\,dm = \int \frac{u_2^2}{2}\cdot\rho\cdot u_2\cdot 2\cdot\pi\cdot r\,dr = \rho\cdot\pi\cdot\int u^3\cdot r\,dr = \rho\cdot\pi\cdot u_{max}^3\cdot\int_0^R \left(1-\frac{r^2}{R^2}\right)^3\cdot r\,dr$$

$$I = \int_0^R \left(1-\frac{r^2}{R^2}\right)^3\cdot r\,dr \qquad I = \frac{1}{8}\cdot R^2$$ and

thus $$I_2 = \rho\cdot\pi\cdot u_{max}^3\cdot\left(\frac{1}{8}\cdot R^2\right)$$ or $$I_2 = \rho\cdot\pi\cdot(2\cdot u_1)^3\cdot\left(\frac{1}{8}\cdot R^2\right) \qquad I_2 = \rho\cdot\pi\cdot u_1^3\cdot R^2$$

Then substituting into (1)

$$\frac{m}{\rho}\cdot(P_1 - P_2) = \rho\cdot\pi\cdot u_1^3\cdot R^2 - \rho\cdot\pi\cdot u_1^3\cdot\left(\frac{1}{2}\cdot R^2\right) = \frac{1}{2}\cdot\left(\rho\cdot\pi\cdot u_1^3\cdot R^2\right)$$

but $$m = \rho\cdot u_1\cdot A_1 = \rho\cdot u_1\cdot\pi\cdot R^2$$

then $$\frac{m}{\rho}\cdot(P_1 - P_2) = \frac{1}{2}\cdot u_1^2\cdot m$$

Finally $$\frac{(P_1 - P_2)}{\frac{1}{2}\cdot\rho\cdot u_1^2} = 1.0$$

Thus there is is a static pressure drop over the development length equal to the inlet dynamic pressure that is due solely to the change in kinetic energy of the fluid. Any additional observed pressure drop would therefore be the result of the internal energy increase due to viscous effects.

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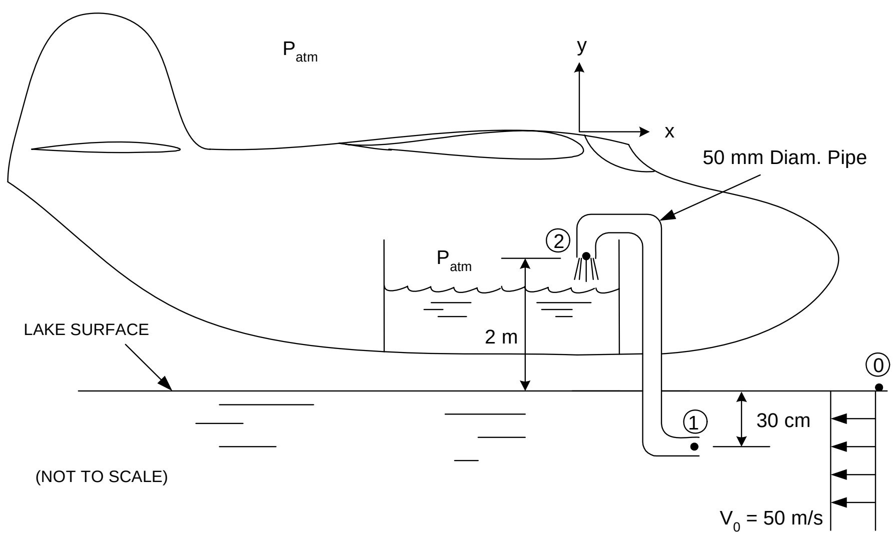
Problem Set #2

7. Water bomber



Use the frame of reference of the aircraft to make the flow appear steady. In that frame of reference the lake water at point 0 appears to have a velocity of 50 m/s to the left.

$\rho := 1000$ kg/m^3 $\qquad g := 9.81$ m/s^2 $\qquad d_{pipe} := 0.05$ m

$V_{ac} := 50$ m/s

Using the free surface of the lake as the datum for elevation, then

$y_0 := 0.0$ m $\qquad y_1 := -0.3$ m $\qquad y_2 := 2$ m

(a) Pressure and flow velocity at point 1 (just inside mouth of the pipe)

