Math 1229A/B

Unit 1:
Vectors

(text reference: Section 1.1)

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1 Vectors

You are familiar with the set of real numbers. **Real numbers** means all the numbers you've ever heard of or can imagine (unless you've learnt about, or at least heard of, *imaginary* numbers – they're not *real* numbers). Real numbers includes all the integers (including both positive and negative, and of course 0), fractions (called *rational numbers*), decimals that can't be expressed as fractions (the *irrational numbers*). All the numbers along the real number line from $-\infty$ to ∞. We call the set of real numbers $\Re$.

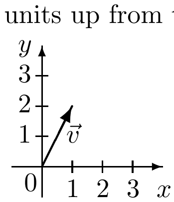
You're also familiar with the x-y plane. You've drawn graphs in this plane. There's the x-axis (horizontal) and the y-axis (vertical), which cross at the *origin*. Each axis is basically a real number line. Any point in this plane can be expressed as an ordered pair, (x, y), giving its *x-coordinate* and its *y-coordinate*. Each coordinate can be any real number. We call the x-y plane **2-space** and the set containing all of the points in this plane is called $\Re^2$. (We pronounce that "R2", i.e. "Artoo".) So $\Re^2$ can be thought of as the set of all ordered pairs of real numbers.

You've probably also seen something even more complicated, with 3 axes: the x-axis, the y-axis and the z-axis. Each axis is perpendicular to both of the others, which makes it hard to draw on a piece of paper or a blackboard. They're often drawn with the y-axis horizontal, the z-axis vertical, and the x-axis off at a funny angle, to represent that it's coming straight out of the page at you. The 3 axes represent the 3 dimensions of "space", i.e. reality. Like the room you're sitting in. There's not only up and down, and left and right, but also near and far, or here and there, or ... well, you know, that third dimension, which we might call depth. In Math, we call the region defined by these 3 axes **3-space**. And points in 3-space are represented by *ordered triples*, (x, y, z), giving the x-, y- and z-coordinates of the point. As before, each of these coordinates can be any real number. The set containing all of the points in 3-space is called $\Re^3$ (pronounced "R3"), and we can think of this as the set of all ordered triples of real numbers.

Now that you've got that straight, let's confuse things. Sometimes when we write an ordered pair or an ordered triple, it doesn't represent a point. Instead, it represents a *directed line segment*, called a **vector**. And when the pair or triple represents a vector, the numbers in it (or symbols representing numbers) aren't called coordinates, they're called **components**.

That seems like it will be confusing, using the same notation to denote two different things. It's not too bad, though, because you can tell by the *context* whether a pair or triplet is a point or a vector. And because we write the *names* differently, depending whether it's a point or a vector. Points are named with normal capital letters, usually P or something nearby in the alphabet, and sometimes with subscripts. So we might have the points $P(p_1, p_2)$ and $Q(q_1, q_2)$, or the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$. With vectors, we don't use capital letters, and we do something to show that it's a vector. In our textbook, they use **boldface** type for the name of a vector. For instance, the vector $\mathbf{v} = (v_1, v_2)$ or the vector $\mathbf{u} = (u_1, u_2, u_3)$. "Sure", I hear you thinking, "that's easy enough for you, but what about me, when I'm writing with a pen or pencil?". Well, that's why we're going to use a different convention in these notes. One that's much more obvious. When a letter is the name of a vector, we'll put an arrow over it. Like this: $\vec{v} = (v_1, v_2)$.

So why is it that we represent a vector, which we said is a directed line segment, using something that looks like a point? Well, it's just a convention. It's shorthand. When we say $\vec{v} = (1, 2)$, what we mean is that the vector $\vec{v}$ is the directed line segment that starts at the origin and ends at the point $(1, 2)$. So here's a picture of the vector $\vec{v} = (1, 2)$. It starts at the point $(0, 0)$, and then ends at a place that's 1 unit to the right and 2 units up from there.



> **Definition:** The **vector** $\vec{v} = (v_1, v_2)$ **in** $\Re^{\mathbf{2}}$ is the directed line segment that goes from the origin (i.e. the point $(0,0)$) to the point $V(v_1, v_2)$. Similarly, the **vector** $\vec{v} = (v_1, v_2, v_3)$ **in** $\Re^{\mathbf{3}}$ is the directed line segment that goes from the point $(0,0,0)$ to the point $V(v_1, v_2, v_3)$. The point V is called the **endpoint** of the vector $\vec{v}$.

If we want to say that $\vec{v}$ is a vector in $\Re^2$, i.e. a vector which has two components, then we say $\vec{v} \in \Re^2$. You've probably seen that symbol before, for instance for saying that a particular object is an element of a particular set. Similarly, we can say $\vec{v} \in \Re^3$ to state that $\vec{v}$ is a vector in $\Re^3$. Notice that earlier we said that $\Re^2$ was the set of all points in the x-y plane. But now we're saying that a vector is in that set. Hmm. If $\Re^2$ is a set, does it contain points or vectors? Well, actually, we can think of it either way. We can describe 2-space as the set of all points in the x-y plane, or as the set of all vectors in the x-y plane. Or we can define it simply as the set of all ordered pairs (x, y). That's probably the best way to think of it. Because an ordered pair can (as we've already discussed) represent either a point or a vector, depending on the context. Likewise, we'll think of $\Re^3$ as simply the set of all ordered triples (x, y, z).

There is a vector in $\Re^2$ corresponding to each point in the x-y plane. Likewise, there is a vector in $\Re^3$ corresponding to each point in 3-space. But wait a minute! What about the point (0,0) or (0,0,0)? That's where the directed line segment starts. So can there be a directed line segment that goes from that point to itself? Well, yes. Although you won't be able to see it, and you won't care, or be able to determine, what direction it goes in. That is, in spite of the fact that "the line segment from the point (0,0) to the point (0,0)" seems nonsensical, because there's no line segment there, we *do* consider there to be a vector $\vec{v} = (0,0)$. It actually comes in very handy. We call it the "zero vector", and give it the name $\vec{0}$.

> **Definition:** A **zero vector** is a vector whose components are all 0. The **zero vector in** $\Re^{\mathbf{2}}$ is the vector $\vec{0} = (0,0)$. Similarly, $\vec{0} = (0,0,0)$ is the **zero vector in** $\Re^{\mathbf{3}}$.

Whenever we define a new mathematical construct, we need to define what "equality" means for that construct. Even if it seems pretty obvious. So we need to define what it means to say that two vectors are equal. We've already used that concept, in attaching names to vectors. For instance when we say $\vec{v} = (v_1, v_2)$, we're saying that the vector whose name is $\vec{v}$ is equal to the vector in $\Re^2$ whose first component is v_1 and whose second component is v_2. Likewise, when we say $\vec{0} = (0,0,0)$, we're saying that the vector whose name is $\vec{0}$ is equal to the vector in $\Re^3$ whose components are all 0. But of course, what we really meant there is "this is the name I'm going to call that vector by", rather than "here are 2 different vectors, and they're equal". But often we do need to equate 2 vectors in that sense, too. Or to say that the vector you get when you do certain vector arithmetic operations (which we'll learn about shortly) is equal to a specified vector. So what do we mean when we say, for instance, that $\vec{u} = \vec{v}$?

> **Definition:** Two vectors are **equal** if they are vectors in the same space and their corresponding components are equal. That is, two vectors in $\Re^2$ are equal if they have the same first component and also have the same second component. Similarly, two vectors in $\Re^3$ are equal if they have the same first component and have the same second component and have the same third component. In mathematical notation, we have
>
> If $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$, then $\vec{u} = \vec{v}$ if and only if $u_1 = v_1$ and $u_2 = v_2$.
>
> and similarly
>
> If $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$, then $\vec{u} = \vec{v}$ if and only if $u_1 = v_1$ and $u_2 = v_2$ and $u_3 = v_3$.

Notice that in order for 2 vectors to be equal they *must* be vectors in the same space. If $\vec{u} \in \Re^2$ and $\vec{v} \in \Re^3$ then they can **never** be equal vectors, no matter what their components are.

Example 1.1. If $\vec{u} = (a, 2)$ and $\vec{v} = (-1, b)$, where it is known that $\vec{u} = \vec{v}$, what are the values of a and b?

Solution:
Since $\vec{u} = \vec{v}$, then $(a, 2) = (-1, b)$. And for these vectors to be equal, their respective components must be equal. Since the first component of $\vec{v}$ is -1 and $\vec{u} = \vec{v}$, then the first component of $\vec{u}$ must also be -1. And a is the first component of $\vec{u}$, so it must be true that $a = -1$. Likewise, since the second component of $\vec{u}$ is 2 and $\vec{v} = \vec{u}$, then the second component of $\vec{v}$ must also be 2. But the second component of $\vec{v}$ is b, and so $b = 2$.

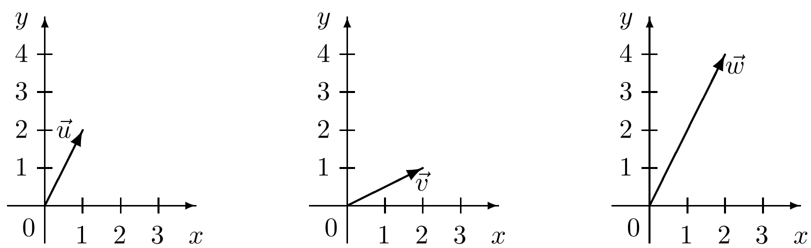
All vectors start at the origin. There are infinitely many lines that pass through the origin. For any vector (other than the zero vector), there's exactly one line through the origin that the vector lies on. And often it's important to realize whether or not 2 vectors lie on the same line. We have a word for that. **Collinear** just means "same line".

Definition: Two vectors in the same space are **collinear** if they lie on the same line.

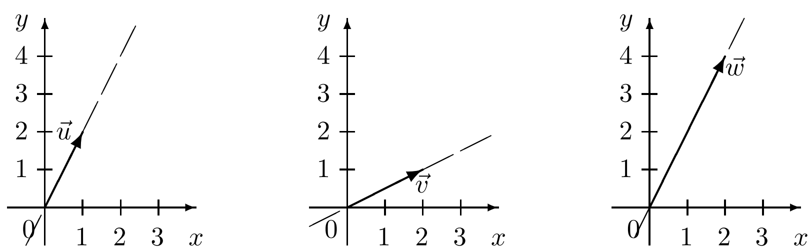
Of course, if two vectors lie on the same line, they must be parallel to one another. So if two vectors are collinear, they are also parallel, and vice versa.

Example 1.2. Consider the vectors $\vec{u} = (1, 2)$, $\vec{v} = (2, 1)$ and $\vec{w} = (2, 4)$. Draw each of these vectors. Show that $\vec{u}$ and $\vec{v}$ are not collinear, but that $\vec{u}$ and $\vec{w}$ are.

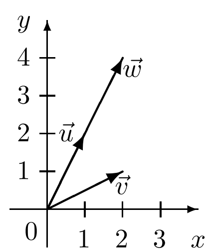
Solution:
First we'll draw each vector on its own set of axes.



We can also show the line that each vector lies on:



It's pretty clear that the line that $\vec{u}$ lies on is not the same as the line that $\vec{v}$ lies on. For $\vec{u}$ and $\vec{w}$, they look pretty much the same, but how can we be sure? We draw them all on the same axes:



From this last diagram, even without the lines drawn in we can see that vectors $\vec{u}$ and $\vec{v}$ are certainly not collinear, and also that vector $\vec{w}$ lies right on top of $\vec{u}$, because they *are* collinear.

All vectors start at the origin, so all vectors (in the same space) touch one another. And yet, we talk about the *distance* between vectors. By this, we mean the furthest distance between any 2 points with one point being on each vector. This always occurs at the endpoints of the vectors.

Definition: The **distance between** two vectors $\vec{u}$ and $\vec{v}$ is defined to be the *distance between their endpoints* and is denoted $d(\vec{u},\vec{v})$. In $\Re^2$ we have:

$$d(\vec{u},\vec{v}) = \sqrt{(v_1-u_1)^2+(v_2-u_2)^2}$$

Similarly, in $\Re^3$ we have:

$$d(\vec{u},\vec{v}) = \sqrt{(v_1-u_1)^2+(v_2-u_2)^2+(v_3-u_3)^2}$$

The formula for the distance between two vectors in $\Re^2$ is just an application of the Pythagorean Theorem, found by considering the line segment joining the two endpoints to be the hypotenuse of a right-angled triangle. The height of the triangle is the vertical distance between the two points (i.e. the difference between their y-coordinates, or the second components of the vectors) and likewise the length of the base of the triangle is the horizontal distance between the two points (i.e. the difference between their x-coordinates, or the first components of the vectors). The formula for the distance between two vectors in $\Re^3$ is based on the same idea, but in 3 dimensions. Notice that because the terms inside the square root are each squared, the distance between $\vec{u}$ and $\vec{v}$ is the same as the distance between $\vec{v}$ and $\vec{u}$. That is, $d(\vec{u},\vec{v}) = d(\vec{v},\vec{u})$. So it doesn't matter which vector is mentioned first.

Example 1.3. For the vectors $\vec{u}=(1,2)$, $\vec{v}=(2,1)$ and $\vec{w}=(2,4)$, find
(a) $d(\vec{u},\vec{v})$ (b) $d(\vec{w},\vec{u})$

Solution:
(a) We have $u_1=1$, $u_2=2$, $v_1=2$ and $v_2=1$. (That is, when we refer to the components of $\vec{u}$ as u_1 and u_2, we simply mean whatever numbers are the first and second components, respectively, of the vector. Similarly for a vector with three components.) So for the distance between $\vec{u}$ and $\vec{v}$ we get

$$d(\vec{u},\vec{v}) = \sqrt{(v_1-u_1)^2+(v_2-u_2)^2} = \sqrt{(2-1)^2+(1-2)^2} = \sqrt{(1)^2+(-1)^2} = \sqrt{1+1} = \sqrt{2}$$

(b) For the distance between $\vec{w}$ and $\vec{u}$ we do a similar calculation:

$$d(\vec{w},\vec{u}) = \sqrt{(u_1-w_1)^2+(u_2-w_2)^2} = \sqrt{(1-2)^2+(2-4)^2} = \sqrt{(-1)^2+(-2)^2} = \sqrt{1+4} = \sqrt{5}$$

Notice that the formula says to take, for each component, the square of the *second-mentioned* vector minus the *first-mentioned* vector. So even though the formula said $(v_1-u_1)^2$, in calculating $d(\vec{w},\vec{u})$, we put u_1 before the minus sign, not after. (That is, in this calculation, $\vec{u}$ was filling the role played by $\vec{v}$ in the formula.) In this particular formula, it doesn't matter because, as previously mentioned, the distance is the same, whether we think of it as the distance between $\vec{w}$ and $\vec{u}$ or as the distance between $\vec{u}$ and $\vec{w}$. (Is the distance between *here* and *there* any different than the distance between *there* and *here*? Well, I suppose sometimes, when there are one-way streets involved. But not usually.) In other situations, though, it will be important to use the right vector in the right place in the formula.

Example 1.4. Find the distance between $\vec{u}=(1,2,3)$ and $\vec{v}=(-1,0,1)$.

Solution:
We do the same sort of calculation as before, but now with a third component. We get:

$$\begin{aligned} d(\vec{u},\vec{v}) &= \sqrt{(v_1-u_1)^2+(v_2-u_2)^2+(v_3-u_3)^2} = \sqrt{(-1-1)^2+(0-2)^2+(1-3)^2} \\ &= \sqrt{(-2)^2+(-2)^2+(-2)^2} = \sqrt{4+4+4} = \sqrt{4\times 3} = \sqrt{4}\sqrt{3} = 2\sqrt{3} \end{aligned}$$

The length of a vector is defined as the distance between where it starts and where it ends. That is, the distance from the origin (where all vectors start) to the endpoint of the vector. There are some other words we sometimes use which mean exactly the same thing. We sometimes talk about the *magnitude* or the *norm* of a vector. These terms both mean the length of the vector.

> **Definition:** The **length** of a vector $\vec{v}$, also called the **magnitude** or the **norm** of the vector, is denoted by $||\vec{v}||$ and is defined to be
>
> $$||\vec{v}|| = d(0, \vec{v})$$
>
> Therefore for $\vec{v} \in \Re^2$ we have:
>
> $$||\vec{v}|| = \sqrt{(v_1)^2 + (v_2)^2}$$
>
> and for $\vec{v} \in \Re^3$ we have:
>
> $$||\vec{v}|| = \sqrt{(v_1)^2 + (v_2)^2 + (v_3)^2}$$
>
> Any vector whose length is 1 is called a **unit vector**.

