Université d'Ottawa · University of Ottawa
Faculté des sciences
Mathématiques et de statistique
Faculty of Science
Mathematics and Statistics

Calculus III for Engineers
MAT 2322A - Fall 2014
Midterm I
Professor: Catalin Rada
Time limit: 80 minutes. Closed books.

Name: Sol BOTH Versions ID Number:

Instructions

- This exam has 8 pages and you have 80 minutes to complete it.
- This is a closed book exam. Furthermore, all cell phones, pagers or any other electronic or communication devices are forbidden. **The only calculators which are allowed are Texas Instruments TI-30, TI-34, Casio fx-260 and fx-300, scientific and non programmable.**
- Read each question carefully before answering.
- Questions 1 to 3 are multiple choice questions. These questions are worth 2 points each and no partial marks are possible.
- Questions 4 to 6 are long answer questions. Questions 4 and 6 are worth 6 marks each, and question 5 is worth 7 marks, so organize your time accordingly. **A correct answer requires a full, clearly-written and detailed solution.** Answer each question in the space provided, using backs of pages or the extra pages at the end if necessary.
- Do not unstaple the test.
- Good luck!

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1. If $f(x,y) = 2x^2 + 3xy - 6y^2$, which of the following expressions corresponds to the tangent plane to the graph of f at the point $(1, 1, -1)$?

A. $z = 7x - 9y - 1$

B. $z = (4x + 3y)(x - 1) + (3x - 12y)(y - 1) - 1$

C. $z = 7\vec{i} - 9\vec{j}$

D. $z = -1$

E. $z = 7(x - 1) - 9(y - 1) - 1$

F. This function is not differentiable at the indicated point, so the tangent plane does not exist.

2. If $f(x,y) = e^{2x^2-3y^2}$, and $\vec{u}$ is the unit vector along $\vec{i}+\vec{j}$, which of the following corresponds to the directional derivative $D_{\vec{u}}f(1,2)$?

A. $-8e^{-10}$

B. $-4\sqrt{2}e^{-10}$

C. $2\sqrt{2}e^{-10}\,\vec{i} - 6\sqrt{2}e^{-10}\,\vec{j}$

D. $-3\sqrt{2}e^{-10}$

E. 0

F. $-2e^{-10}$

3. If $f(x, y) = x^2y^3$ and R is the square region defined by $0 \leq x \leq 1$, $0 \leq y \leq 2$, what is the value of the double integral $\iint_R f\, dA$?

A. 0

B. 1/3

C. 2/3

D. 1

E. 4/3

F. 5/3

4. Find and classify the critical points of the function $f(x, y) = e^y\,(2y^2 - x^2)$.

5. Find the global maximum and the global minimum of the function $f(x, y) = x^2 + xy + y^2$ on the region

$$A = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4 \}.$$

Clearly identify all steps of the optimization algorithm.

6. Let R be the bounded region enclosed between the parabola $y = x^2 + 2$, the parabola $y = -x^2$, and the lines $x = -1$ and $x = 3$ in the (x, y)-plane. Compute the double integral $\iint_R xy\,dA$.

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(B) $z = 7(x-1) - 9(y-1) - 1$

C. $z = 7x - 9y - 1$

D. $z = (4x + 3y)(x-1) + (3x - 12y)(y-1) - 1$

E. $z = 7\vec{\imath} - 9\vec{\jmath}$

F. This function is not differentiable at the indicated point, so the tangent plane does not exist.

2. If $f(x,y) = e^{2x^2 - 3y^2}$, and $\vec{u}$ is the unit vector along $\vec{\imath} + \vec{\jmath}$, which of the following corresponds to the directional derivative $D_{\vec{u}} f(1,2)$?

A. $2\sqrt{2}e^{-10}\,\vec{\imath} - 6\sqrt{2}e^{-10}\,\vec{\jmath}$

B. $-3\sqrt{2}e^{-10}$

C. 0

D. $-8e^{-10}$

(E) $-4\sqrt{2}e^{-10}$

F. $-2e^{-10}$

3. If $f(x, y) = x^2y^3$ and R is the square region defined by $0 \leq x \leq 1$, $0 \leq y \leq 2$, what is the value of the double integral $\iint_R f\, dA$?