Consider a streamline from 0 to 2. To allow for friction and other losses, apply SFEE:

$$P_0 + \frac{1}{2}\cdot\rho\cdot V_0^{\,2} + \rho\cdot g\cdot y_0 - \Delta P_L = P_2 + \frac{1}{2}\cdot\rho\cdot V_2^{\,2} + \rho\cdot g\cdot y_2 \qquad (1)$$

where $V_0 := 50$ m/s $\qquad P_0 = P_2 = P_{atm}$

If we neglect losses

$$\frac{1}{2}\cdot\rho\cdot V_0^{\ 2} + \rho\cdot g\cdot y_0 = \frac{1}{2}\cdot\rho\cdot V_2^{\ 2} + \rho\cdot g\cdot y_2$$

and solving for V_2

$$V_2 := \left(V_0^{\ 2} + 2\cdot y_0\cdot g - 2\cdot y_2\cdot g\right)^{\frac{1}{2}}$$

$V_2 = 49.606$ m/s

We can find the corresponding pressure at 1 by applying Bernoulli from 1 to 2 (again neglecting losses)

$$P_1 + \frac{1}{2}\cdot\rho\cdot V_1^{\ 2} + \rho\cdot g\cdot y_1 = P_2 + \frac{1}{2}\cdot\rho\cdot V_2^{\ 2} + \rho\cdot g\cdot y_2$$

and $P_2 = P_{atm}$ $V_1 = V_2$ from continuity

thus $P_1 - P_{atm} = P_{1g} = \rho\cdot g\cdot\left(y_2 - y_1\right)$

$P_{1g} := \rho\cdot g\cdot\left(y_2 - y_1\right)$ $P_{1g} = 22563$ Pa(g)

The velocity inside the pipe obviously quite high, and this suggests that the losses there may be significant. Reanalyzing with some reasonable assumptions about the losses:

$$\Delta P_L = f\cdot\frac{L}{d_{pipe}}\cdot\frac{1}{2}\cdot\rho\cdot V_{pipe}^{\ 2} + \Sigma K\cdot\frac{1}{2}\cdot\rho\cdot V_{pipe}^{\ 2} \qquad (2)$$

f = friction factor for pipe flow
ΣK = loss coefficients associated with other losses (eg. elbows)

Estimate $L := 3$ m

Using the no-loss pipe velocity to get a first estimate for the Re

For water at 15 C $\mu := 1.2\cdot 10^{-3}$ Ns/m^2

$V_{pipe} := V_2$ $\qquad V_{pipe} = 49.606$ m/s

$$Re := \frac{\rho \cdot V_{pipe} \cdot d_{pipe}}{\mu} \qquad Re = 2.067 \times 10^6$$

Thus the flow is turbulent, and assuming a smooth pipe, from Moody chart $\quad f := 0.011$

Assuming 3 reasonably long-radius elbows $\quad \Sigma K := 3 \cdot 0.3 \qquad \Sigma K = 0.9$

Substituting (2) into (1)

$$P_0 + \frac{1}{2} \cdot \rho \cdot V_0^{\,2} + \rho \cdot g \cdot y_0 - \left(f \cdot \frac{L}{d_{pipe}} \cdot \frac{1}{2} \cdot \rho \cdot V_{pipe}^{\,2} + \Sigma K \cdot \frac{1}{2} \cdot \rho \cdot V_{pipe}^{\,2} \right) = P_2 + \frac{1}{2} \cdot \rho \cdot V_2^{\,2} + \rho \cdot g \cdot y_2 \qquad (3)$$

For the present system geometry $\quad V_{pipe} = V_2 \quad$ and $\quad P_2 = P_0 = P_{atm} \qquad y_0 = 0$

Then (3) can be written

$$\frac{1}{2} \cdot \rho \cdot V_0^{\,2} - \left(f \cdot \frac{L}{d_{pipe}} \cdot \frac{1}{2} \cdot \rho \cdot V_2^{\,2} + \Sigma K \cdot \frac{1}{2} \cdot \rho \cdot V_2^{\,2} \right) = \frac{1}{2} \cdot \rho \cdot V_2^{\,2} + \rho \cdot g \cdot y_2$$

Solving for V_2

$$V_2 := \sqrt{\frac{V_0^{\,2} - g \cdot y_2}{1 + f \cdot \frac{L}{d_{pipe}} + \Sigma K}} \qquad (4) \qquad V_2 = 31.127 \text{ m/s}$$

Using this for updated estimate of Re

$$Re := \frac{\rho \cdot V_2 \cdot d_{pipe}}{\mu} \qquad Re = 1.297 \times 10^6$$

Revised $\quad f := 0.0115$

Revised value for V_2

$$V_2 := \sqrt{\frac{V_0^{\,2} - g \cdot y_2}{1 + f \cdot \frac{L}{d_{pipe}} + \Sigma K}}$$