Notice that the only way to get $||\vec{v}|| = 0$ is by having each component of $\vec{v}$ be 0. So $||\vec{v}|| = 0$ if and only if $\vec{v} = \vec{0}$.

Example 1.5. Find the length of $\vec{u} = (1, 2)$, the magnitude of $\vec{v} = (2, 1)$ and the norm of $\vec{w} = (2, 4)$.

Solution:
Length, magnitude and norm all mean the same thing, so we just use the same formula three times, once for each vector. We get

$$\begin{aligned} ||\vec{u}|| &= \sqrt{1^2 + 2^2} = \sqrt{1 + 4} = \sqrt{5} \\ ||\vec{v}|| &= \sqrt{2^2 + 1^2} = \sqrt{4 + 1} = \sqrt{5} \\ ||\vec{w}|| &= \sqrt{2^2 + 4^2} = \sqrt{4 + 16} = \sqrt{20} \end{aligned}$$

Notice that $\sqrt{20} = \sqrt{(4)(5)} = \sqrt{4}\sqrt{5} = 2\sqrt{5}$. The length of $\vec{w}$ is twice the length of $\vec{u}$. That is, $||\vec{w}|| = 2||\vec{u}||$.

Example 1.6. Find $\left|\left|\left(\frac{3}{5}, 0, -\frac{4}{5}\right)\right|\right|$.

Solution:

$$\begin{aligned} \left|\left|\left(\frac{3}{5}, 0, -\frac{4}{5}\right)\right|\right| &= \sqrt{\left(\frac{3}{5}\right)^2 + (0)^2 + \left(-\frac{4}{5}\right)^2} \\ &= \sqrt{\left(\frac{3^2}{5^2}\right) + 0 + \left(\frac{(-4)^2}{5^2}\right)} \\ &= \sqrt{\frac{9}{25} + \frac{16}{25}} \\ &= \sqrt{\frac{9 + 16}{25}} = \sqrt{\frac{25}{25}} = \sqrt{1} = 1 \end{aligned}$$

Notice: Since $\left|\left|\left(\frac{3}{5}, 0, -\frac{4}{5}\right)\right|\right| = 1$ then $\left(\frac{3}{5}, 0, -\frac{4}{5}\right)$ is a unit vector.

Example 1.7. If $\vec{u} = \left(\frac{2}{3}, \frac{2}{3}, k\right)$ is a unit vector, what is the value of k?

Solution:
We need $||\vec{u}|| = 1$ in order for $\vec{u}$ to be a unit vector. Since $||\vec{u}||$ is given by

$$||\vec{u}|| = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + k^2} = \sqrt{\frac{4}{9} + \frac{4}{9} + k^2} = \sqrt{\frac{8}{9} + k^2}$$

then we must have

$$\sqrt{\frac{8}{9} + k^2} = 1$$

$$\text{so} \qquad \left(\sqrt{\frac{8}{9} + k^2}\right)^2 = 1^2 \quad \rightarrow \quad \frac{8}{9} + k^2 = 1$$

$$\text{therefore} \quad k^2 = 1 - \frac{8}{9} = \frac{9}{9} - \frac{8}{9} = \frac{9-8}{9} = \frac{1}{9}$$

$$\text{and so} \qquad k = \pm\sqrt{\frac{1}{9}} = \pm\frac{\sqrt{1}}{\sqrt{9}} = \pm\frac{1}{3}$$

(That is, since both $\left(\frac{1}{3}\right)^2 = \frac{1}{9}$ and $\left(-\frac{1}{3}\right)^2 = \frac{1}{9}$, then knowing that $k^2 = \frac{1}{9}$ tells us that k is one of these 2 values, but doesn't tell us which one it is.)

We see that k could be either $\frac{1}{3}$ or $-\frac{1}{3}$. That is, both $\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$ and $\left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$ are unit vectors, and $\vec{u}$ could be either of these vectors.

Definition: A **scalar** is just a number. Thus any element of $\Re$ is a scalar.

A vector can be multiplied, or scaled, by a number. When a vector (or other entity, other than simply a number) is multiplied by a scalar, we call it *scalar multiplication.* When a vector is multiplied by a scalar, the effect is that *each component* of the vector is multiplied by that scalar.

Definition: Let $c \in \Re$ be any scalar and $\vec{v} \in \Re^2$ or $\Re^3$ be any vector. The **scalar multiple** $c\vec{v}$ is obtained by:

$$c\vec{v} = (cv_1, cv_2) \text{ if } \vec{v} \in \Re^2, \text{ or } c\vec{v} = (cv_1, cv_2, cv_3) \text{ if } \vec{v} \in \Re^3$$

For instance, for the vectors in Examples 1.2 and 1.3 we had

$$\vec{w} = (2, 4) = (2 \times 1, 2 \times 2) = 2(1, 2) = 2\vec{u}$$

The vector $\vec{w}$ is two times the vector $\vec{u}$, which is why we found that it was twice as long, i.e. has length twice the length of $\vec{u}$.

Every scalar multiple of a vector lies along the same line as that vector. And from the origin, there are (only) two directions you can go along a line that passes through the origin. So there are two directions which a scalar multiple of a vector could go: the same direction as the vector, or the *opposite* direction. For instance, if $\vec{v}$ goes straight up, then some scalar multiples of $\vec{v}$ go straight up, and others go straight down. Or if the scalar is 0, then the scalar multiple doesn't go anywhere at all. Scalars bigger than 0 give vectors with the same direction, while scalars less than 0 give vectors with the opposite direction. And whether a new vector goes twice as far as $\vec{v}$ in the same direction as $\vec{v}$, or goes twice as far as $\vec{v}$ in the opposite direction, it will still be true that the new vector is twice as long as $\vec{v}$. That is, both $2\vec{v}$ and $-2\vec{v}$ have magnitude $2||\vec{v}||$, but $2\vec{v}$ goes in the same direction as $\vec{v}$, while $-2\vec{v}$ goes in the opposite direction.

Theorem 1.1. *Let $\vec{v}$ be any vector, either in $\Re^2$ or $\Re^3$, and consider any $c \in \Re$. The vectors $\vec{v}$ and $c\vec{v}$ are collinear, and*

1. *if $c > 0$ then $c\vec{v}$ has the same direction as $\vec{v}$;*

2. *if $c < 0$ then $c\vec{v}$ has the opposite direction to $\vec{v}$.*

Also, $||c\vec{v}|| = |c|\,||\vec{v}||$.

Saying that two vectors are collinear is the same as saying that they are parallel. So $\vec{v}$ and $c\vec{v}$ are always parallel to one another, no matter what the value of the scalar c is. And the last part of the theorem says that you can find the magnitude of the scalar multiple of a vector by multiplying the magnitude of the vector by the *absolute value* of the scalar. So for instance, as observed before, both $2\vec{v}$ and $-2\vec{v}$ have magnitude $2||\vec{v}||$.

Notice that the zero vector is collinear to every vector, because for any $\vec{v}$ we have $0\vec{v} = \vec{0}$, so the zero vector is a scalar multiple of every vector (with the same number of components).

Example 1.8. Let $\vec{u} = (2, -3)$ and $\vec{v} = (0, -1, 2)$. Find the vectors $2\vec{u}$, $-0.5\vec{u}$, $-3\vec{v}$ and $\frac{10}{3}\vec{v}$, and find the magnitude of each of these vectors.

Solution:

$$\begin{aligned}
2\vec{u} &= 2(2,-3) = (2\times 2, 2\times(-3)) = (4,-6) \\
-0.5\vec{u} &= -0.5(2,-3) = ((-0.5)\times 2, (-0.5)\times(-3)) = (-1, 1.5) \\
-3\vec{v} &= -3(0,-1,2) = ((-3)(0), (-3)(-1), (-3)(2)) = (0, 3, -6) \\
\frac{10}{3}\vec{v} &= \frac{10}{3}(0,-1,2) = \left(\frac{10}{3}\times 0, \frac{10}{3}\times(-1), \frac{10}{3}\times 2\right) = \left(0, -\frac{10}{3}, \frac{20}{3}\right)
\end{aligned}$$

We could find the magnitudes of these new vectors using the appropriate formula for each. But it's easier to just find the magnitudes of $\vec{u}$ and $\vec{v}$ and use those to find the magnitudes of the given vectors.

$$\begin{aligned}
||\vec{u}|| &= ||(2,-3)|| = \sqrt{(2)^2 + (-3)^2} = \sqrt{4+9} = \sqrt{13} \\
\text{so } ||2\vec{u}|| &= |2|\,||\vec{u}|| = 2\sqrt{13} \\
\text{and } ||-0.5\vec{u}|| &= |-0.5|\,||\vec{u}|| = 0.5\sqrt{13} = \frac{\sqrt{13}}{2} = \frac{\sqrt{13}}{\sqrt{4}} = \sqrt{\frac{13}{4}} = \sqrt{3.25} \\
\text{Also } ||\vec{v}|| &= ||(0,-1,2)|| = \sqrt{(0)^2 + (-1)^2 + (2)^2} = \sqrt{0+1+4} = \sqrt{5} \\
\text{so } ||-3\vec{v}|| &= |-3|\,||\vec{v}|| = 3\sqrt{5} \\
\text{and } \left|\left|\frac{10}{3}\vec{v}\right|\right| &= \left|\frac{10}{3}\right|\,||\vec{v}|| = \frac{10}{3}\sqrt{5} = \frac{10\sqrt{5}}{3}
\end{aligned}$$

Example 1.9. Let $\vec{u} = (4, -3)$ and $\vec{v} = (5, 0, -2)$. Find a unit vector in the same direction as $\vec{u}$, and a unit vector in the opposite direction to $\vec{v}$.

Solution:
The magnitude of $\vec{u}$ is

$$||\vec{u}|| = ||(4, -3)|| = \sqrt{(4)^2 + (-3)^2} = \sqrt{16 + 9} = \sqrt{25} = 5$$

We get a vector in the same direction as $\vec{u}$ by taking a scalar multiple of $\vec{u}$, using a positive scalar. Let c be the positive scalar for which $c\vec{u}$ is a unit vector. Then we need $||c\vec{u}|| = 1$. But we know that $||c\vec{u}|| = |c|\,||\vec{u}||$, where in this case $||\vec{u}|| = 5$, and since c is positive, then $|c| = c$. So we need

$$c||\vec{u}|| = 1 \qquad \Rightarrow \qquad 5c = 1 \qquad \Rightarrow \qquad c = \frac{1}{5}$$

Therefore, a unit vector in the same direction as $\vec{u}$ is the vector

$$c\vec{u} = \frac{1}{5}(4, -3) = \left(\frac{4}{5}, -\frac{3}{5}\right)$$

Notice: For any vector $\vec{u}$, if $c\vec{u}$ is a unit vector, then $|c| = \dfrac{1}{||\vec{u}||}$.

We take a similar approach for finding a unit vector in the opposite direction to $\vec{v}$. As we've already seen, for a unit vector we need to scale $\vec{v}$ by a constant whose magnitude (i.e. absolute value) is $\frac{1}{||\vec{v}||}$. And for the vector to have the opposite direction to that of $\vec{v}$, the constant must be negative. So to get a unit vector in the opposite direction to $\vec{v}$, we multiply $\vec{v}$ by the scalar $-\frac{1}{||\vec{v}||}$. We have

$$||\vec{v}|| = ||(5, 0, -2)|| = \sqrt{(5)^2 + (0)^2 + (-2)^2} = \sqrt{25 + 0 + 4} = \sqrt{29}$$

and so a unit vector in the opposite direction to $\vec{v}$ is the vector $c\vec{v}$ with $c = -\frac{1}{||\vec{v}||}$, which gives

$$-\frac{1}{||\vec{v}||}\vec{v} = -\frac{1}{\sqrt{29}}(5, 0, -2) = \left(-\frac{5}{\sqrt{29}}, 0, \frac{2}{\sqrt{29}}\right)$$

In Theorem 1.1 we observed that for any vector $\vec{v}$ and any scalar c, the vectors $\vec{v}$ and $c\vec{v}$ are collinear. But it is also true that if 2 vectors are collinear then it *must* be true that they are scalar multiples of one another.

Theorem 1.2. *Vectors $\vec{u}$ and $\vec{v}$ are collinear if and only if there is some scalar value c such that $\vec{v} = c\vec{u}$.*

Example 1.10. Let $\vec{u} = (-2, 7)$ and $\vec{v} = (4, k)$. If it is known that $\vec{u}$ and $\vec{v}$ are collinear, what is the value of k?

Solution:
We know that if 2 vectors are collinear, one can be written as a scalar multiple of the other. So knowing that $\vec{u}$ and $\vec{v}$ are collinear tells us that there is some value c for which $\vec{v} = c\vec{u}$. We have

$$c\vec{u} = c(-2, 7) = (-2c, 7c)$$

Therefore to have $c\vec{u} = \vec{v}$, we need $(-2c, 7c) = (4, k)$. And of course these vectors are only equal if their corresponding components are the same. So we must have $-2c = 4$ and $7c = k$. We use the first of these to solve for c, and then substitute that in to find k. We get

$$-2c = 4 \to c = \frac{4}{-2} = -2 \to k = 7c = 7(-2) = -14$$

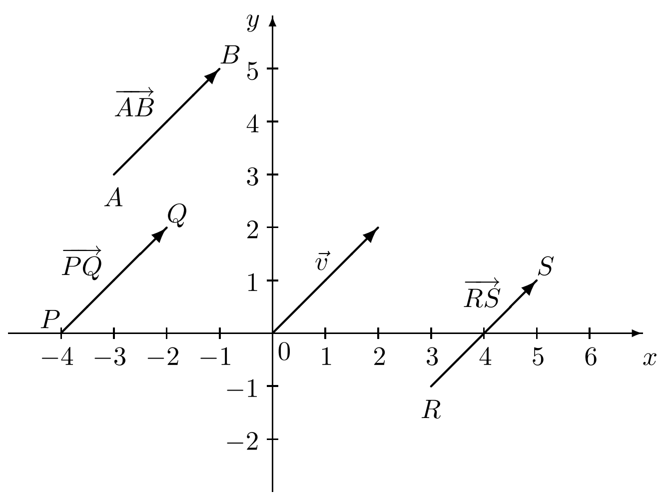
The only vector $(4, k)$ which is collinear with $\vec{u}$ is the vector $(4, -14)$.

Translation of Vectors

A vector is a directed line segment which starts at the origin. A directed line segment from some point P to some point Q, where P is not the origin, is not called a vector. (So all vectors are directed line segments, but only some directed line segments are vectors.) We use $\overrightarrow{PQ}$ to denote such a directed line segment.

> **Definition:** Two directed line segments which have the same magnitude and the same direction are called **equivalent**.

This means that any non-zero vector $\vec{v}$ is equivalent to many other directed line segments – every directed line segment which is parallel to $\vec{v}$ in the same direction as $\vec{v}$ and is the same length as $\vec{v}$. For instance, the vector $\vec{v} = (2, 2)$, which goes up to the right with slope 1, and is $\sqrt{8}$ units long, is equivalent to the directed line segment from the point $P(-4, 0)$ to the point $Q(-2, 2)$, and is also equivalent to the directed line segment from the point $R(3, -1)$ to the point $S(5, 1)$, and to the directed line segment from the point $A(-3, 3)$ to the point $B(-1, 5)$, and so forth.



In some contexts, we want to replace a directed line segment by the vector which is equivalent to it, or replace a vector by an equivalent directed line segment which starts somewhere other than at the origin. We refer to this as translating the vector, or the directed line segment.

> **Definition:** The process of replacing a directed line segment $\overrightarrow{AB}$ with the equivalent vector is called **translating $\overrightarrow{AB}$ to the origin**. Similarly, the process of replacing the vector $\vec{v}$ with an equivalent directed line segment which starts at some point P is called **translating $\vec{v}$ to P**.

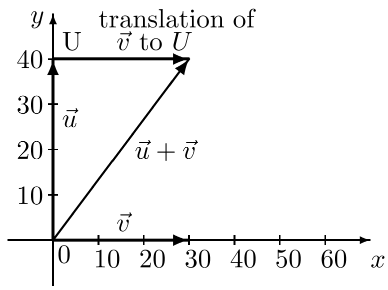
Notice that a vector or other directed line segment obtained by translation is always parallel to, and the same length as, the original directed line segment. This is because by definition, translating involves an equivalent directed line segment, i.e. one which has the same direction and the same length.