A. 5/3

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D. 2/3

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SOLUTIONS

Q1

$$z-(-1) = f_x(1,1)(x-1) + f_y(1,1)(y-1)$$

$$z+1 = 7(x-1) - 9(y-1) \text{ or}$$

$$\boxed{z = 7(x-1) - 9(y-1) - 1}$$

because: $\begin{cases} f_x = 4x + 3y \Rightarrow f_x(1,1) = 7 \\ f_y = 3x - 12y \Rightarrow f_y(1,1) = -9 \end{cases}$

Q2

$$V = i + j = (1,0) + (0,1) = (1,1) \Rightarrow |V| = \sqrt{1^2+1^2} = \sqrt{2}$$

$$u = \frac{1}{\sqrt{2}} V = \frac{1}{\sqrt{2}}(1,1) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$f_x = e^{2x^2-3y^2}(4x) \; ; \quad f_y = e^{2x^2-3y^2}(-6y)$$

$$D_u f(1,2) = \left(e^{-10} \cdot 4, \; e^{-10}(-12)\right) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}\left(e^{-10} \cdot 4 + e^{-10}(-12)\right) = \boxed{\frac{1}{\sqrt{2}} e^{-10}(-8) = -4\sqrt{2}e^{-10}}$$

Q3

$$\iint_R f\,dA = \int_0^1 \left(\int_0^2 x^2 y^3\,dy \right) dx =$$

$$= \int_0^1 \left(x^2 \frac{y^4}{4} \Big|_0^2 \right) dx = \int_0^1 \left(x^2 \cdot \frac{2^4}{4} \right) dx$$

$$= \frac{2^4}{4} \cdot \frac{x^3}{3} \Big|_0^1 = \frac{16 \cdot 1^3}{4 \cdot 3} = \frac{16}{12} = \boxed{\frac{4}{3}}$$

Q4

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} e^y(-2x) = 0 \\ e^y(2y^2 - x^2) + e^y(4y) = 0 \end{cases} \Rightarrow$$

$$\begin{cases} x = 0 \\ 2y^2 - x^2 + 4y = 0 \end{cases} \to \begin{cases} x = 0 \\ 2y(y+2) = 0 \end{cases} \quad \begin{cases} (0,0) \\ (0,-2) \end{cases} \quad \text{C.P.}$$

$$\boxed{f_{xy} = e^y(-2x)} \quad \boxed{f_{xx} = e^y(-2)} \quad \boxed{f_{yy} = e^y[2y^2 - x^2 + 8y + 4]}$$

FOR $(0,0)$: $D(0,0) = (-2) \cdot 4 - 0^2 = -8 < 0$

SO $(0,0)$ IS A SADDLE POINT.

FOR $(0,-2)$: $D(0,-2) = [e^{-2}(-2)]\, e^{-2}(8 - 0 - 16 + 4) - 0^2 =$

$= e^{-4} \cdot 8 > 0$

$f_{xx}(0,-2) = e^{-2}(-2) < 0 \Rightarrow (0,-2)$ IS A LOCAL MAXIMUM

(here $f = e^y(2y^2 - x^2)$)

Q4

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \quad \begin{cases} e^x(2x^2-y^2) + e^x(4x) = 0 \\ e^x(-2y) = 0 \end{cases} \Rightarrow$$

$$\begin{cases} e^x(2x^2-y^2+4x) = 0 \\ y = 0 \end{cases} \quad \begin{cases} 2x(x+2) = 0 \\ y = 0 \end{cases} \quad \begin{cases} (0,0) \\ (-2,0) \end{cases} \text{ c.p.}$$

$$f_{xy} = e^x(-2y) \qquad f_{yy} = e^x(-2) \qquad f_{xx} = e^x[2x^2-y^2+8x+4]$$

FOR (0,0): $D = 4 \cdot (-2) - 0^2 < 0 \Rightarrow (0,0)$ is A SADDLE point

For: (-2,0): $D = e^{-2}(-4)e^{-2}(-2) - 0^2 > 0.$

$f_{xx}(-2,0) = e^{-2}(-4) < 0 \Rightarrow (-2,0)$ is a LOCAL Maximum.

Here $f = e^x(2x^2-y^2)$

Q5 $f = x^2 + xy + y^2$ on $x^2+y^2 \le 4$

C.P.: $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} 2x+y=0 \\ x+2y=0 \end{cases} \Rightarrow \begin{cases} y=-2x \\ x+2(-2x)=0 \end{cases}$

$\begin{cases} -3x=0 \\ y=-2x \end{cases} \to \begin{cases} x=0 \\ y=-2\cdot 0=0 \end{cases} \Rightarrow \boxed{(0,0) \in A}$

$\boxed{f(0,0) = 0^2 + 0\cdot 0 + 0^2 = 0}$

Boundary: $\begin{cases} \nabla f = \lambda \nabla g \\ x^2+y^2=4 \end{cases} \Rightarrow \begin{cases} (2x+y,\ x+2y) = \lambda(2x, 2y) \\ x^2+y^2=4 \end{cases}$