$V_2 = 30.946$ m/s

From continuity this must also be the velocity at 1

$V_1 := V_2$ $V_1 = 30.946$ m/s

Then can find the revised P_1 either by applying SFEE from 1 to 2, or more simply by applying Bernoulli from 0 to 1

$$P_0 + \frac{1}{2} \cdot \rho \cdot V_0^{\,2} + \rho \cdot g \cdot y_0 = P_1 + \frac{1}{2} \cdot \rho \cdot V_1^{\,2} + \rho \cdot g \cdot y_1$$

where $P_0 = P_{atm}$

$$P_1 - P_{atm} = \frac{1}{2} \cdot \rho \cdot V_0^{\,2} + \rho \cdot g \cdot y_0 - \left(\frac{1}{2} \cdot \rho \cdot V_1^{\,2} + \rho \cdot g \cdot y_1\right)$$

or $P_{1g} := \frac{1}{2} \cdot \rho \cdot V_0^{\,2} + \rho \cdot g \cdot y_0 - \left(\frac{1}{2} \cdot \rho \cdot V_1^{\,2} + \rho \cdot g \cdot y_1\right)$

$P_{1g} = 7.741 \times 10^5$ Pa(g) or 774 kPa(g)

This looks high (abslolute pressure of nearly 9 atm). As a check examine the stagnation pressure (which is the highest pressure that can be generated here) at the depth of the scoop:

$$P_{stag} := \rho \cdot g \cdot y_1 + \frac{1}{2} \cdot \rho \cdot V_0^{\,2}$$

$P_{stag} = 1.247 \times 10^6$ or 1247 kPa(g)

Thus the value of P_1 looks appropriate.

(b) Time required to pick up 1500 kg of water $M := 1500$ kg

From continuity, mass flow rate in pipe

With $V_1 = 30.946$ m/s (from above)

$m = \rho \cdot A_{pipe} \cdot V_1$ $A_{pipe} := \frac{\pi}{4} \cdot d_{pipe}^{\;2}$ $A_{pipe} = 1.963 \times 10^{-3}$ m^2

$m := \rho \cdot A_{pipe} \cdot V_1$ $m = 60.763$ kg/s

$\Delta t := \frac{M}{m}$ $\Delta t = 24.686$ sec

Neglecting losses $V'_1 := 49.6$ m/s

$m' = \rho \cdot A_{pipe} \cdot V'_1$

$m' := \rho \cdot A_{pipe} \cdot V'_1$ $m' = 97.389$ kg/s

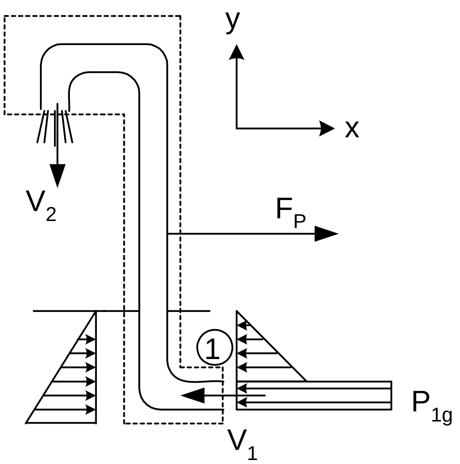
$\Delta t := \frac{M}{m'}$ $\Delta t = 15.402$ sec

(c) Additional thrust and power:

Consider the force on the intake while it is picking up water. This force must be reacted by the aircraft, and therefore represents an additional "drag". This drag force has two main contributions:
(1) The force needed to change the momentum of the water that is captured by the pick-up system.
(2) The hydrodynamic drag of the submerged pipe as it moves through the water.

I am particularly interested in the analysis of contribution (1) since it is the largest of the two. But source (2) should be mentioned, or will become a topic of discussion during the oral. Contribution (2) can be estlimated fairly easily from the drag coefficient of a cylinder in crossflow.