Addition of Vectors

We can add one vector to another, as long as they are in the same space, i.e both in $\Re^2$ or both in $\Re^3$. The way we do this is quite obvious. If I told you that $\vec{u}$ is the vector that travels 40 metres

North, and that $\vec{v}$ is the vector that travels 30 metres East, what would you suppose the vector $\vec{u}+\vec{v}$ would be? You'd probably guess that it's equivalent to going 40 metres North and then 30 metres East. And if you thought about it a bit more, you might realize that it should be the vector that goes directly from where you're starting to where you're ending up, instead of taking the less direct route. Because vectors don't turn corners. Each starts at the origin and goes in a straight line to its endpoint.

To add vector $\vec{v}$ to vector $\vec{u}$, we translate $\vec{v}$ to the endpoint of $\vec{u}$, which we can call U. This is like travelling 40 metres North, and *then* travelling 30 metres East. So the sum vector, $\vec{u}+\vec{v}$, is the vector that starts at the start of $\vec{u}$, which is the origin of course, and goes to the endpoint of the translation of $\vec{v}$ to U.



Adding two vectors when they're written in component form is even easier. All we need to do is to add the corresponding components.

> **Definition:** For any vectors $\vec{u}=(u_1,u_2)$ and $\vec{v}=(v_1,v_2)$ in $\Re^2$, the vector $\vec{u}+\vec{v}$, the **sum** of $\vec{u}$ and $\vec{v}$, is given by
>
> $$\vec{u}+\vec{v}=(u_1,u_2)+(v_1,v_2)=(u_1+v_1,u_2+v_2)$$
>
> Similarly, for vectors in $\Re^3$, if $\vec{u}=(u_1,u_2,u_3)$ and $\vec{v}=(v_1,v_2,v_3)$, the **sum** of $\vec{u}$ and $\vec{v}$ is
>
> $$\vec{u}+\vec{v}=(u_1,u_2,u_3)+(v_1,v_2,v_3)=(u_1+v_1,u_2+v_2,u_3+v_3)$$

For instance, in the picture above, we have $\vec{u}=(0,40)$ (go 40 metres due North) and $\vec{v}=(30,0)$ (go 30 metres due east), and we get

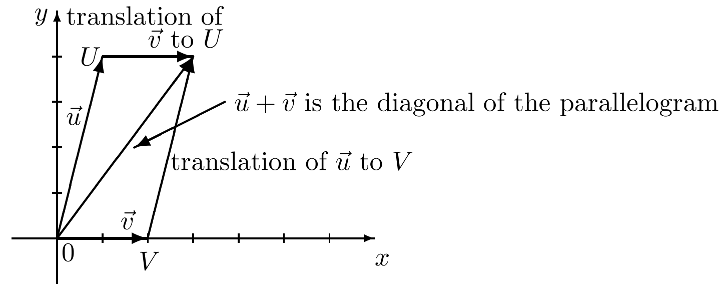
$$\vec{u}+\vec{v}=(0,40)+(30,0)=(0+30,40+0)=(30,40)$$

Example 1.11. Let $\vec{u}=(12,-4,7)$ and $\vec{v}=(-3,5,8)$. Find $\vec{u}+\vec{v}$.

Solution:

$$\vec{u}+\vec{v}=(12,-4,7)+(-3,5,8)=(12+(-3),(-4)+5,7+8)=(9,1,15)$$

There's another way to think about the sum of 2 vectors. We've seen that if we translate $\vec{v}$ to the endpoint of $\vec{u}$ (i.e. to U), the vector $\vec{u}+\vec{v}$ is the vector that goes from the start of $\vec{u}$ (i.e. the origin) to the endpoint of the translation of $\vec{v}$ to U. And of course the translation of $\vec{v}$ is parallel to the vector $\vec{v}$. If we also translate $\vec{u}$ to the endpoint of $\vec{v}$ (i.e. to V), then that will be a directed line segment which is parallel to $\vec{u}$. And those two sets of parallel lines form a parallelogram. That is, the two translations have the same endpoint. So the vector $\vec{u}+\vec{v}$ also goes from the start of $\vec{v}$ to the end of the translation of $\vec{u}$ to V. This vector is the diagonal of the parallelogram. (See diagram next page.)



Now let's think about something a bit different. What do you suppose we mean by the *negative* of a vector? For instance, if $\vec{u}$ is the vector which starts at the origin and goes 40 metres due North, what would the negative of this vector be? What direction do you suppose it goes? How long do you think it would be?

> **Definition:** The **negative** of a vector is the vector which is the same length, but has the opposite direction. We write the negative of $\vec{v}$ as $-\vec{v}$.

We know that a vector with the opposite direction to a vector $\vec{v}$ is collinear with $\vec{v}$ and hence is a scalar multiple of $\vec{v}$. For the direction to be opposite, the scalar must be negative. And for the length to be the same, the components of the vector musn't change in size, only in sign. That is, $-\vec{v} = (-1)\vec{v}$. So for instance for the vectors in Example 1.11 we get

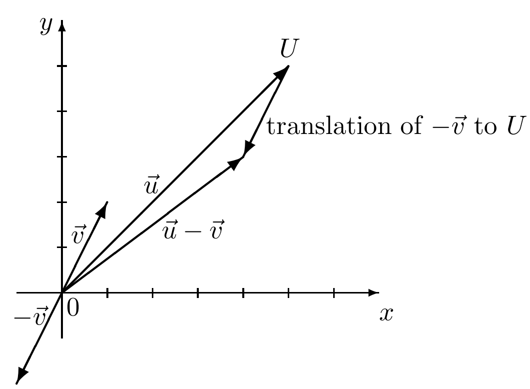
$$-\vec{u} = (-1)(12, -4, 7) = ((-1)12, (-1)(-4), (-1)(7)) = (-12, 4, -7)$$

and similarly $-\vec{v} = -(-3, 5, 8) = (3, -5, -8)$.

Theorem 1.3. *In component form, $-\vec{u}$ is the vector obtained by changing the sign of each component of $\vec{u}$. That is, $-\vec{u} = (-u_1, -u_2)$ in $\Re^2$, or $-\vec{u} = (-u_1, -u_2, -u_3)$ in $\Re^3$.*

Subtracting one vector from another

In general in mathematics, subtraction is the same as adding the negative. For instance, with numbers, we can think of $5 - 2$ as $5 + (-2)$. And the same is true with subtraction of vectors. We define that $\vec{u} - \vec{v}$ means $\vec{u} + (-\vec{v})$. In terms of directed line segments, this means that we form the vector difference $\vec{u} - \vec{v}$ by adding $-\vec{v}$ to $\vec{u}$, i.e. by translating the vector $-\vec{v}$ to the endpoint of $\vec{u}$, i.e. to U. And of course $-\vec{v}$ is simply the vector with the same magnitude as $\vec{v}$ but in the opposite direction. And then the vector $\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$ is the vector that goes from the origin, i.e. from the start of $\vec{u}$, directly to the endpoint of the translation of $-\vec{v}$ to U.



Of course, since subtracting $\vec{v}$ from $\vec{u}$ is the same as adding $-\vec{v}$ to $\vec{u}$, and since adding two vectors in component form simply involves adding corresponding components, then when we subtract one vector from another in component form, we just add the negative of each component, i.e., subtract corresponding components.

Definition: For any vectors $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$ in $\Re^2$, the **difference** vector $\vec{u} - \vec{v}$ is given by

$$\vec{u} - \vec{v} = (u_1, u_2) - (v_1, v_2) = (u_1 - v_1, u_2 - v_2)$$

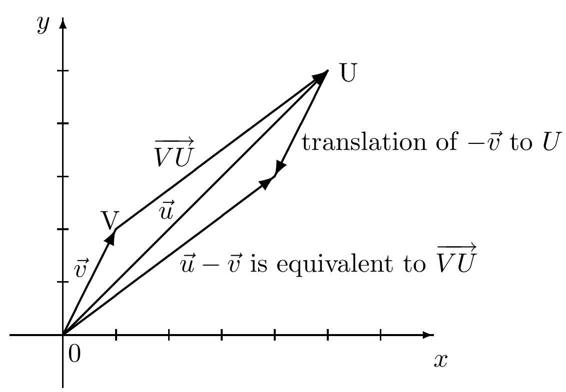
And for vectors in $\Re^3$, the **difference** vector $\vec{u} - \vec{v}$ is given by

$$\vec{u} - \vec{v} = (u_1, u_2, u_3) - (v_1, v_2, v_3) = (u_1 - v_1, u_2 - v_2, u_3 - v_3)$$

For instance, for $\vec{u} = (1, 2)$ and $\vec{v} = (3, 1)$ we get $\vec{u} - \vec{v} = (1, 2) - (3, 1) = (1 - 3, 2 - 1) = (-2, 1)$. Similarly, for $\vec{u} = (1, 0, 2)$ and $\vec{v} = (1, 1, -1)$ we have

$$\vec{u} - \vec{v} = (1, 0, 2) - (1, 1, -1) = (1 - 1, 0 - 1, 2 - (-1)) = (0, -1, 3)$$

Just as there was with addition of vectors, there's another way we can think of the subtraction of one vector from another. When we find a difference, i.e. one vector minus another, the resulting vector, when translated to the endpoint of the second vector mentioned, has as its endpoint the first vector mentioned. That is, if we translate $\vec{u} - \vec{v}$ to V, the endpoint of $\vec{v}$, we see that it goes to U, the endpoint of $\vec{u}$. Or we can think of it the other way around. If we draw the vectors $\vec{u}$ and $\vec{v}$ (each starting at the origin, of course), and draw the directed line segment $\overrightarrow{VU}$, from the endpoint of $\vec{v}$ to the endpoint of $\vec{u}$, and then translate $\overrightarrow{VU}$ to the origin, this translated vector **is** the vector $\vec{u} - \vec{v}$. We can see this in the diagram below.



Some Special Vectors

The unit vectors which run along the axes are very useful, and so they have special names. Consider $\Re^2$. The unit vector that runs along the positive x-axis, i.e. that runs from the origin for one unit in the positive horizontal direction (right), is called $\vec{\imath}$. And the unit vector that runs along the positive y-axis, i.e. that runs from the origin for one unit in the positive vertical direction (up), is called $\vec{\jmath}$. Similarly, in $\Re^3$ the unit vector along the positive x-axis is $\vec{\imath}$, the unit vector along the positive y-axis is $\vec{\jmath}$ and we also have the unit vector that runs along the positive z-axis, which is called $\vec{k}$.

Of course, the vector that starts at the origin, i.e. the point $(0, 0)$ or $(0, 0, 0)$, and runs for one unit along the positive part of one of the axes ends at the point which has 1 as the coordinate corresponding to the axis it moved along, and the other coordinate(s) is (are) still 0. For instance, the vector $\vec{\imath}$ in $\Re^2$ starts at the origin and ends at the point one unit to the right, which is the point (1,0).

Definition: The special unit vectors running along the positive axes in $\Re^2$ are:

$$\begin{aligned} \vec{\imath} &= (1,0) \\ \text{and } \vec{\jmath} &= (0,1) \end{aligned}$$

Similarly, the special unit vectors running along the positive axes in $\Re^3$ are:

$$\begin{aligned} \vec{\imath} &= (1,0,0) \\ \vec{\jmath} &= (0,1,0) \\ \text{and } \vec{k} &= (0,0,1) \end{aligned}$$

Any other vector $\vec{v}$ can be expressed in terms of these vectors. We multiply each of these special vectors by a scalar which is the corresponding component of $\vec{v}$ and then add them up. That is, we can express (v_1, v_2) as $v_1\vec{\imath} + v_2\vec{\jmath}$. Likewise, any vector (v_1, v_2, v_3) can be expressed as $v_1\vec{\imath} + v_2\vec{\jmath} + v_3\vec{k}$. So for instance $(2, -1) = 2\vec{\imath} - \vec{\jmath}$ and $(-3, 5, 17) = -3\vec{\imath} + 5\vec{\jmath} + 17\vec{k}$.

In this Unit we have learnt several vector operations: addition, subtraction and scalar multiplication. There are various properties of these operations which hold because of the way they are defined. (Some of them we've already seen; others we haven't talked about – but they're fairly obvious.) You should be aware of, and able to use, all of these properties, which are enumerated in the following Theorem.

Theorem 1.4. *Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be any vectors, all in $\Re^2$ or all in $\Re^3$. Let $\vec{0}$ be the zero vector in that same space. Let c and d be any scalars. Then the following properties hold:*

(a) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
That is, addition of vectors is commutative.

(b) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
That is, addition of vectors is associative.

(c) $\vec{u} + \vec{0} = \vec{u}$
That is, adding the zero vector to any vector leaves the vector unchanged.

(d) $\vec{u} + (-\vec{u}) = \vec{0}$
That is, the sum of a vector and its negative is the zero vector.

(e) $cd(\vec{u}) = c(d\vec{u})$
That is, to form the scalar multiple of a vector, where the scalar is a product of two scalars, it doesn't matter if the scalars are applied one at a time, or if the scalars are multiplied together before the vector is multiplied by them.

(f) $(c + d)\vec{u} = c\vec{u} + d\vec{u}$
That is, scalar multiplication of a vector is distributive over addition of scalars. So if the scalar by which a vector is to be multiplied is considered as the sum of two scalars, it doesn't matter if the vector is multiplied by each scalar separately, and then these new vectors added together, or if the two scalars are added together and then the vector is multiplied by that sum.

(g) $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
That is, scalar multiplication of a vector is distributive over addition of vectors. So if the sum of two vectors is to be multiplied by a scalar, it doesn't matter whether the vectors are multiplied by the scalar separately, and then these new vectors added together, or whether the two vectors are added together and then the sum vector is multiplied by the scalar.

(h) $1\vec{u} = \vec{u}$
That is, multiplying any vector by the scalar 1 leaves the vector unchanged.

(i) $(-1)\vec{u} = -\vec{u}$
That is, the negative of a vector (i.e. the vector with the same magnitude but opposite in direction) is the vector obtained by multiplying the vector by the scalar -1.

(j) $0\vec{u} = \vec{0}$
That is, multiplying any vector by the scalar 0 produces the zero vector.

Math 1229A/B

Unit 2:
Products of Vectors

(text reference: Section 1.2)

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2 Products of Vectors

In Unit 1 we learnt about a variety of arithmetic operations we can do with vectors. But the only kind of multiplication we learnt about is scalar multiplication of a vector, i.e. when we take a vector, and multipliply it be a scalar (i.e. a number) to obtain a new vector. We didn't have anything that was like multiplying two vectors together, i.e. forming a *product* of two vectors. In this Unit, we learn about two different kinds of vector products.

The first of these is the *dot product*. This is a product in which two vectors are combined to produce something new ... but what they produce isn't a new vector, instead it's a *scalar*. So the inputs to the calculation are 2 vectors, i.e. two elements of $\Re^2$, or of $\Re^3$, but the output of the calculation is a number, i.e. some element of $\Re$. Because this product produces a scalar as its result, the term *scalar product* is often used instead of dot product. In Physics, when a force is applied to a mass (i.e. an object), to displace the object, both the force which is applied and the displacement of the object (mass) are vectors. The amount of *work* done is a scalar quantity, calculated as the dot product of the two vectors.

The second product is called the *cross product*. Again we have two vectors being combined to produce something new, but this time what they produce is a new vector. (Just like when you take a product of 2 numbers, what you get is a new number.) The cross product is therefore also known as the *vector product*. The new vector that we get is perpendicular to *both* of the other vectors, which means it is perpendicular to the whole plane in which those 2 vectors lie. Because of that, this product is not defined for "vectors in the plane", i.e. for vectors in $\Re^2$. We must be dealing with 3-dimensional vectors, i.e. vectors in $\Re^3$, in order to be able to express a vector that is perpendicular to both. So the cross product is *only* defined for 2 vectors which are both in $\Re^3$. Again, there's a Physics application of this kind of product. This time, think about applying a force to a wrench in order to turn a bolt. The force applied is a vector, and the wrench represents another vector (actually, the vector runs from the centre of the bolt to the point on the wrench at which the force is applied). And the *turning effect* the force has on the bolt, called the *moment*, is the vector (i.e. cross) product of those two vectors. Its direction is perpendicular to both the force and the wrench – the bolt moves up or down. (But don't worry about the Physics of these things. We're not going to be talking about Physics at all. They're mentioned here just to show you that these products do have real-world applications.)