$\begin{cases} 2x+y = \lambda 2x \\ x+2y = \lambda 2y \\ x^2+y^2=4 \end{cases}$ Add $\to$ $3(x+y) = (x+y)2\lambda \Rightarrow (x+y)(3-2\lambda)=0$

(I) $\lambda = \frac{3}{2} \Rightarrow \begin{cases} 2x+y=3x \\ x+2y=3y \end{cases} \Rightarrow \boxed{x=y}$. From $x^2+y^2=4 \Rightarrow$

$x^2=2 \Rightarrow x=\pm\sqrt{2} \Rightarrow (\sqrt{2},\sqrt{2});\ (-\sqrt{2},-\sqrt{2})$

(II) $x+y=0 \to x=-y$. so $\begin{cases} x=\lambda 2x \\ y=\lambda 2y \\ x^2+y^2=4. \end{cases}$

if $x=0 \Rightarrow y=0 \Rightarrow 0=4$ contradiction. So $x\neq 0, \Rightarrow \lambda=\frac{1}{2}$

$2x^2=4 \Rightarrow x=\pm\sqrt{2} \Rightarrow y=\mp\sqrt{2} \Rightarrow (\sqrt{2},-\sqrt{2});\ (-\sqrt{2},\sqrt{2})$.

$\boxed{f(\sqrt{2},\sqrt{2}) = 2+2+2=6}$ $\boxed{f(-\sqrt{2},-\sqrt{2}) = 6}$

$f(\sqrt{2},-\sqrt{2}) = 2-2+2=2$ $f(-\sqrt{2},\sqrt{2}) = 2-2+2=2$

GLOBAL MAX: 6 AT $(\sqrt{2},\sqrt{2})$ or $(-\sqrt{2},-\sqrt{2})$; GLOBAL MIN: 0 @ (0,0)

Q5 $f = x^2 + xy + y^2$ on $x^2+y^2 \le 9$

CP $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} 2x+y=0 \\ x+2y=0 \end{cases} \Rightarrow \begin{cases} y=-2x \\ x+2(-2x)=0 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=-2 \cdot 0 = 0 \end{cases} \Rightarrow$

$(0,0) \in A$ is the C.P. inside of A AND $f(0,0)=0$

Boundary: $\begin{cases} \nabla f = \lambda \nabla g \\ x^2+y^2=9 \end{cases} \Rightarrow \begin{cases} (2x+y, x+2y) = \lambda(2x, 2y) \\ x^2+y^2=9 \end{cases} \Rightarrow$

$\begin{cases} 2x+y = \lambda 2x \\ x+2y = \lambda 2y \\ x^2+y^2=9 \end{cases}$ ADD UP $\Rightarrow 3(x+y) = \lambda 2(x+y) \Rightarrow (x+y)(3-\lambda 2)=0$

(I) $\lambda = 3/2 \Rightarrow \begin{cases} 2x+y=3x \\ x+2y=3y \end{cases} \Rightarrow x=y$. From $x^2+y^2=9 \Rightarrow$

$x^2 = \frac{9}{2} \Rightarrow x = \pm \frac{3}{\sqrt{2}} \Rightarrow \left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right); \left(\frac{-3}{\sqrt{2}}, \frac{-3}{\sqrt{2}}\right)$.

(II) $x+y=0 \Rightarrow x=-y$. So $\begin{cases} x = \lambda 2x \\ y = \lambda 2y \\ x^2+y^2=9 \end{cases}$. if $x=0 \Rightarrow y=0 \Rightarrow$

$\Rightarrow 0=9$, contradiction. So $x \neq 0 \Rightarrow \lambda = \frac{1}{2}$.

$2x^2=9 \Rightarrow x^2 = \frac{9}{2} \Rightarrow x = \frac{\pm 3}{\sqrt{2}}$. So $\left(\frac{3}{\sqrt{2}}, \frac{-3}{\sqrt{2}}\right); \left(\frac{-3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$

$f\left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) = \frac{27}{2} = f\left(-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right)$