Enclosing the pipe in a control volume:



F_P = force the fuselage exerts on the pipe to hold it in place - this is the additional effective drag that the aircraft feels

Subtracting atmospheric pressure from all surfaces and assuming the contributions from the water are the same on the upstream and downstream faces of the submerged pipe (this amounts to neglecting the hydrodynamic drag of the submerged pipe)

$$\Sigma F_x = m \cdot (V_{xout} - V_{xin})$$

$m = 60.763$ kg/s

$V_{xout} := 0$ $\quad V_{xin} := -V_1$ $\quad V_{xin} = -30.946$ m/s

and $\quad \Sigma F_x = F_P - P_{1g} \cdot A_{pipe}$

Thus $\quad F_P := m \cdot (V_{xout} - V_{xin}) + P_{1g} \cdot A_{pipe}$

$F_P = 3.4 \times 10^3$ N $\quad$ this is the additional thrust that the propellers must produce during the pick up

Corresponding additional engine power

$\text{Power} := F_P \cdot V_{ac}$ $\quad \text{Power} = 1.7 \times 10^5$ W

$\frac{\text{Power}}{745} = 228.211$ HP

Again neglecting losses

$$\Sigma F_x = m' \cdot (V_{xout} - V_{xin})$$

$m' = 97.389$ kg/s

$V_{xout} := 0$ $\quad V_{xin} := -V'_1$ $\quad V_{xin} = -49.6$ m/s

and $\quad \Sigma F_x = F_P - P'_{1g} \cdot A_{pipe}$ $\quad P'_{1g} := 22500$ Pa(g)

Thus $\quad F_P := m \cdot (V_{xout} - V_{xin}) + P'_{1g} \cdot A_{pipe}$

$F_P = 3.058 \times 10^3$ N $\quad$ this is the additional thrust that the propellers must produce during the pick up

Corresponding additional engine power

$Power := F_P \cdot V_{ac}$ $\quad Power = 1.529 \times 10^5$ W

$\frac{Power}{745} = 205.236$ HP

(d) Modifications to decrease time for the pick up (i.e. increase the flow rate of water in the pipe)

From equation (3)

$$P_0 + \frac{1}{2} \cdot \rho \cdot V_0^2 + \rho \cdot g \cdot y_0 - \left(f \cdot \frac{L}{d_{pipe}} \cdot \frac{1}{2} \cdot \rho \cdot V_{pipe}^2 + \Sigma K \cdot \frac{1}{2} \cdot \rho \cdot V_{pipe}^2 \right) = P_2 + \frac{1}{2} \cdot \rho \cdot V_2^2 + \rho \cdot g \cdot y_2 \qquad (3)$$

Since the intake diameter can not be altered, increasing flow rate means increasing V_{pipe}

Solving for V_{pipe}

$$V_{pipe} = \sqrt{\frac{\left(\frac{2 \cdot P_0}{\rho} + V_0^2 \right) - \left(\frac{2 \cdot P_2}{\rho} + V_2^2 \right) + 2 \cdot g \cdot (y_0 - y_2)}{f \cdot \frac{L}{d_{pipe}} + \Sigma K}} \qquad (5)$$

We have no ability to change: P_0, V_0, or (y_0-y_2)

Then the following changes could increase the flow rate:

(1) Reduce V_2 by attaching a diffuser to the discharge end of the pipe

eg. suppose we have a diffuser with area ratio 2 to the discharge. Then

$$V_2 = \frac{1}{2} \cdot V_{pipe}$$

and (3) can be solved for V_{pipe} to give

$$V_{pipe} = \sqrt{\frac{\left(\frac{2 \cdot P_0}{\rho} + V_0^2\right) - \left(\frac{2 \cdot P_2}{\rho}\right) + 2 \cdot g \cdot (y_0 - y_2)}{f \cdot \frac{L}{d_{pipe}} + \Sigma K + \frac{1}{4}}} \qquad (6)$$

If $P_0 = P_2 = P_{atm}$, then this reduces to

$$V_{pipe} := \sqrt{\frac{\left(V_0^2\right) + 2 \cdot g \cdot (y_0 - y_2)}{f \cdot \frac{L}{d_{pipe}} + \Sigma K + \frac{1}{4}}} \qquad (7)$$

$V_{pipe} = 36.57$ m/s

Thus this would increase the flow rate by about 20%, a significant gain for a relatively simple modification.

(2) The value of P_2 could be reduced if we enclose the tank and then vent it to some point on the fuselage where the static pressure is low. It would make sense to combine this modification with that in (1).

Say we can find a point on the fuselage where the local pressure is 1.0q below atmospheric

$$P_2 - P_0 = \frac{-1}{2} \cdot \rho_{air} \cdot V_{ac}^2$$

$\rho_{air} := 1.2$ kg/m^3

Then from (6)

$$V_{pipe} := \sqrt{\frac{V_0{}^2 + \frac{\rho_{air}}{\rho}\cdot V_{ac}{}^2 + 2\cdot g\cdot (y_0 - y_2)}{f\cdot\frac{L}{d_{pipe}} + \Sigma K + \frac{1}{4}}} \qquad V_{pipe} = 36.592 \quad \text{m/s}$$

This gives very little gain.