The Dot Product

Definition: Consider any vectors $\vec{u}$ and $\vec{v}$, either both from $\Re^2$ or both from $\Re^3$. The **dot product** of $\vec{u}$ and $\vec{v}$, written $\vec{u} \bullet \vec{v}$, is the number obtained when the corresponding components of $\vec{u}$ and $\vec{v}$ are multiplied together and then these products are summed. That is, we have:

$$\text{If } \vec{u},\vec{v} \in \Re^2 \text{ then } \quad \vec{u} \bullet \vec{v} = u_1v_1 + u_2v_2$$

$$\text{If } \vec{u},\vec{v} \in \Re^3 \text{ then } \quad \vec{u} \bullet \vec{v} = u_1v_1 + u_2v_2 + u_3v_3$$

Example 2.1.
(a) Calculate $(1,2) \bullet (3,-1)$ and $(2,-1,4) \bullet (3,2,-2)$.
(b) If $\vec{u} = (0,3,-2)$ and $\vec{v} = (1,0,5)$, find $\vec{v} \bullet \vec{u}$ and $(-\vec{u}) \bullet \vec{u}$.
(c) If $\vec{u} = (1,2)$, $\vec{v} = (2,-1)$ and $\vec{w} = (1,-2)$, find $(\vec{u} - \vec{v}) \bullet \vec{w}$.

Solution:
(a) For $\vec{u} = (u_1, u_2) = (1, 2)$ and $\vec{v} = (v_1, v_2) = (3, -1)$, we calculate the product of the first components, and the product of the second components, and add the two products together. That is,

$$(1, 2) \bullet (3, -1) = [1 \times 3] + [2 \times (-1)] = 3 + (-2) = 3 - 2 = 1$$

Likewise, if we consider the vectors to be $\vec{x} = (x_1, x_2, x_3) = (2, -1, 4)$ and $\vec{y} = (y_1, y_2, y_3) = (3, 2, -2)$, we find the sum of $x_1 \times y_1$, $x_2 \times y_2$ and $x_3 \times y_3$:

$$(2, -1, 4) \bullet (3, 2, -2) = (2)(3) + (-1)(2) + (4)(-2) = 6 + (-2) + (-8) = 6 - 2 - 8 = -4$$

(b) For $\vec{v} \bullet \vec{u}$, we just use the formula (recognizing that the order of the vectors is reversed this time).

$$\vec{v} \bullet \vec{u} = (1, 0, 5) \bullet (0, 3, -2) = (1)(0) + (0)(3) + (5)(-2) = 0 + 0 + (-10) = -10$$

For $(-\vec{u}) \bullet \vec{u}$, we first find $-\vec{u} = -(0, 3, -2) = (0, -3, 2)$ and then take the dot product asked for:

$$(-\vec{u}) \bullet \vec{u} = (0, -3, 2) \bullet (0, 3, -2) = (0)(0) + (-3)(3) + (2)(-2) = 0 + (-9) + (-4) = -9 - 4 = -13$$

(c) For $(\vec{u} - \vec{v}) \bullet \vec{w}$, we first need to find $\vec{u} - \vec{v}$, and then dot that vector with $\vec{w}$. We get

$$\begin{aligned} \vec{u} - \vec{v} &= (1, 2) - (2, -1) = (1 - 2, 2 - (-1)) = (-1, 3) \\ \text{so } (\vec{u} - \vec{v}) \bullet \vec{w} &= (-1, 3) \bullet (1, -2) = (-1)(1) + (3)(-2) = -1 + (-6) = -7 \end{aligned}$$

Because all we're doing is multiplying and adding, and order isn't important in those operations, then order also isn't important in calculating a dot product. That is, the order of the components is of course important, but the order of the vectors isn't. Whether we calculate $\vec{u} \bullet \vec{v}$ or calculate $\vec{v} \bullet \vec{u}$, the same numbers are being multiplied together, giving the same products being added up, so the answer is the same. That is, the dot product operation is commutative. Likewise, it doesn't matter when, or to what, a scalar multiplier is applied. Whether we multiply $\vec{u}$ by a scalar before forming the dot product, or multiply $\vec{v}$ by that scalar instead, or even wait and multiply the dot product value by that scalar, it all comes out the same. The same with addition. We can distribute dot product over addition of vectors, or we can "factor out" a dot product from a sum of dot products (as long as the same vector is dotted) and the answer never changes. Of course, if there are 0's in all the products that are being added, the final result is just 0, so the dot product of any vector with the 0 vector is just 0. And compare the dot product formula, when dotting a vector with *itself*, to the formula for finding the magnitude of a vector. The only thing missing is the square root sign. So the magnitude of $\vec{u}$ can be thought of as $\sqrt{\vec{u} \bullet \vec{u}}$. (Remember, magnitude must be positive, so we take the positive square root.) Or the dot product of $\vec{u}$ with itself can be thought of as the square of the magnitude of $\vec{u}$ ... whichever way you want to look at it. These ideas are summarized in the following theorem.

Theorem 2.1. *Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be any 3 vectors from $\Re^2$, or any 3 vectors from $\Re^3$, and let c be any scalar. Then the following properties hold:*

1. *(commutative property)* $\vec{u} \bullet \vec{v} = \vec{v} \bullet \vec{u}$
2. $c(\vec{u} \bullet \vec{v}) = (c\vec{u}) \bullet \vec{v} = \vec{u} \bullet (c\vec{v})$
3. *(distributive property)* $\vec{u} \bullet (\vec{v} + \vec{w}) = (\vec{u} \bullet \vec{v}) + (\vec{u} \bullet \vec{w})$
4. $\vec{u} \bullet \vec{0} = 0$
5. $\vec{u} \bullet \vec{u} = ||\vec{u}||^2$

For instance, in Example 2.1 (b), we found that $\vec{v} \bullet \vec{u} = -10$. By property 1. above, we could have calculated $\vec{u} \bullet \vec{v}$ instead, and we would have got the same answer:

$$\vec{u} \bullet \vec{v} = (0,3,-2) \bullet (1,0,5) = (0)(1) + (3)(0) + (-2)(5) = 0 + 0 + (-10) = -10$$

Also, we found that $(-\vec{u}) \bullet \vec{u} = -13$. Property 2. tells us that we didn't need to find $-\vec{u}$ first. We could instead have calculated $\vec{u} \bullet \vec{u}$, and then taken the negative (i.e. applied a -1 multiplier) afterwards. Also, having found that $-(\vec{u} \bullet \vec{u}) = -13$, so that $\vec{u} \bullet \vec{u} = 13$, Property 5. tells us that $||\vec{u}|| = \sqrt{13}$. And in Example 2.1 (c), instead of finding $(\vec{u} - \vec{v}) \bullet \vec{w}$, we could have used Property 3. as follows:

$$(\vec{u}-\vec{v})\bullet\vec{w} = (\vec{u}\bullet\vec{w})-(\vec{v}\bullet\vec{w}) = [(1,2)\bullet(1,-2)]-[(2,-1)\bullet(1,-2)] = (1+(-4))-(2+2) = 1-4-4 = -7$$

Notice: In that calculation, we actually used other properties, of dot product and of vector arithmetic, too. Because the dot product is commutative (Property 1.), we don't need to worry about whether we have the form $\vec{x} \bullet (\vec{y} + \vec{z})$ (as shown in Property 3.), or the form $(\vec{y} + \vec{z}) \bullet \vec{x}$ (as we had above) when we distribute the dot product over vector addition. Also, because we know that vector subtraction is really just the same as addition of the negative of the vector, we can also distribute the dot product over vector subtraction. That is, $(\vec{u} - \vec{v}) \bullet \vec{w} = (\vec{u} + (-1)\vec{v}) \bullet \vec{w}$. And then when we have distributed the dot product, we can also, by Property 2., pull the -1 multiplier outside, so that again we're adding the negative, of $\vec{v} \bullet \vec{w}$ this time, and thus just subtracting.

Also Notice: We learnt in Unit 1 that for any vector $\vec{u}$, $0\vec{u} = \vec{0}$. That is, if any vector is multiplied by the *scalar* value 0, the result is *the zero vector*. And now, in the above theorem, we are told that $\vec{u} \bullet \vec{0} = 0$. That is, when any vector is dotted with *the zero vector*, the result of that dot product, which as always is *a scalar*, is the number 0. Make sure you understand the use of, and difference between, the scalar value 0 and the vector $\vec{0}$, here and elsewhere.

Example 2.2. If $\vec{v} = -\frac{2}{3}\vec{u}$, and $\vec{u} \bullet \vec{v} = -6$, find $||\vec{v}||$.

Solution:
Approach 1: Since $\vec{v} = -\frac{2}{3}\vec{u}$, then we have

$$\vec{u} \bullet \vec{v} = \vec{u} \bullet \left(-\frac{2}{3}\vec{u}\right) = \left(-\frac{2}{3}\right) \vec{u} \bullet \vec{u} = -\frac{2}{3}(||\vec{u}||)^2$$

so knowing that $\vec{u} \bullet \vec{v} = -6$ also tells us that $-\frac{2}{3}||\vec{u}||^2 = -6$, so we get

$$||\vec{u}||^2 = -6 \div \left(-\frac{2}{3}\right) = -6 \times -\frac{3}{2} = 3 \times 3 = 9$$

Therefore $||\vec{u}|| = \sqrt{9} = 3$ and so

$$||\vec{v}|| = \left|\left|-\frac{2}{3}\vec{u}\right|\right| = \left|-\frac{2}{3}\right| ||\vec{u}|| = \frac{2}{3} \times 3 = 2$$

Approach 2: (quicker – don't find $||\vec{u}||$ first) Knowing that $\vec{v} = -\frac{2}{3}\vec{u}$ also means that we have $\vec{u} = \vec{v} \div -\frac{2}{3} = \vec{v} \times -\frac{3}{2} = -\frac{3}{2}\vec{v}$. And then we have

$$\vec{u} \bullet \vec{v} = \left(-\frac{3}{2}\vec{v}\right) \bullet \vec{v} = -\frac{3}{2}(\vec{v} \bullet \vec{v}) = -\frac{3}{2}||\vec{v}||^2$$

and so using the fact that $\vec{u} \bullet \vec{v} = -6$ we get

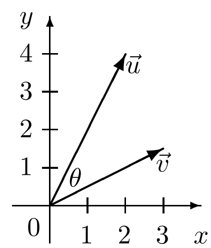
$$-\frac{3}{2}||\vec{v}||^2 = -6 \quad \Rightarrow \quad ||\vec{v}||^2 = -6 \times -\frac{2}{3} = 4 \quad \Rightarrow \quad ||\vec{v}|| = \sqrt{4} = 2$$

The angle between 2 vectors

Let $\vec{u}$ and $\vec{v}$ be any 2 vectors, either both in $\Re^2$ or both in $\Re^3$. Then $\vec{u}$ and $\vec{v}$ are both directed line segments which each go from the origin to some endpoint. Their *tails*, which come together at the origin, form an angle. Actually, they form 2 angles – the angle formed by "going the short way", and the angle formed by "going the long way", where the 2 angles together form a full circle. (If $\vec{u}$ and $\vec{v}$ are opposite in direction, then "the short way" and "the long way" are the same distance, halfway around a circle.) When we say *the angle between* the vectors, we mean the angle the 2 vectors form at the origin, "going the short way".

> **Definition:** For any vectors $\vec{u}$ and $\vec{v}$, either both in $\Re^2$ or both in $\Re^3$, **the angle between $\vec{u}$ and $\vec{v}$** means the angle no more than 180° (or π radians) formed by the directed line segment representations of these vectors. We often use θ to represent this angle.

For instance, for the vectors $\vec{u}$ and $\vec{v}$ depicted here, the angle between $\vec{u}$ and $\vec{v}$ is the angle θ shown:



Geometrically, the scalar value that is the dot product of two vectors gives the value of a specific calculation involving the magnitudes of the vectors and the angle between them. And we can use this fact to express a formula for the cosine value of the angle between the vectors.

You have probably taken some trigonometry before, which deals with angles (particularly in triangles). Since our focus in this course is primarily on vectors expressed in component form, we are not particularly concerned with trigonometry in this course, so you don't need to remember anything you learned previously about this. We don't even need to think about what the cosine value of an angle means or represents. And we certainly don't want to learn/review enough trigonometry to understand where the following theorem comes from. However, you should realize that any 2 vectors *do* form an angle. And the formula in the theorem below expresses how the dot product and magnitudes of the vectors tell us about this angle.

Theorem 2.2. *Consider any vectors $\vec{u}$ and $\vec{v}$, either both in $\Re^2$ or both in $\Re^3$. Let θ be the angle between $\vec{u}$ and $\vec{v}$. Then*

$$\vec{u} \bullet \vec{v} = ||\vec{u}||\ ||\vec{v}|| \cos\theta$$

and provided that $\vec{u} \neq \vec{0}$ and $\vec{v} \neq \vec{0}$, this relationship can be rearranged to find the value of $\cos\theta$ *as*

$$\cos\theta = \frac{\vec{u} \bullet \vec{v}}{||\vec{u}||\ ||\vec{v}||}$$

That is, for any non-zero vectors $\vec{u}$ and $\vec{v}$, the angle between $\vec{u}$ and $\vec{v}$ is the angle θ whose cosine value is the value given by this formula. (There's only one angle no bigger than 180° which has that cosine value.)

Notice: If $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$, then one of the terms in the denominator is 0, so we are trying to divide by 0, which isn't possible. It doesn't make any sense to talk about the angle between the zero vector and another vector, so in that case θ is undefined and therefore so is $\cos\theta$.

Also Notice: This formula for calculating $\cos\theta$, and another formula involving $\sin\theta$ later in this unit, are the only times we'll be using any of the ideas of trigonometry in this course.

Example 2.3. Find the value of $\cos\theta$ where θ is the angle between $\vec{u}$ and $\vec{v}$ in each of the following:
(a) $\vec{u} = (5,4)$ and $\vec{v} = (2,-3)$ (b) $\vec{u} = (1,2,3)$ and $\vec{v} = (3,-2,1)$
(c) $\vec{u} = (1,2)$ and $\vec{v} = (2,-1)$ (c) $\vec{u} = (2,3)$ and $\vec{v} = (4,6)$

Solution:
For each of these, we use the formula $\cos\theta = \dfrac{\vec{u}\bullet\vec{v}}{||\vec{u}||\,||\vec{v}||}$, so we need to find $||\vec{u}||$, $||\vec{v}||$ and $\vec{u}\bullet\vec{v}$.

(a) For $\vec{u} = (5,4)$ and $\vec{v} = (2,-3)$ we get:

$$\begin{aligned} ||\vec{u}|| &= \sqrt{5^2+4^2} = \sqrt{25+16} = \sqrt{41} \\ ||\vec{v}|| &= \sqrt{2^2+(-3)^2} = \sqrt{4+9} = \sqrt{13} \\ \text{and } \vec{u}\bullet\vec{v} &= (5,4)\bullet(2,-3) = (5)(2)+(4)(-3) = 10+(-12) = 10-12 = -2 \\ \text{so } \cos\theta &= \frac{-2}{(\sqrt{41})(\sqrt{13})} = -\frac{2}{\sqrt{41}\sqrt{13}} \end{aligned}$$

(b) We have $\vec{u} = (1,2,3)$ and $\vec{v} = (3,-2,1)$, so we get:

$$\begin{aligned} ||\vec{u}|| &= \sqrt{1^2+2^2+3^2} = \sqrt{1+4+9} = \sqrt{14} \\ ||\vec{v}|| &= \sqrt{3^2+(-2)^2+1^2} = \sqrt{9+4+1} = \sqrt{14} \\ \vec{u}\bullet\vec{v} &= (1,2,3)\bullet(3,-2,1) = (1)(3)+(2)(-2)+(3)(1) = 3+(-4)+3 = 6-4 = 2 \\ \text{and } \cos\theta &= \frac{2}{(\sqrt{14})(\sqrt{14})} = \frac{2}{14} = \frac{1}{7} \end{aligned}$$

(c) When $\vec{u} = (1,2)$ and $\vec{v} = (2,-1)$ we get:

$$\begin{aligned} ||\vec{u}|| &= \sqrt{1^2+2^2} = \sqrt{1+4} = \sqrt{5} \\ ||\vec{v}|| &= \sqrt{2^2+(-1)^2} = \sqrt{4+1} = \sqrt{5} \\ \vec{u}\bullet\vec{v} &= (1,2)\bullet(2,-1) = (1)(2)+(2)(-1) = 2+(-2) = 2-2 = 0 \\ \text{so } \cos\theta &= \frac{0}{(\sqrt{5})(\sqrt{5})} = \frac{0}{5} = 0 \end{aligned}$$

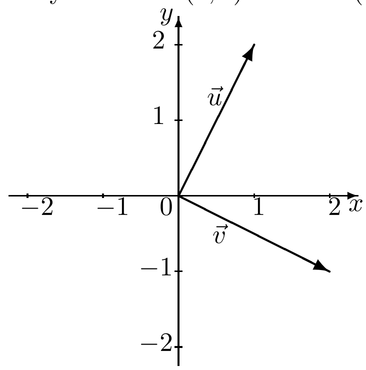
(d) For $\vec{u} = (2,3)$ and $\vec{v} = (4,6)$ we have:

$$\begin{aligned} ||\vec{u}|| &= \sqrt{2^2+3^2} = \sqrt{4+9} = \sqrt{13} \\ ||\vec{v}|| &= \sqrt{4^2+6^2} = \sqrt{16+36} = \sqrt{52} \\ \text{and } \vec{u}\bullet\vec{v} &= (2,3)\bullet(4,6) = (2)(4)+(3)(6) = 8+18 = 26 \\ \text{so } \cos\theta &= \frac{26}{(\sqrt{13})(\sqrt{52})} = \frac{26}{\sqrt{13\times 52}} = \frac{26}{\sqrt{13\times 13\times 4}} = \frac{26}{\sqrt{13^2\times 2^2}} = \frac{26}{13\times 2} = \frac{26}{26} = 1 \end{aligned}$$

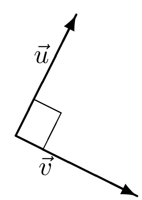
Notice: Since $(4,6) = 2(2,3)$ we have $\vec{v} = 2\vec{u}$ and so $\vec{u}$ and $\vec{v}$ have the same direction. That means that the angle between them is 0 (i.e. 0° or 0 radians), and $\cos 0 = 1$. Similarly, if $\vec{w} = (-4,-6)$ we have $\vec{w} = -2\vec{u}$ and so $\vec{w}$ has the opposite direction to $\vec{u}$. In this case, the calculations are the same as those shown above except that the numerator is negative so the final answer is $\cos\theta = -1$. Whenever $\vec{u}$ and $\vec{v}$ have opposite directions, the angle between them is $\theta = 180^\circ$ (or π radians), which has cosine value -1.