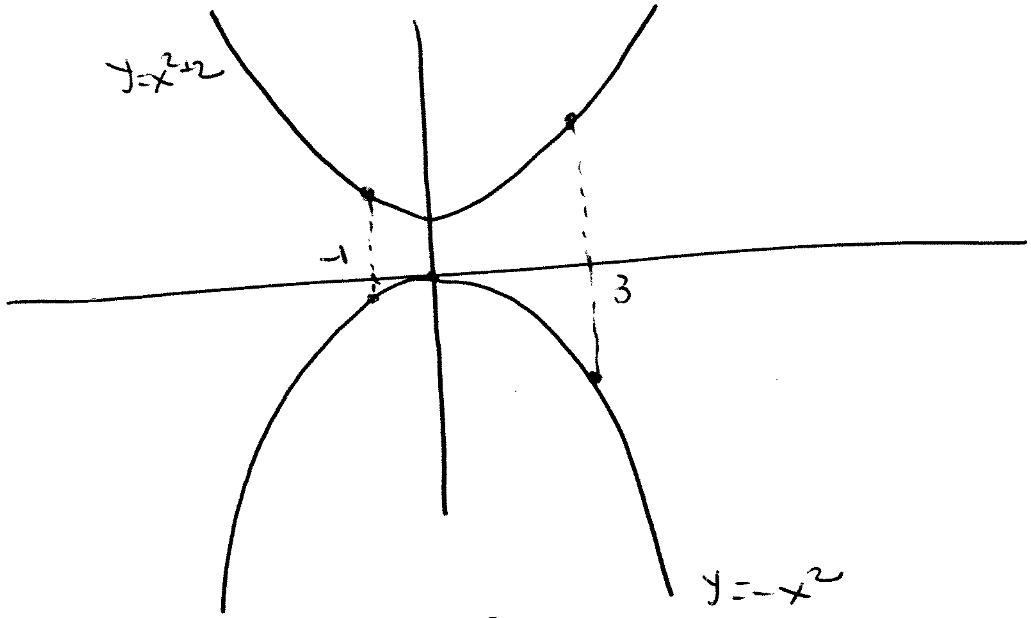
$f\left(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right) = \frac{9}{2} = f\left(-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$

GLOBAL MAX: $\frac{27}{2}$ @ either $\left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$ or $\left(\frac{-3}{\sqrt{2}}, \frac{-3}{\sqrt{2}}\right)$

GLOBAL MIN: 0 @ (0,0)

Q6

$y = x^2+2, \quad y = -x^2, \quad x=-1, \quad x=3$



$$\iint_R xy \, dA = \int_{x=-1}^{x=3} \left[\int_{-x^2}^{x^2+2} xy \, dy \right] dx =$$

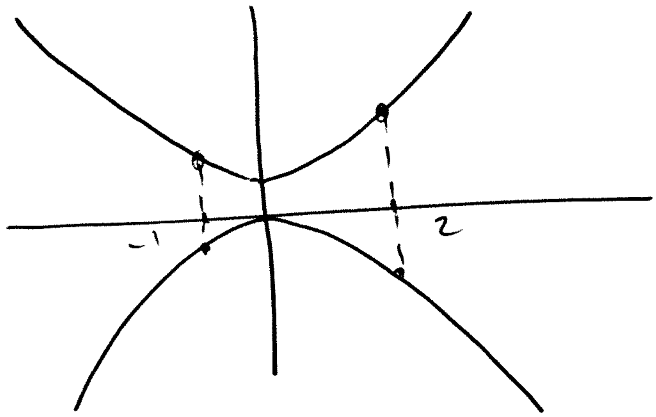
$$= \int_{-1}^{3} \left[x\frac{y^2}{2} \Big|_{-x^2}^{x^2+2} \right] dx = \int_{-1}^{3} \left[\frac{x}{2}(x^2+2)^2 - \frac{x}{2}(-x^2)^2 \right] dx$$

$$= \int_{-1}^{3} \frac{x}{2}\left(x^4+4x^2+4\right) - \frac{x}{2}\left(x^4\right) dx = \int_{-1}^{3} 2x^3+2x \, dx$$

$$= \left(2\cdot\frac{x^4}{4} + x^2\right)\Big|_{-1}^{3} = \frac{81}{2} + 9 - \frac{1}{2} - 1 = 40-8 =$$

$$= \boxed{32}$$

(Q6) $y = x^2+1$; $y = -x^2$; $x=-1$; $x=2$



$$\iint_R xy\,dA =$$

$$= \int_{x=-1}^{x=2} \left[\int_{y=-x^2}^{y=x^2+1} xy \; dy \right] dx =$$

$$= \int_{-1}^{2} \left[x\frac{y^2}{2} \Big|_{-x^2}^{x^2+1} \right] dx = \int_{-1}^{2} \left[\frac{x}{2}(x^2+1)^2 - \frac{x}{2}(-x^2)^2 \right] dx$$

$$= \int_{-1}^{2} \left[\frac{x}{2}(x^4+2x^2+1) - \frac{x}{2}(x^4) \right] dx = \int_{-1}^{2} x^3 + \frac{x}{2} \; dx$$

$$= \frac{x^4}{4} + \frac{x^2}{4} \Big|_{-1}^{2} = \frac{16}{4} + \frac{4}{4} - \frac{1}{4} - \frac{1}{4}$$

$$= \frac{18}{4} = \boxed{\frac{9}{2}}$$