(3) We could try to reduce the losses in the duct. Consider the magnitude of the losses:

$$f\cdot\frac{L}{d_{pipe}} = 0.69 \qquad \Sigma K = 0.9$$

If the diffuser is inserted earlier in the duct (as close to the intake as the scoop-retraction system will allow) then we can shorten the length of duct exposed to the high q, as well as reduce the q seen by the two downstream elbows. Suppose we have new effective values:

$$L := 1.5 \quad \text{m} \qquad \Sigma K := 0.5$$

Then from (7)

$$V_{pipe} := \sqrt{\frac{\left(V_0{}^2\right) + 2\cdot g\cdot (y_0 - y_2)}{f\cdot\frac{L}{d_{pipe}} + \Sigma K + \frac{1}{4}}} \qquad V_{pipe} = 47.405 \quad \text{m/s}$$

This is a very effective modification.

(4) Add a pump to the pipe system. Crude and expensive.

MECH 3310 Biofluid Mechanics
Problem Set #2

8. Power law fluid model for blood

$$\tau = K_{PL} \cdot (\gamma)^n$$ with $n < 1$ (typical value of 0.75)

$$\tau = K_{PL} \cdot \left(\frac{du}{dr}\right)^n \qquad (1)$$

(a) Units for K_{PL}

K_{PL} has the units of $\dfrac{\tau}{\left(\frac{du}{dr}\right)^n}$ and thus depends on the choice of n

In fact, if the value of n is adjusted, the value of K_{PL} that gives best agreement with experimental τ versus γ is likely also changed. K_{PL} does not really seem to have inherent meaning as a fluid property. In summary, K_{PL} is an empirical factor with ambiguous physical meaning and magnitude and units that depend on the choice of the other (non-dimensional) factor n.

(b) Volume flow rate

From force balance on CV (which does not depend on the properties of the fluid)

$$\frac{dP}{dx} = -2 \cdot \frac{\tau}{r} \qquad \text{then} \qquad \tau = \frac{-r}{2} \cdot \frac{dP}{dx} \qquad (2)$$

Substitute (2) into (1)

$$K_{PL} \cdot \left(\frac{du}{dr}\right)^n = \frac{-r}{2} \cdot \frac{dP}{dx}$$

or

$$\frac{du}{dr} = -\left(\frac{1}{2 \cdot K_{PL}} \cdot \frac{dP}{dx}\right)^{\frac{1}{n}} \cdot r^{\frac{1}{n}} \qquad (3)$$

Separate variables and integrate to get u(r)

$$u = -\left(\frac{1}{2 \cdot K_{PL}} \cdot \frac{dP}{dx}\right)^{\frac{1}{n}} \int_0^r r^{\frac{1}{n}} \, dr$$

which gives

$$u = \frac{n}{n+1}\cdot\left(\frac{1}{2\cdot K_{PL}}\cdot\frac{dP}{dx}\right)^{\frac{1}{n}}\cdot\left(R^{\frac{n+1}{n}} - r^{\frac{n+1}{n}}\right)$$

or

$$u = \frac{n}{n+1}\cdot\left(\frac{1}{2\cdot K_{PL}}\cdot\frac{dP}{dx}\right)^{\frac{1}{n}}\cdot R^{\frac{n+1}{n}}\cdot\left[1-\left(\frac{r}{R}\right)^{\frac{n+1}{n}}\right]$$

for which

$$u_{CL} = \frac{n}{n+1}\cdot\left(\frac{1}{2\cdot K_{PL}}\cdot\frac{dP}{dx}\right)^{\frac{1}{n}}\cdot R^{\frac{n+1}{n}}$$

Then

$$u = u_{CL}\cdot\left[1-\left(\frac{r}{R}\right)^{\frac{n+1}{n}}\right]$$

or

$$\frac{u}{u_{CL}} = \left[1-\left(\frac{r}{R}\right)^{\frac{n+1}{n}}\right]$$

Volume flow rate

$$Q = V\cdot A$$

$$Q = \int u\cdot 2\cdot\pi\cdot r\,dr \qquad Q = u_{CL}\cdot 2\cdot\pi\cdot\int_0^R \left[1-\left(\frac{r}{R}\right)^{\frac{n+1}{n}}\right]\cdot r\,dr$$

which integrates to

$$Q = u_{CL}\cdot 2\cdot\pi\cdot\left[\frac{1}{2}\cdot R^2\cdot\frac{(n+1)}{(3\cdot n+1)}\right] = u_{CL}\cdot\pi\cdot R^2\cdot\frac{(n+1)}{(3\cdot n+1)} \qquad (4)$$