Let's think some more about what we had in part (c) of that example. We had two non-zero vectors whose dot product was 0, which meant that also the cosine of the angle between them was 0. What angle has cosine value 0? Do you remember?

You probably don't (and we said earlier that you wouldn't need to remember such things), so let's look at a picture of these vectors. They were $\vec{u} = (1, 2)$ and $\vec{v} = (2, -1)$. We have:



Let's see that again, without the axes:



It's a right angle! $\vec{u}$ and $\vec{v}$ are perpendicular! *Notice:* In $\Re^2$, the vector $(b, -a)$ is always perpendicular to the vector (a, b). And it's true ... the angle (smaller than 180°) whose cosine value is zero is $\theta = 90°$ (or $\frac{\pi}{2}$ radians). But when we're talking about the relationship between two vectors, *perpendicular* isn't the word we usually use. We say that two lines are perpendicular, or that two planes are perpendicular, but we say that two vectors are *orthogonal*.

> **Definition:** Two vectors are said to be **orthogonal** if the angle between them is 90° (i.e. $\frac{\pi}{2}$ radians).

Notice: We would also use the word orthogonal if we were talking about two directed line segments. Two directed line segments are orthogonal if the vectors obtained by translating them to the origin are orthogonal.

Also Notice: Orthogonal means the same thing as perpendicular, but the usage is a bit different. Another word that also means the same thing is *normal*. We use *perpendicular* when comparing two lines or planes, *orthogonal* when comparing two vectors (or directed line segments), and *normal* when comparing a vector to a line or a plane. That is, we say that the vector $\vec{u}$ is *normal to* a particular line or plane if the vector, or the corresponding directed line segment when translated to some point on that line or plane, meets the line or plane at an angle of 90°.

Theorem 2.3. *Let $\vec{u}$ and $\vec{v}$ be two vectors, either both in $\Re^2$ or both in $\Re^3$. Then:*

$$\vec{u} \text{ and } \vec{v} \text{ are orthogonal if and only if } \vec{u} \bullet \vec{v} = 0$$

Notice that for $\vec{u} = (a, b)$ and $\vec{v} = (b, -a)$ we get

$$\vec{u} \bullet \vec{v} = (a, b) \bullet (b, -a) = (a)(b) + (b)(-a) = ab + (-ba) = ab - ab = 0$$

So as mentioned above, for any real values of a and b, the 2-dimensional vectors (a, b) and $(b, -a)$ are orthogonal to one another. And any scalar multiple of $(b, -a)$ is also orthogonal to (a, b). However, in $\Re^3$ there's no easy way to recognize that 2 vectors are orthogonal, other than by calculating their dot product. (For $\Re^2$, you'll want to remember that you can get a vector orthogonal to (a, b) simply by switching the components and changing the sign of one of them.)

Example 2.4. Prove that $\vec{u} = (1, 3, -2)$ and $\vec{v} = (5, 1, 4)$ are orthogonal.

Solution:

$$\vec{u} \bullet \vec{v} = (1, 3, -2) \bullet (5, 1, 4) = (1)(5) + (3)(1) + (-2)(4) = 5 + 3 + (-8) = 0$$

Since $\vec{u} \bullet \vec{v} = 0$, then $\vec{u}$ and $\vec{v}$ must be orthogonal.

Example 2.5. Let $\vec{u} = (2, 0, 4)$ and $\vec{v} = (1, 1, k)$. If $\vec{u}$ and $\vec{v}$ are orthogonal, what is the value of k?

Solution:
If $\vec{u}$ and $\vec{v}$ are orthogonal, then it must be true that $\vec{u} \bullet \vec{v} = 0$. But we can calculate $\vec{u} \bullet \vec{v}$:

$$\vec{u} \bullet \vec{v} = (2, 0, 4) \bullet (1, 1, k) = (2)(1) + (0)(1) + 4(k) = 2 + 0 + 4k = 2 + 4k$$

So it must be true that $2 + 4k = 0$, which gives $4k = -2$ and so $k = \frac{-2}{4} = -\frac{1}{2}$.

The Cross Product

As previously stated, the **Cross Product** of two vectors $\vec{u}$ and $\vec{v}$ in $\Re^3$ is a new vector in $\Re^3$ which is orthogonal to both $\vec{u}$ and $\vec{v}$. This vector operation is ***only*** defined for vectors in $\Re^3$.

Definition: Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ be any two vectors in $\Re^3$. The **cross product** of $\vec{u}$ and $\vec{v}$, denoted $\vec{u} \times \vec{v}$, is the vector

$$\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$$

Example 2.6. Find the cross product of $\vec{u} = (1, 2, 3)$ and $\vec{v} = (4, 5, 6)$.

Solution:
We have $u_1 = 1$, $u_2 = 2$ and $u_3 = 3$, with $v_1 = 4$, $v_2 = 5$ and $v_3 = 6$, and so we get:

$$\begin{aligned}
\vec{u} \times \vec{v} &= (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1) \\
&= ((2)(6) - (3)(5), (3)(4) - (1)(6), (1)(5) - (2)(4)) \\
&= (12 - 15, 12 - 6, 5 - 8) \\
&= (-3, 6, -3)
\end{aligned}$$

You will have noticed that the formula is kind of nasty ... easy to get confused on. Fortunately, there are some easier ways to remember how to find the cross product. That is, procedures that are less easily confused that will get you to the answer. The text shows one such procedure, involving something called a *determinant*, on p. 17. You should have a look at that. Here, we'll look at another procedure that the text doesn't show, and which you might find easier (until later in the course when we learn about determinants).

A Procedure for finding $\vec{u} \times \vec{v}$:

Step 1: Write down the components of the first vector, twice.
That is, we write

$$u_1 \quad u_2 \quad u_3 \quad u_1 \quad u_2 \quad u_3$$

Step 2: On the next line, lined up beneath them, write down the components of the second vector, twice.
So now we have

$$\begin{array}{cccccc} u_1 & u_2 & u_3 & u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 & v_1 & v_2 & v_3 \end{array}$$

Step 3: Now, cross off the first and last numbers on each line.
We get

$$\begin{array}{cccccc} \cancel{u}_1 & u_2 & u_3 & u_1 & u_2 & \cancel{u}_3 \\ \cancel{v}_1 & v_2 & v_3 & v_1 & v_2 & \cancel{v}_3 \end{array}$$

Step 4: Always going left-to-right, and using only the numbers that aren't crossed off, find the vectors of *down-products* and of *up-products*, by multiplying.
For the *down-products* we go down to the right:

$$\begin{array}{cccccccccc} \cancel{u}_1 & u_2 & & u_3 & & u_1 & & u_2 & \cancel{u}_3 \\ & & \searrow & & \searrow & & \searrow & & \\ \cancel{v}_1 & v_2 & & v_3 & & v_1 & & v_2 & \cancel{v}_3 \end{array}$$

and when we multiply we get (u_2v_3, u_3v_1, u_1v_2).
Similarly, for the *up-products* we go up to the right:

$$\begin{array}{cccccccccc} \cancel{u}_1 & u_2 & & u_3 & & u_1 & & u_2 & \cancel{u}_3 \\ & & \nearrow & & \nearrow & & \nearrow & & \\ \cancel{v}_1 & v_2 & & v_3 & & v_1 & & v_2 & \cancel{v}_3 \end{array}$$

and multiplying we get (v_2u_3, v_3u_1, v_1u_2).

Step 4: $\vec{u} \times \vec{v}$ is given by *down-products* $-$ *up-products*.
That is, we use

$$\begin{aligned} \vec{u} \times \vec{v} &= \textit{down-products} - \textit{up-products} \\ &= (u_2v_3, u_3v_1, u_1v_2) - (v_2u_3, v_3u_1, v_1u_2) \\ &= (u_2v_3 - v_2u_3, u_3v_1 - v_3u_1, u_1v_2 - v_1u_2) \end{aligned}$$

Notice: If you compare this to the definition, you'll see that we've switched the order of the things being multiplied together corresonding to the *up-products*, i.e. the things after the minus signs. But because multiplication of numbers is commutative, that doesn't matter. And since we won't be writing down a formula like this, but rather just numbers, we don't care.

Example 2.7. Use this procedure to find the cross product in Example 2.6.

Solution:
We have $\vec{u} = (1, 2, 3)$ and $\vec{v} = (4, 5, 6)$. Applying the procedure step-by-step, we get:

Step 1: We write down the components of $\vec{u}$, twice:

$$1 \quad 2 \quad 3 \quad 1 \quad 2 \quad 3$$

Step 2: Then on the line below, we write down the components of $\vec{v}$, twice:

$$\begin{array}{cccccc} 1 & 2 & 3 & 1 & 2 & 3 \\ 4 & 5 & 6 & 4 & 5 & 6 \end{array}$$

Step 3: And we cross off the first and last numbers on each line:

$$\begin{array}{cccccc} \cancel{1} & 2 & 3 & 1 & 2 & \cancel{3} \\ \cancel{4} & 5 & 6 & 4 & 5 & \cancel{6} \end{array}$$

Step 4: Now we multiply down to the right for the vector of *down-products* and up to the right for the vector of *up-products*:

$$\begin{array}{cccccccccc} \cancel{1} & 2 & & 3 & & 1 & & 2 & \cancel{3} \\ & & \searrow & & \searrow & & \searrow & & \\ \cancel{4} & 5 & & 6 & & 4 & & 5 & \cancel{6} \end{array}$$

gives

$$\textit{down-products} = (2 \times 6, 3 \times 4, 1 \times 5) = (12, 12, 5)$$

and then

$$\begin{array}{cccccccccc} \cancel{1} & 2 & & 3 & & 1 & & 2 & \cancel{3} \\ & & \nearrow & & \nearrow & & \nearrow & & \\ \cancel{4} & 5 & & 6 & & 4 & & 5 & \cancel{6} \end{array}$$

gives

$$\textit{up-products} = (5 \times 3, 6 \times 1, 4 \times 2) = (15, 6, 8)$$

Step 4: And now we just subtract:

$$\begin{aligned} \vec{u} \times \vec{v} &= \textit{down-products} - \textit{up-products} \\ &= (12, 12, 5) - (15, 6, 8) \\ &= (12 - 15, 12 - 6, 5 - 8) \\ &= (-3, 6, -3) \end{aligned}$$

Example 2.8. If $\vec{u} = (1, 2, 1)$ and $\vec{v} = (-1, 0, 1)$, find $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$.

Solution:
We use the procedure above. This time we'll just carry out the steps, without saying what we're doing for each:

$$\begin{array}{cccccccccc} \cancel{1} & 2 & & 1 & & 1 & & 2 & \cancel{1} \\ & & \times & & \times & & \times & & \\ \cancel{-1} & 0 & & 1 & & -1 & & 0 & \cancel{1} \end{array}$$

$$\begin{aligned} \text{We get } \textit{down-products} &= (2 \times 1, 1 \times (-1), 1 \times 0) = (2, -1, 0) \\ \textit{up-products} &= (0 \times 1, 1 \times 1, (-1) \times 2) = (0, 1, -2) \\ \text{so } \vec{u} \times \vec{v} &= (2, -1, 0) - (0, 1, -2) = (2 - 0, -1 - 1, 0 - (-2)) = (2, -2, 2) \end{aligned}$$

And then we do it again for $\vec{v} \times \vec{u}$, this time writing $\vec{u}$ under $\vec{v}$, instead of $\vec{v}$ under $\vec{u}$. That is, for $\vec{v} \times \vec{u}$, the roles of $\vec{u}$ and $\vec{v}$ are switched. And now that we've had some practice, we'll speed things up a bit by not writing down the vector of *down-products* and then the vector of *up-products*, but just skipping ahead to the line where we write that we're going to do the subtraction:

$$\begin{array}{cccccccccc} \cancel{-1} & 0 & & 1 & & -1 & & 0 & \cancel{1} \\ & & \times & & \times & & \times & & \\ \cancel{1} & 2 & & 1 & & 1 & & 2 & \cancel{1} \end{array}$$

$$\begin{aligned} \text{We get } \vec{v} \times \vec{u} &= \textit{down-products} - \textit{up-products} \\ &= (0 \times 1, 1 \times 1, (-1) \times 2) - (2 \times 1, 1 \times (-1), 1 \times 0) \\ &= (0, 1, -2) - (2, -1, 0) \\ &= (-2, 2, -2) \end{aligned}$$

Oh, look! $\vec{v} \times \vec{u}$ looks a lot like $\vec{u} \times \vec{v}$, except the signs are all switched. That is, we see that $(-2, 2, -2) = -(2, -2, 2)$, so $\vec{v} \times \vec{u} = -(\vec{u} \times \vec{v})$. That's not just a fluke. That's always true! (Go back to the definition, or to the statement of the procedure, and think about what happens if you reverse the roles of $\vec{u}$ and $\vec{v}$. You'll see that in each component, the 2 products have switched places in the subtraction, so each component is the negative of what it was before. That is, in the procedure, we've just switched the down-products and the up-products, so we get the negative of the vector we were getting before.)

We asserted before that the cross product of two vectors is a vector that's orthogonal (i.e. perpendicular) to both vectors. How can we see that this is true? Recall from Theorem 2.3 that if two vectors are orthogonal, their dot product is 0. We can use this to see that $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.

Example 2.9. For the vectors in Example 2.6, show that $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.

Solution:
We know that two vectors are orthogonal if and only if their dot product is 0. We calculate $(\vec{u} \times \vec{v}) \bullet \vec{u}$ and $(\vec{u} \times \vec{v}) \bullet \vec{v}$. In Example 2.6 we had $\vec{u} = (1, 2, 3)$ and $\vec{v} = (4, 5, 6)$, and we found that $\vec{u} \times \vec{v} = (-3, 6, -3)$. We get:

$$\begin{aligned}
(\vec{u} \times \vec{v}) \bullet \vec{u} &= (-3, 6, -3) \bullet (1, 2, 3) = (-3)(1) + 6(2) + (-3)(3) = -3 + 12 + (-9) = -3 + 12 - 9 = 0 \\
(\vec{u} \times \vec{v}) \bullet \vec{v} &= (-3, 6, -3) \bullet (4, 5, 6) = (-3)(4) + 6(5) + (-3)(6) = -12 + 30 + (-18) = 0
\end{aligned}$$

Since $(\vec{u} \times \vec{v}) \bullet \vec{u} = 0$, then we see that the vector $\vec{u} \times \vec{v}$ and the vector $\vec{u}$ are orthogonal, i.e. are perpendicular to one another. Also, since $(\vec{u} \times \vec{v}) \bullet \vec{v} = 0$ as well, then $\vec{u} \times \vec{v}$ and $\vec{v}$ are also orthogonal. That is, the vector $\vec{u} \times \vec{v}$ is orthogonal both to $\vec{u}$ and to $\vec{v}$.

That's just one example, but if you use the cross product formula, it's not hard to see that *for any* vectors $\vec{u}$ and $\vec{v}$, $(\vec{u} \times \vec{v}) \bullet \vec{u} = 0$ and also $(\vec{u} \times \vec{v}) \bullet \vec{v} = 0$. (Try it!)

Theorem 2.4. *For any vectors $\vec{u}$ and $\vec{v}$ in $\Re^3$, the vector $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.*

How can a vector be orthogonal to both $\vec{u}$ and $\vec{v}$ at the same time? If you try to think about that in $\Re^2$, you'll hurt your brain, because it can't be done! But in $\Re^3$ it's easy. There's a particular plane which contains the vectors $\vec{u}$ and $\vec{v}$, and the vector $\vec{u} \times \vec{v}$ lies in a plane perpendicular to that plane. The easiest instance to understand can be seen in the next example.

Example 2.10. Recall the vectors $\vec{\imath}$ and $\vec{\jmath}$ in $\Re^3$. Find $\vec{\imath} \times \vec{\jmath}$.

Solution:
We have $\vec{\imath} = (1, 0, 0)$ and $\vec{\jmath} = (0, 1, 0)$, so we get:

$$\begin{array}{cccccccccc}
\cancel{1} & 0 & & 0 & & 1 & & 0 & \cancel{0} \\
& & \times & & \times & & \times & & \\
\cancel{0} & 1 & & 0 & & 0 & & 1 & \cancel{0}
\end{array}$$

$$\begin{aligned}
\text{We get } \vec{\imath} \times \vec{\jmath} &= \textit{down-products} - \textit{up-products} \\
&= (0 \times 0, 0 \times 0, 1 \times 1) - (1 \times 0, 0 \times 1, 0 \times 0) \\
&= (0, 0, 1) - (0, 0, 0) \\
&= (0, 0, 1)
\end{aligned}$$

But that's the vector $\vec{k}$! So we see that $\vec{\imath} \times \vec{\jmath} = \vec{k}$.

So let's think about that. We know that $\vec{\imath} = (1, 0, 0)$ is a unit vector running along the positive x-axis, and that $\vec{\jmath} = (0, 1, 0)$ is a unit vector running along the positive y-axis, while $\vec{k} = (0, 0, 1)$ is a unit vector running along the positive z-axis. Imagine the x- and y- axes drawn on the page or the computer screen in the usual way (for 2-space), and the z-axis coming up out of the page at you, or toward you out of the computer screen. The z-axis is perpendicular to both the x-axis and the y-axis (and to the whole plane containing the page or screen). And so $\vec{k}$ is orthogonal to both $\vec{\imath}$ and $\vec{\jmath}$. Also, if we calculate $\vec{\jmath} \times \vec{\imath}$, we have $(0, 0, 0) - (0, 0, 1) = (0, 0, -1) = -\vec{k}$. That's a unit vector running along the negative z-axis (down through the page, or into the computer screen). $\vec{\jmath} \times \vec{\imath}$ has the opposite direction to $\vec{\imath} \times \vec{\jmath}$. And it's still orthogonal to both $\vec{\imath}$ and $\vec{\jmath}$.

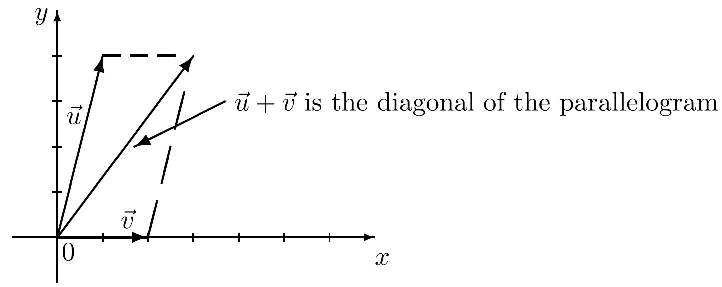
We've seen that the cross product of two vectors is orthogonal to both, and also that the cross product is *not commutative*, because if you switch the order of the vectors in the cross product, we get the opposite vector, i.e. the negative, which runs in the opposite direction. There are various other interesting properties that the cross product has. For instance, just like with the dot product, if there's a scalar multiplier on one of the vectors in a cross product, it can be applied to *either* vector, or simply factored out and applied after finding the cross product vector. And the cross product is distributive over addition (or subtraction) of vectors ... but since cross product is not commutative, we state 2 distributive laws, one for the sum being the first vector in the cross product, and one for the sum being the second. And it should be pretty easy to see why it's true that if either one of the vectors in a cross product is the zero vector, then the cross product vector is the zero vector. (But remember, it's still a vector. It's $\vec{0}$, not 0.) And likewise, it's not hard to see that if you cross any vector with itself, the down-products and up-products vectors are the same, so what you get is, again, the zero vector. Also, there's a formula for the magnitude of the cross product of 2 vectors, and it looks a lot like the formula relating the dot product of the vectors to their magnitudes, except that it involves the sine of the angle between the vectors, rather than the cosine. The following theorem lists all these properties.

Theorem 2.5. *Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be any vectors in $\Re^3$ and let c be any scalar. Let θ denote the angle between $\vec{u}$ and $\vec{v}$, as previously defined. Then the following properties hold:*

1. $\vec{v} \times \vec{u} = -(\vec{u} \times \vec{v})$
2. $c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v})$
3. *(first distributive property)* $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$
4. *(second distributive property)* $(\vec{u} + \vec{v}) \times \vec{w} = (\vec{u} \times \vec{w}) + (\vec{v} \times \vec{w})$
5. $\vec{u} \times \vec{0} = \vec{0} \times \vec{u} = \vec{0}$
6. $\vec{u} \times \vec{u} = \vec{0}$
7. $||\vec{u} \times \vec{v}|| = ||\vec{u}||\,||\vec{v}|| \sin\theta$

All those properties seem pretty theoretical, and this isn't a very theoretical course. They're useful to know, but let's focus on something more practical. What can we use the cross product for? Well, because of the last part of the theorem above, and because of some trigonometry that we don't need to know, it turns out that the magnitude of the cross product of two vectors is the same as the area of the parallelogram whose sides are those vectors, or are any directed line segments equivalent to those vectors.

To see what we mean about the parallelogram whose sides are vectors $\vec{u}$ and $\vec{v}$, consider the following diagram. Note that this appears to depict vectors in $\Re^2$. That's just to keep the diagram clear. Imagine that there's a third axis coming straight out of the page, but that the vectors depicted happen to both (all) have third component 0, so that they lie in the x-y plane.



The area of this parallelogram is of course width times height. The width is $||\vec{u}||$. What is the height? Well, it turns out (for reasons we don't need to concern ourselves with) it's $||\vec{v}|| \sin\theta$. And that means that width times height gives $||\vec{u}||\,||\vec{v}|| \sin\theta$ which, according to the last part of the theorem above, is the same thing as $||\vec{u} \times \vec{v}||$. You don't need to worry about *why* that's the height of the parallelogram, or even *why* the magnitude of the cross product vector is what the theorem says it is. You simply need to know that you can always find the area of a parallelogram in $\Re^3$ as the magnitude of the cross product of vectors equivalent to 2 adjacent sides of the parallelogram, as it says in the following theorem.

Theorem 2.6. *Consider any parallelogram in $\Re^3$. Let $\vec{u}$ and $\vec{v}$ be vectors equivalent to two adjacent sides of the parallelogram. Then the area of the parallelogram is given by $||\vec{u} \times \vec{v}||$.*

Example 2.11. Consider the vectors $\vec{u} = (1, -2, 3)$ and $\vec{v} = (2, 1, 0)$. Find the area of the parallelogram determined by these vectors.

Solution:
When we say "the parallelogram determined by" $\vec{u}$ and $\vec{v}$, we mean the parallelogram which has these vectors as two adjacent sides. According to the theorem, the area of this parallelogram is the magnitude of the cross product vector, so first we need to find that vector. For $\vec{u} = (1, -2, 3)$ and $\vec{v} = (2, 1, 0)$ we have

$$\begin{array}{cccccccc} \cancel{1} & -2 & & 3 & & 1 & -2 & \cancel{3} \\ & & & & & & & \\ \cancel{2} & 1 & & 0 & & 2 & 1 & \cancel{0} \end{array}$$

$$\begin{aligned} \text{so we get } \vec{u} \times \vec{v} &= \textit{down-products} - \textit{up-products} \\ &= ((-2) \times 0, 3 \times 2, 1 \times 1) - (1 \times 3, 0 \times 1, 2 \times (-2)) \\ &= (0, 6, 1) - (3, 0, -4) \\ &= (-3, 6, 5) \end{aligned}$$

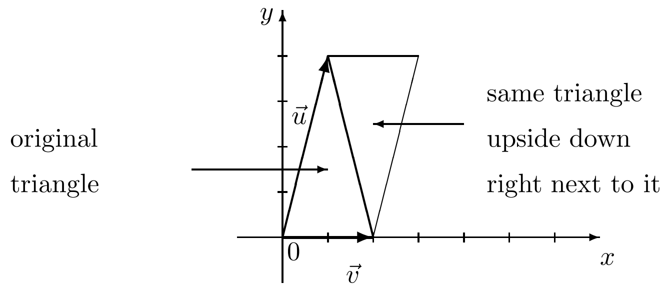
Now we need to find the magnitude of $\vec{u} \times \vec{v} = (-3, 6, 5)$, which is

$$||\vec{u} \times \vec{v}|| = ||(-3, 6, 5)|| = \sqrt{(-3)^2 + (6)^2 + (5)^2} = \sqrt{9 + 36 + 25} = \sqrt{70}$$

Therefore the area of the parallelogram determined by $\vec{u}$ and $\vec{v}$ is $\sqrt{70}$ units.

Any triangle can be thought of as half of a parallelogram. Generally, you just think about taking another copy of the same triangle and putting it upside down right beside the triangle, so that they have a common edge.

For instance, a triangle which has two adjacent sides being some vectors $\vec{u}$ and $\vec{v}$, we have:



This means that the triangle determined by $\vec{u}$ and $\vec{v}$ (i.e. with $\vec{u}$ and $\vec{v}$ as adjacent sides) is one half of the parallelogram determined by $\vec{u}$ and $\vec{v}$.

Theorem 2.7. *The area of any triangle in $\Re^3$ can be found as area $= \frac{1}{2} \times ||\vec{u} \times \vec{v}||$, where $\vec{u}$ and $\vec{v}$ are vectors equivalent to any 2 sides of the triangle.*

Example 2.12. Find the area of the triangle OAB, where O is the origin, A is the point $(2, 5, 1)$ and B is the point $(4, 6, 2)$.

Solution:
The sides of the triangle are $\vec{a} = \overrightarrow{OA} = (2, 5, 1)$, $\vec{b} = \overrightarrow{OB} = (4, 6, 2)$ and $\overrightarrow{BA} = \vec{a} - \vec{b} = (-2, -1, 1)$. (Recall from Unit 1 that the vector obtained by translating $\overrightarrow{UV}$ to the origin is the vector $\vec{v} - \vec{u}$, where $\vec{u}$ and $\vec{v}$ have endpoints U and V, respectively. Also, notice that we don't need to use the vector equivalent to $\overrightarrow{BA}$ in finding the area of the given triangle, because we only need to use 2 of the sides.) So the area of the triangle is one-half the magnitude of the vector $\vec{a} \times \vec{b}$:

$$\begin{array}{ccccccccc} \cancel{2} & 5 & & 1 & & 2 & & 5 & \cancel{1} \\ & & \times & & \times & & \times & & \\ \cancel{4} & 6 & & 2 & & 4 & & 6 & \cancel{2} \end{array}$$

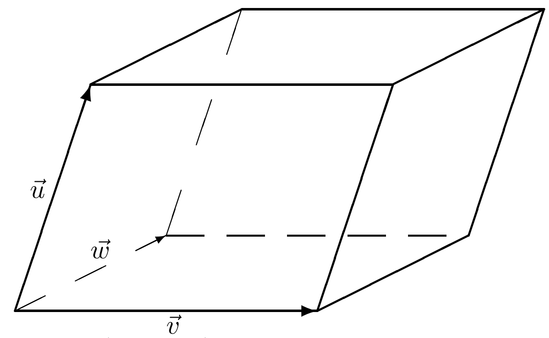
$$\begin{aligned} \text{so we get } \vec{u} \times \vec{v} &= \textit{down-products} - \textit{up-products} \\ &= (5 \times 2, 1 \times 4, 2 \times 6) - (6 \times 1, 2 \times 2, 4 \times 5) \\ &= (10, 4, 12) - (6, 4, 20) \\ &= (4, 0, -8) \end{aligned}$$

$$\begin{aligned} \text{and therefore Area} &= \frac{1}{2}||(4, 0, -8)|| = \frac{\sqrt{(4)^2 + (0)^2 + (-8)^2}}{2} = \frac{\sqrt{16 + 0 + 64}}{2} \\ &= \frac{\sqrt{80}}{2} = \frac{\sqrt{4 \times 20}}{2} = \frac{\sqrt{4 \times 4 \times 5}}{2} = \frac{4\sqrt{5}}{2} = 2\sqrt{5} \end{aligned}$$

There's one more useful application of the cross product, which is to find the volume of *the parallelepiped determined by 3 vectors.* A parallelepiped is a 3-dimensional figure which is basically a sloped cube or box. That is, a cube is a special case of a parallelepiped, in which each face is a square. In general in a parallelepiped, each face is a parallelogram.

Definition: The **parallelepiped determined by vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$** is the 6-faced solid whose faces are the parallelograms determined by $\vec{u}$ and $\vec{v}$, by $\vec{u}$ and $\vec{w}$, and by $\vec{v}$ and $\vec{w}$.

Here's a picture of the parallelepiped determined by some vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$:



Notice that each of the edges is a (directed) line segment equivalent to one of the three vectors.

Theorem 2.8. *The volume of the parallelepiped determined by the vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ is given by*

$$\textit{Volume} = |(\vec{u} \times \vec{v}) \bullet \vec{w}|$$

That is, the volume is the absolute value of the dot product of the vector $\vec{u} \times \vec{v}$ with vector $\vec{w}$. Notice that the absolute value signs are necessary because volume must be positive, whereas the dot product of $\vec{u} \times \vec{v}$ with $\vec{w}$ may be either a positive or a negative number. And given 3 vectors or directed line segments which determine a parallelepiped, there's no "right" designation of which plays the role of $\vec{u}$, which plays the role of $\vec{v}$ and which plays the role of $\vec{w}$ for this calculation. Any configuration of the vectors will give the same value for the volume. (But for some, the value of the dot product is negative, so we need to take the absolute value.)

Example 2.13. Find the volume of the parallelepiped determined by the directed line segments $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ for the points $A(1,0,1)$, $B(2,1,1)$, $C(2,2,-1)$ and $D(1,1,2)$.

Solution:
We first use the fact that when any directed line segment $\overrightarrow{UV}$ is translated to the origin, the result is the vector $\vec{v} - \vec{u}$, where $\vec{u}$ and $\vec{v}$ have endpoints U and V, respectively. We get:

$$\begin{aligned}
\overrightarrow{AB} &= \vec{b} - \vec{a} = (2,1,1) - (1,0,1) = (1,1,0) \\
\overrightarrow{AC} &= \vec{c} - \vec{a} = (2,2,-1) - (1,0,1) = (1,2,-2) \\
\overrightarrow{AD} &= \vec{d} - \vec{a} = (1,1,2) - (1,0,1) = (0,1,1)
\end{aligned}$$

The volume of the parallelepiped determined by the 3 directed line segments is the same as the volume of the parallelepiped determined by these 3 vectors. (The parallelepiped determined by the vectors is just the same parallelepiped, but moved so that the corner which was at point A is now at the origin.) We can designate the 3 vectors above as $\vec{u}$, $\vec{v}$ and $\vec{w}$ in any way. Let's assign the names in the order the vectors are listed in. That is, let's use $\vec{u} = (1,1,0)$, $\vec{v} = (1,2,-2)$ and $\vec{w} = (0,1,1)$. The next step is to find $\vec{u} \times \vec{v}$:

$$\begin{array}{ccccccccc}
\cancel{1} & 1 & & 0 & & 1 & & 1 & \cancel{0} \\
 & & \times & & \times & & \times & & \\
\cancel{1} & 2 & & -2 & & 1 & & 2 & \cancel{-2}
\end{array}$$

$$\begin{aligned}
\text{so } \vec{u} \times \vec{v} &= (1 \times (-2), 0 \times 1, 1 \times 2) - (2 \times 0, (-2) \times 1, 1 \times 1) \\
&= (-2, 0, 2) - (0, -2, 1) \\
&= (-2, 2, 1)
\end{aligned}$$

Now, we dot this cross product vector with $\vec{w}$:

$$(\vec{u} \times \vec{v}) \bullet \vec{w} = (-2, 2, 1) \bullet (0, 1, 1) = (-2)(0) + (2)(1) + (1)(1) = 0 + 2 + 1 = 3$$

The volume of the parallelepiped is $|(\vec{u} \times \vec{v}) \bullet \vec{w}| = |3| = 3$.