(c) Relationship between centreline (maximum) velocity and the mean velocity

$$Q = u_{mean}\cdot A = u_{mean}\cdot\pi\cdot R^2$$

Equating to (4)

$$u_{mean}\cdot\pi\cdot R^2 = u_{CL}\cdot\pi\cdot R^2\cdot\frac{(n+1)}{(3\cdot n+1)}$$

$$u_{CL} = u_{mean}\cdot\frac{3\cdot n+1}{n+1} \qquad (5)$$

For Newtonian fluid with $n := 1$

$$C := \frac{3 \cdot n + 1}{n + 1} \qquad C = 2$$

and $u_{CL} = 2 \cdot u_{mean}$ as previously derived

For power law fluid with $n := 0.75$

$$C := \frac{3 \cdot n + 1}{n + 1} \qquad C = 1.857$$

and $u_{CL} = 1.857 \cdot u_{mean}$

and the power law profile is slightly fuller

Using (5) we can rewrite the velocity profile

$$u = u_{CL} \cdot \left[1 - \left(\frac{r}{R} \right)^{\frac{n+1}{n}} \right] \qquad \text{then} \qquad u = u_{mean} \cdot \frac{3 \cdot n + 1}{n + 1} \cdot \left[1 - \left(\frac{r}{R} \right)^{\frac{n+1}{n}} \right]$$

or

$$\frac{u}{u_{mean}} = \frac{3 \cdot n + 1}{n + 1} \cdot \left[1 - \left(\frac{r}{R} \right)^{\frac{n+1}{n}} \right]$$

If we plot this as a function of n, we can compare the velocity profiles normalized to the same u_{mean} i.e. for flows that have the same total volume flow rate

Let $U = \frac{u}{u_{mean}}$ $\qquad r' = \frac{r}{R}$

then

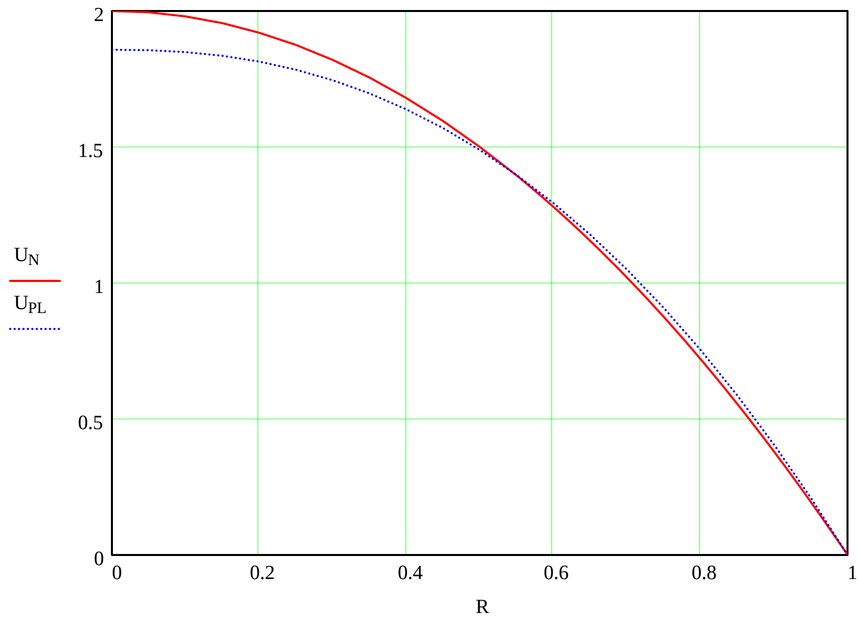
$$U = \frac{3 \cdot n + 1}{n + 1} \cdot \left[1 - (r')^{\frac{n+1}{n}} \right]$$

$$F(n, r') := \frac{3 \cdot n + 1}{n + 1} \cdot \left[1 - (r')^{\frac{n+1}{n}} \right]$$

$i := 1 .. 21 \qquad R_i := (i - 1) \cdot 0.05$

Newtonian velocity distribution $U_{N_i} := F(1.0, R_i)$

Power law distribution $U_{PL_i} := F(0.75, R_i)$



and the power law profile is seen to be slightly fuller.