Notice: If we had designated the vectors in a different way, we would still get the same answer. For instance, suppose we used $\vec{u} = (1, 1, 0)$, $\vec{v} = (0, 1, 1)$ and $\vec{w} = (1, 2, -2)$. (That is, suppose we switched the roles of the vectors we previously called $\vec{v}$ and $\vec{w}$.) Then we would get:

$$\begin{array}{ccccccccc}
\cancel{1} & 1 & & 0 & & 1 & & 1 & \cancel{0} \\
 & & \times & & \times & & \times & & \\
\cancel{0} & 1 & & 1 & & 0 & & 1 & \cancel{1}
\end{array}$$

$$\begin{aligned}
\text{which gives } \vec{u} \times \vec{v} &= (1 \times 1, 0 \times 0, 1 \times 1) - (1 \times 0, 1 \times 1, 0 \times 1) = (1, 0, 1) - (0, 1, 0) = (1, -1, 1) \\
\text{and so Volume} &= |(1, -1, 1) \bullet (1, 2, -2)| = |(1)(1) + (-1)(2) + 1(-2)| = |1 - 2 - 2| = |-3| = 3
\end{aligned}$$

Math 1229A/B

Unit 3:

Lines and Planes

(text reference: Section 1.3)

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3 Lines and Planes

Lines in $\Re^2$

You are already familiar with equations of lines. In previous courses you will have written equations of lines in slope-point form, in slope-intercept form, and probably also in standard form for a line in $\Re^2$. Recall that:

$y - y_1 = m(x - x_1)$	is the *slope-point* form equation of the line through point (x_1, y_1) with slope m
$y = mx + b$	is the *slope-intercept* form equation of the line with slope m and y-intercept b
$ax + by = c$	is the *standard form* equation which either of the others can be rearranged to

In this course, we don't use the slope-point or slope-intercept forms of equations of lines. Instead, we use various other, vector-based, forms of equations. But we do still use the standard form.

You already know that given any 2 distinct points, whether in $\Re^2$ or in $\Re^3$, there is exactly one line which passes through both points. Suppose we have 2 points, P and Q. Let ℓ be the line that passes through these two points. Both point P and point Q lie on line ℓ, and so do all the points between them. In fact, the directed line segment $\overrightarrow{PQ}$ lies on line ℓ. When this directed line segment is translated to the origin, the resulting vector most likely doesn't lie on line ℓ (unless the origin happens to lie on line ℓ), but if not, it does lie on a line which is *parallel* to line ℓ. It lies on the line parallel to ℓ which passes through the origin. So this vector does give us some information about the line. (Similar to the information given by knowing the slope of a line in $\Re^2$, although it's not quite the same information.)

If a vector $\vec{v}$ lies on a particular line, or on a line parallel to that line, we say that $\vec{v}$ is *parallel to*, or is *collinear with* that line. And we call $\vec{v}$ a *direction vector* for the line. Not that the line actually has a direction associated with it. It doesn't. It extends in both directions, but has no particular "forwards along the line" or "backwards along the line" associated with it. So don't read too much meaning into the term *direction vector*. If $\vec{v}$ is a direction vector for line ℓ, then so is $-\vec{v}$. And so is every other scalar multiple of $\vec{v}$, except for $0\vec{v}$. Because of course $0\vec{v} = \vec{0}$ which has no direction information. But every other scalar multiple of $\vec{v}$ starts at the origin, and goes either the same or the opposite direction as $\vec{v}$ and therefore also lies on the line parallel to ℓ which passes through the origin. So any such vector would be considered a *direction vector* for line ℓ.

Definition: Any non-zero vector which is parallel to a line ℓ is called a **direction vector** for line ℓ.

For instance, consider the line $x + y = 2$. The points $(1, 1)$, $(2, 0)$, $(0, 2)$, $(-1, 3)$, $(3, -1)$, $(-2, 4)$, $(4, -2)$, etc., all lie on this line. So do $(1/2, 3/2)$ and $(3/2, 1/2)$ and infinitely many other points. Pick any 2 of these points, and find the vector which is the translation to the origin of the directed line segment between them, and you have a direction vector for the line. And we know that for any points P and Q, letting $\vec{p} = \overrightarrow{OP}$ denote the vector from the origin to point P and $\vec{q} = \overrightarrow{OQ}$ denote the vector from the origin to point Q, the vector $\vec{v} = \vec{q} - \vec{p}$ is the translation of directed line segment $\overrightarrow{PQ}$ to the origin. So for instance for points $P(1, 1)$ and $Q(0, 2)$, we have $\vec{p} = (1, 1)$ and $\vec{q} = (0, 2)$, and we see that $\vec{v} = \vec{q} - \vec{p} = (0, 2) - (1, 1) = (-1, 1)$ is a direction vector for the line $x + y = 2$. And other choices of P and Q give other direction vectors which are scalar multiples of this one. (Go ahead, pick some other points, and see what vectors you get.)

Point-Parallel Form

If we know a direction vector, $\vec{v}$, for a line, and any one point, P, on the line, we can use them to write an equation of the line. Because if we take the vector from the origin to the specified point, $\vec{p} = \overrightarrow{OP}$, and travel from there any non-zero scalar multiple of the direction vector (i.e. form the vector sum of the vector $\vec{p}$ and some scalar multiple of the direction vector), we travel along the line and end up at some other point on the line (i.e. the sum vector goes from the origin to some point on the line). And any point on the line can be reached by doing this. It's just a matter of choosing the right scalar multiple. So let $Q(x, y)$ be any point on the line ℓ which goes through point P and has direction vector $\vec{v}$. Then $\vec{q} = \overrightarrow{OQ} = \vec{p} + t\vec{v}$, for some value of t. And if we write the vectors in component form, it looks like we're adding points together, although of course we're not. If we let $P(x_1, y_1)$ denote the known point on the line, and use $\vec{v} = (v_1, v_2)$ to denote the direction vector, we get $(x, y) = (x_1, y_1) + t(v_1, v_2)$ as an equation which describes all points (x, y) on the line ℓ.

The way we actually write the line is a little different. We use $\vec{x}(t)$ to denote the vector (x, y), i.e. the vector from the origin to an unspecified point on the line. (It is written as $\vec{x}(t)$ to denote that the specific point obtained depends on, i.e. is a function of, the particular t value used.) Since we're writing an equation of the line using a point on the line and a vector which is parallel to the line, we refer to the equation as being in *point-parallel* form.

> **Definition:** The **point-parallel** form equation of the line ℓ which passes through point P and has direction vector $\vec{v}$ is given by
>
> $$\begin{aligned} \vec{x}(t) &= \vec{p} + t\vec{v} \\ \text{i.e. } \vec{x}(t) &= (p_1, p_2) + t(v_1, v_2) \end{aligned}$$

Example 3.1. Write a point-parallel form equation for each of the following lines:
(a) The line through $P(3, 1)$ and $Q(0, 6)$.
(b) The line through $P(1, 2)$ with direction vector $\vec{v} = (2, -1)$.
(c) The line through the origin with direction vector $\vec{v} = (0, 1)$.

Solution:
(a) The directed line segment $\overrightarrow{PQ}$ lies on, and hence is parallel to, the line through P and Q. We have the vector $\vec{p} = (3, 1)$ which goes from the origin to the point P, and the vector $\vec{q} = (0, 6)$ which goes from the origin to the point Q, and so when we translate $\overrightarrow{PQ}$ to the origin, we get

$$\vec{v} = \vec{q} - \vec{p} = (0, 6) - (3, 1) = (-3, 5)$$

as a direction vector for the line. And now, we can use *either* P or Q as the point which we know to be on the line. If we use P, we get the point-parallel form equation:

$$\vec{x}(t) = \vec{p} + t\vec{v} \qquad \Rightarrow \qquad \vec{x}(t) = (3, 1) + t(-3, 5)$$

(b) This time, we don't need to find the direction vector. We have $\vec{p} = (1, 2)$, so we use this and $\vec{v} = (2, -1)$ in the form $\vec{x}(t) = \vec{p} + t\vec{v}$ to get the point-parallel form equation

$$\vec{x}(t) = (1, 2) + t(2, -1)$$

(c) Again, all we need to do is plug the point and the direction vector into the point-parallel form. The point, of course, is the origin, i.e. $(0, 0)$, and the direction vector is $\vec{v} = (0, 1)$. We get:

$$\vec{x}(t) = (0, 0) + t(0, 1)$$

Parametric Equations

There's another form of an equation of a line, which is really more than one equation, that follows directly from the point-parallel form. Remember, the $\vec{x}(t)$ on the left hand side of the point-parallel equation is simply saying that the vector $\vec{x} = (x, y)$, corresponding to any point (x, y) on the line, is determined by the choice of value of the **parameter** t. So as we saw before, the point-parallel equation $\vec{x}(t) = (p_1, p_2) + t(v_1, v_2)$ really says that for any point (x, y) on the line, $(x, y) = (p_1, p_2) + t(v_1, v_2)$. Now, this is a statement about vectors, but (as previously noted) it looks like we're doing arithmetic with points. We're not really, because that wouldn't make any sense, but if we break it down to individual components of vectors, we get statements which are equally true of coordinates of points. Consider the first components of the vectors in the equation. We can express the vector arithmetic being done for that component as $x = p_1 + tv_1$. And if we think about x as the x-coordinate of an unspecified point on the line, and p_1 as the x-coordinate of the point P, and v_1 as the x-coordinate of the endpoint of the direction vector, then the statement $x = p_1 + tv_1$ simply says that you can get the x-coordinate of a point on the line by adding some multiple of the x-coordinate of the endpoint of the direction vector to the x-coordinate of the known point. And then, if you do the same thing with the y-coordinates, using *the same* value of the multiplier, t, the formula $y = p_2 + tv_2$ gives the y-coordinate of the same point on the line. So the point-parallel form equation also gives us *two* equations, which together describe any point on the line. And because it describes the point in terms of the effect of the parameter t, we call these *parametric equations* of the line.

> **Definition:** The line ℓ with point-parallel form equation $\vec{x}(t) = (p_1, p_2) + t(v_1, v_2)$ has **parametric equations**
>
> $$\begin{aligned} x &= p_1 + tv_1 \\ y &= p_2 + tv_2 \end{aligned}$$

Notice: Parametric equations of lines in $\Re^2$ **always** come in pairs. You can't have only one parametric equation, telling about just one component/coordinate. That doesn't describe the line. Also, if you use parametric equations to find points on the line, you have to remember to use the *same* value of t in both equations.

Example 3.2. Find parametric equations for each of the lines in Example 3.1.

Solution:
(a) We have the point-parallel form equation $\vec{x}(t) = (3, 1) + t(-3, 5)$. We get the right hand side of the x equation from the first components, and the right hand side of the y equation from the second components. Of course, we don't usually write something like $t(-3)$ or $t(5)$, or even $t5$. We would write this product as $-3t$ or $5t$. And we know that adding a negative is the same as subtracting, so the minus sign in the $-3t$ can replace the plus sign. We get:

$$\begin{aligned} x &= 3 - 3t \\ y &= 1 + 5t \end{aligned}$$

(b) This time we have the point-parallel form equation $\vec{x}(t) = (1, 2) + t(2, -1)$, which gives the parametric equations:

$$\begin{aligned} x &= 1 + 2t \\ y &= 2 - t \end{aligned}$$

(c) And now, we use $\vec{x}(t) = (0, 0) + t(0, 1)$. But unless it's the only thing there, we don't need to write a 0. And we never need to write a 1 multiplier. We get:

$$\begin{aligned} x &= 0 + 0t \qquad \Rightarrow \qquad x = 0 \\ y &= 0 + 1t \qquad \Rightarrow \qquad y = t \end{aligned}$$

Example 3.3. Write a point-parallel form equation for the line with parametric equations

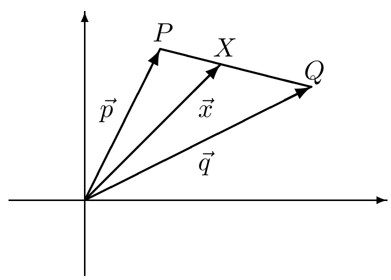
$$\begin{aligned} x &= 1+5t \\ y &= 2 \end{aligned}$$

Solution:
We use the x equation to find the first components for our point-parallel equation, and the y equation to find the second components. We need to recognize that in each equation, the number on the right hand side that isn't multiplied by t is the coordinate of the known point, P, and that the number that *is* multiplied by t is the component of the direction vector, $\vec{v}$. So from the first equation, i.e. the x-equation, we see that $p_1 = 1$ and $v_1 = 5$. And from the second equation, since there's no t multiplying it, the 2 must be p_2. So where's the t? It's invisible, which means it must have a 0 multiplier. That is, $v_2 = 0$. So we have the point $P(1,2)$ and the direction vector $\vec{v} = (5,0)$, which when we put it in the form $\vec{x}(t) = \vec{p} + t\vec{v}$ gives the point-parallel form equation

$$\vec{x}(t) = (1,2) + t(5,0)$$

Two-Point Form

Now, suppose that we have two points, P and Q, and consider the vectors $\vec{p}$ and $\vec{q}$, from the origin to the points. Let X be any point on the line segment joining P and Q. Of course, we can consider X to be a point on the *directed* line segment $\overrightarrow{PQ}$. How can we describe the vector $\vec{x}$, from the origin to point X? Let's look at a picture.



Consider the directed line segment $\overrightarrow{PX}$. Suppose that we travel along the vector $\vec{p}$ and then along the directed line segment $\overrightarrow{PX}$. Then we started at the origin and ended up at the point X, the same as if we travelled along the vector $\vec{x}$. In terms of vector sums, we travelled $\vec{p} + \vec{u}$ where $\vec{u}$ is the translation of $\overrightarrow{PX}$ to the origin. So we have $\vec{x} = \vec{p} + \vec{u}$.

But let's think more about $\overrightarrow{PX}$ and $\vec{u}$. $\overrightarrow{PX}$ is a piece of the directed line segment $\overrightarrow{PQ}$. For instance, if X was exactly one-third of the way along $\overrightarrow{PQ}$, then $\overrightarrow{PX}$ would be equivalent to one-third of $\overrightarrow{PQ}$. And we know that $\overrightarrow{PQ} = \vec{q} - \vec{p}$. So we could say (if X happened to be exactly one-third of the way along $\overrightarrow{PQ}$) that $\vec{u} = (1/3)(\vec{q} - \vec{p})$. Now, we don't necessarily have X being one-third of the way along. We're considering *any* point X on $\overrightarrow{PQ}$. But then we do know that X is $100t\%$ of the way along $\overrightarrow{PQ}$, for some value t between 0 and 1. (For instance, if X is 20% of the way along, then $t = .2$.) And then this means that we have $\overrightarrow{PX} = t\overrightarrow{PQ}$ so that $\vec{u} = t(\vec{q} - \vec{p})$. Therefore we also have

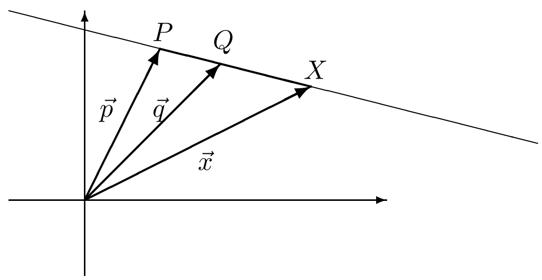
$$\vec{p} + \vec{u} = \vec{p} + t(\vec{q} - \vec{p}) = \vec{p} + t\vec{q} - t\vec{p} = (1-t)\vec{p} + t\vec{q}$$

That is, for any point X along the line segment joining point P and point Q, we have

$$\vec{x} = (1-t)\vec{p} + t\vec{q}$$

where t is some value between 0 and 1.

But nothing that we did here really required being in between P and Q. We could do something similar for *any point on the line containing* P *and* Q. The only difference is that t would no longer necessarily be between 0 and 1. That is, for any point X on the line containing the points P and Q, we could travel from the origin to point P, and then travel some scalar multiple of the vector $\vec{u}$ to end up at the point X. For instance, consider the diagram shown below. As before, we have $\vec{x} = \vec{p} + t\vec{u}$, but now t is bigger than 1. Or if we needed to go the other direction along the line, from P, then t would be negative.



And so for any point X on the line, we have

$$\vec{x} = \vec{p} + t\vec{u} = \vec{p} + t(\vec{q} - \vec{p}) = \vec{p} + t\vec{q} - t\vec{p} = (1-t)\vec{p} + t\vec{q}$$

for some value t. That is, we can express the line containing two points P and Q as the line containing all points $X(x, y)$ such that $\vec{x} = (1-t)\vec{p} + t\vec{q}$ for some value t. And so we have another form of equation for the line. We call this the *two-point* form, and as before, we write $\vec{x}(t)$ instead of just $\vec{x}$.

Definition: The **two-point** form of equation for the line through points P and Q is:

$$\vec{x}(t) = (1-t)\vec{p} + t\vec{q}$$

Example 3.4. Write equations in two-point form for each of the lines in Example 3.1.

Solution:
(a) The line here is the line through the points $P(3,1)$ and $Q(0,6)$, so we have $\vec{p} = (3,1)$ and $\vec{q} = (0,6)$ and we get the two-point form

$$\vec{x}(t) = (1-t)(3,1) + t(0,6)$$

(b) This time, we have the line through $P(1,2)$ with direction vector $\vec{v} = (2,-1)$. We don't know two points on the line, so we need to find a second point. We saw in Example 3.1(b) that a point-parallel form equation for this line is $\vec{x}(t) = (1,2) + t(2,-1)$. We can choose any value of t other than 0 to get another point on the line. (*Notice:* We don't want to use $t = 0$, because that will just give us the point we already know. But any other value of t will do.) For instance, using $t = 1$ we have $\vec{x}(1) = (1,2) + 1(2,-1) = (1,2) + (2,-1) = (3,1)$, so we see that $Q(3,1)$ is another point on the same line. Now that we know two points on the line, we can find a two-point form equation. Notice, though, that since we have already been using t as the parameter for the point-parallel form equation, we should use a different name for the parameter in the two-point form equation. (Especially since we gave t a specific value. We wouldn't want to get confused and think the parameter in the two-point form equation was supposed to have that value too.) Notice also that it doesn't matter in the least what letter we use to represent the parameter (which is just a scalar multiplier). So we can use s instead. We get:

$$\vec{x}(s) = (1-s)(1,2) + s(3,1)$$

(c) This time, we have the line through the origin with direction vector $(0, 1)$. We know that the point $(0, 0)$ is on the line, and clearly the point $(0, 1)$ is also on the line (because the vector $(0, 1)$ is on the line, since the line does pass through the origin). So a two-point form equation for this line is

$$\vec{x}(t) = (1 - t)(0, 0) + t(0, 1)$$

Point-Normal Form

When two lines meet at right angles, we call them **perpendicular**. (You knew that.) And we have already learnt that when two vectors are perpendicular, there's another word we use. Instead of saying they're perpendicular, we say they are **orthogonal**. When we're talking about a vector and a line, there's yet another word that we use. (This was mentioned earlier, but is now defined.)

> **Definition:** A vector which is perpendicular to a particular line in $\Re^2$ is said to be **normal** to the line and is called a **normal** for that line, or a **normal vector** for the line.

So *orthogonal* and *normal* really just mean *perpendicular*, but the three words are used in different contexts.

If $\vec{n}$ is a normal vector for a particular line, then it is orthogonal to any direction vector for the line. We have already seen that if we know a direction vector for a line, and one point on the line, we can write a vector equation for the line, in *point-parallel form*. Similarly, if we know a normal vector for a line in $\Re^2$, and one point on the line, we can write a vector equation for the line. We call it the **point-normal form** of the line. The equation comes from the fact that the dot product of two orthogonal vectors is 0.

Suppose $\vec{n}$ is a normal vector for a particular line, and P is a point on that line. Let X be any other point on the line. Then the directed line segment $\overrightarrow{PX}$ lies on the line, and so the vector $\vec{x} - \vec{p}$, which is equivalent to $\overrightarrow{PX}$, is parallel to the line. But then $\vec{n}$ is orthogonal to $\vec{x} - \vec{p}$, and so $\vec{n} \bullet (\vec{x} - \vec{p}) = 0$. So if $\vec{n} = (n_1, n_2)$ and the point is $P(p_1, p_2)$, then we have $(n_1, n_2) \bullet (\vec{x} - (p_1, p_2)) = 0$. This equation is the form we call point-normal. As always, it describes all the points $X(x, y)$ which lie on the line.

> **Definition:** Let ℓ be any line in $\Re^2$. If $\vec{n} = (n_1, n_2)$ is a normal for line ℓ, and $P(p_1, p_2)$ is a point on line ℓ, then an equation for line ℓ in **point-normal form** is:
>
> $$(n_1, n_2) \bullet (\vec{x} - (p_1, p_2)) = 0$$

Example 3.5. Write an equation in point-normal form for the line through $P(1, 2)$ with normal $\vec{n} = (-1, 1)$.

Solution:
We get the equation:

$$(-1, 1) \bullet (\vec{x} - (1, 2)) = 0$$

Recall that in $\Re^2$, the vector $(b, -a)$ is orthogonal to the vector (a, b), because $(b, -a) \bullet (a, b) = 0$. This means that whenever we know a direction vector for a line in $\Re^2$ (i.e. a vector which is parallel to the line) then we can easily find a normal for the line (i.e. a vector which is perpendicular to the line). And vice versa. So it's easy to find the point-parallel form of a line from the point-normal form, and also to find the point-normal form from the point-parallel form.

Example 3.6.
(a) Find an equation in point-normal form for the line $\vec{x}(t) = (0,1) + t(2,-1)$.
(b) Write an equation in point-parallel form for the line from Example 3.5.
(c) Write a point-normal form equation for the line with parametric equations

$$x = 3 + t \qquad \text{and} \qquad y = 2t - 4$$

Solution:
(a) We have $\vec{x}(t) = (0,1) + t(2,-1)$, which we recognize as a point-parallel form equation for the line through point $(0,1)$ parallel to the vector $(2,-1)$. Since the vector $(2,-1)$ is parallel to the line, then the vector $(1,2)$, obtained by switching the components and changing one of the signs, is perpendicular to the line. That is, $\vec{n} = (1,2)$ is a normal for this line. So a point-normal form equation for the line is

$$(1,2) \bullet (\vec{x} - (0,1)) = 0$$

(b) In Example 3.5 we found the point-normal form equation $(-1,1) \bullet (\vec{x} - (1,2)) = 0$ for a particular line. Since $(-1,1)$ is a normal for this line, and the vector $(1,1)$ is orthogonal to $(-1,1)$, then the vector $(1,1)$ is parallel to the line, i.e. is a direction vector for the line. And of course $(1,2)$ is a point on the line. So a point-parallel form equation for this line is

$$\vec{x}(t) = (1,2) + t(1,1)$$

(c) From the parametric equations of the line we can identify both a point on the line and a direction vector for the line. Remember, the multiplier on t is the component of the direction vector, while the number without a t is the coordinate of the known point. Keeping this in mind allows us to correctly identify both the known point and the direction vector from the parametric equations, even when they look a bit different than we expect.

Here, the parametric equations are given as

$$\begin{aligned} x &= 3 + t \\ y &= 2t - 4 \end{aligned}$$

We're more accustomed to seeing the form we have in the x equation. The form in the y equation, with the t term coming before the non-t term, is different. This is just done to avoid having a "leading negative". Equations look less tidy when the first thing on one side of the equation is a negative sign, so mathematicians often avoid writing things that way. That is, the given parametric equations are just a tidier form of

$$\begin{aligned} x &= 3 + t \\ y &= -4 + 2t \end{aligned}$$

In this form we see that the corresponding point-parallel form equation is $\vec{x}(t) = (3,-4) + t(1,2)$. So $(1,2)$ is a direction vector for the line and therefore $(2,-1)$ is a normal for the line. Thus we can write a point-normal form equation as

$$(2,-1) \bullet (\vec{x} - (3,-4)) = 0$$

Standard Form

Using the point-normal form of a line, we can get another form of equation for a line in $\Re^2$ – one which is already familiar to you. We get it by writing the vector $\vec{x}$ as $\vec{x} = (x, y)$ and distributing the dot product over the bracket in the point-normal form equation. (The vector (x, y), as always, just represents any vector whose endpoint (x, y) is a point on the line. That is, (x, y) is any unspecified point on the line.) For any line with normal vector (n_1, n_2) containing point (p_1, p_2) we get

$$\begin{aligned}(n_1, n_2) \bullet (\vec{x} - (p_1, p_2)) = 0 \quad &\Rightarrow \quad (n_1, n_2) \bullet ((x, y) - (p_1, p_2)) = 0\\ &\Rightarrow \quad [(n_1, n_2) \bullet (x, y)] - [(n_1, n_2) \bullet (p_1, p_2)] = 0\\ &\Rightarrow \quad n_1 x + n_2 y = (n_1, n_2) \bullet (p_1, p_2)\end{aligned}$$

But if (n_1, n_2) is a known normal vector to the line, and (p_1, p_2) is a known point on the line (so that n_1, n_2, p_1 and p_2 are all just numbers and we know which numbers they are), then $(n_1, n_2) \bullet (p_1, p_2)$ is just a number, i.e. is a known scalar, which we could call c. So we have $n_1 x + n_2 y = c$. Or, to make this general form look more familiar to you, we could use a and b in place of n_1 and n_2 and write $ax + by = c$. Ah. You've seen that before, haven't you? That's the standard form of an equation of a line.

Definition: The equation $ax + by = c$ is called the **standard form** equation of a line.

And what we've seen is that in this kind of equation, the coefficients a and b are the components of a normal vector for the line. And we also saw how to find the constant c.

Theorem 3.1. *If $ax + by = c$ is a standard form equation for line ℓ, then $\vec{n} = (a, b)$ is a normal vector for line ℓ. Also, if $P(p_1, p_2)$ is a point on the line, then $c = \vec{n} \bullet \vec{p}$.*

So if we know a point-normal equation for a line, i.e. if we know a normal for the line and we know a point on the line, then it's easy to find a standard form equation for the line. We simply use the components of the normal as the coefficients of x and y, and use the normal vector and the point to find the right hand side value, c. Likewise, if we have an equation of a line in standard form, we can easily find a normal to the line, because the coefficients of x and y are the components of a normal vector to the line. And then we just need to find any point on the line, to write a point-normal form equation of the line.

Example 3.7. Write a standard form equation for the line in Example 3.5.

Solution:
In Example 3.5 we had the line through $P(1, 2)$ with normal vector $\vec{n} = (-1, 1)$. We use the components of the normal vector as the coefficients of x and y in the standard form equation, so we have $(-1)x + (1)y = c$, or $-x + y = c$. We find the value of c using $c = \vec{n} \bullet \vec{p}$. We get

$$c = (-1, 1) \bullet (1, 2) = (-1)(1) + (1)(2) = -1 + 2 = 1$$

So the standard form equation is $-x + y = 1$. However, we don't usually write something like this with a leading negative. So we multiply the whole equation (i.e. both sides of the equation) by -1 to get rid of it. We get $x - y = -1$. (*Notice:* $(1, -1) = -(-1, 1) = -\vec{n}$ is parallel to (collinear with) $\vec{n}$, and is therefore another normal vector for this line.)

Example 3.8. Write a point-normal form equation for the line $x - 2y = 5$.

Solution:
We use the coefficients of x and y as the components of a normal vector for the line. Of course, $x - 2y = 1x + (-2)y$, so the coefficients are 1 and -2. That is, we get $\vec{n} = (1, -2)$ as a normal vector for the line. Now, we just need to find any point on the line. We plug in any convenient x-value and solve for y. Or we plug in any convenient y-value and solve for x. For instance, when $y = 0$ we have $x - 2(0) = 5$, so $x - 0 = 5$. That is, we see that when $y = 0$ we must have $x = 5$. So $P(5, 0)$ is a point on the line. Now we can write the point-normal form equation:

$$(1, -2) \bullet (\vec{x} - (5, 0)) = 0$$

Example 3.9. Write a standard form equation of the line $\vec{x}(t) = (3, 2) + t(2, 7)$.

Solution:
From the given point-parallel form equation, we see that $P(3, 2)$ is a point on the line and $\vec{v} = (2, 7)$ is a direction vector for the line, i.e. is parallel to the line. And so $\vec{n} = (7, -2)$ is a normal vector to the line, so the standard form equation has $7x - 2y = c$ for some value c. And we can find c using

$$c = (7, -2) \bullet (3, 2) = 7(3) + (-2)(2) = 21 - 4 = 17$$

Therefore the standard form equation is $7x - 2y = 17$.

Lines in $\Re^3$

Of course, we can have lines in 3-space, as well as in the plane. And there's a lot that's the same in $\Re^3$ as it was in $\Re^2$, so we use the same terminology and notation.

For instance, when we move from 2 dimensions to 3, it's still true that given any 2 points, there is exactly one line that passes through both those points. And the vector equivalent to the directed line segment between those points is parallel to that line, so we still call it a direction vector for the line. That is, we define the term direction vector the same way in $\Re^3$ as we did in $\Re^2$.

Definition: If $\vec{v} \in \Re^3$ is parallel to some line ℓ in $\Re^3$, we say that $\vec{v}$ is a **direction vector** for ℓ.

As in $\Re^2$, we can use a direction vector for a line (i.e. a vector parallel to the line) and any one point on the line to write a point-parallel equation for the line. And from that we can write parametric equations. Or we could write a 2-point form equation, instead.

The only difference is that now the points have 3 coordinates and the vectors have 3 components. Of course, for parametric equations this means that we have a third equation, corresponding to the z components of the vectors.

These observations are summarized in the following definitions.

Definition: Let $P(p_1, p_2, p_3)$ and $Q(q_1, q_2, q_3)$ be any points in $\Re^3$ and let $\vec{v} = (v_1, v_2, v_3)$ be any vector in $\Re^3$. Then:

1. If ℓ is the line which passes through P parallel to $\vec{v}$ (so that $\vec{v}$ is a direction vector for ℓ), then
$$\vec{x}(t) = (p_1, p_2, p_3) + t(v_1, v_2, v_3)$$
is an equation for line ℓ in **point-parallel form**.

2. If line ℓ passes through point P and $\vec{v}$ is a direction vector for ℓ, then **parametric equations** of line ℓ are:
$$\begin{aligned} x &= p_1 + tv_1 \\ y &= p_2 + tv_2 \\ z &= p_3 + tv_3 \end{aligned}$$

3. If points P and Q are both on line ℓ then a **two-point** form equation for ℓ is
$$\vec{x}(t) = (1-t)(p_1, p_2, p_3) + t(q_1, q_2, q_3)$$

Example 3.10. Let ℓ be the line which passes through the points $P(1, 2, 3)$ and $Q(1, -1, 1)$. Write equations of line ℓ in two-point form and in point-parallel form.

Solution:
In two-point form, we get the equation for ℓ:
$$\vec{x}(t) = (1-t)(1, 2, 3) + t(1, -1, 1)$$

For a point-parallel form equation of line ℓ we first need to find a direction vector for ℓ. The directed line segment $\overrightarrow{PQ}$ is equivalent to
$$\vec{v} = \vec{q} - \vec{p} = (1, -1, 1) - (1, 2, 3) = (0, -3, -2)$$
which is parallel to (and hence is a direction vector for) ℓ. Using this direction vector and the point P which we know is on the line, we get
$$\vec{x}(t) = (1, 2, 3) + t(0, -3, -2)$$
(Of course, we could have used point Q instead of point P to write the point-parallel form equation. Likewise, we could have used $\vec{p} - \vec{q} = (0, 3, 2)$ as the direction vector. And in the two-point form equation, we could have switched the roles of P and Q.)

Example 3.11. Write parametric equations for the line through the point $(0, 1, -1)$ which is parallel to $\vec{v} = (2, 1, 0)$.

Solution:
An equation of the line in point-parallel form is $\vec{x}(t) = (0, 1, -1) + t(2, 1, 0)$. This tells us that a point (x, y, z) is on this line if there is some value of t for which $(x, y, z) = (0, 1, -1) + t(2, 1, 0)$. So it must be true that, for the same value of t, we have
$$\begin{aligned} x &= 0 + 2t \\ y &= 1 + 1t \\ z &= -1 + 0t \end{aligned}$$

That is, we can write parametric equations of the line as

$$\begin{aligned} x &= 2t \\ y &= 1+t \\ z &= -1 \end{aligned}$$

Example 3.12. ℓ_1 is the line $\vec{x}(t) = (1-t)(2,1,-1) + t(0,1,2)$. ℓ_2 is the line with parametric equations $x = 2t - 2$, $y = 1$, $z = 5 - 3t$. Are ℓ_1 and ℓ_2 the same line?

Solution:
Hmm. That's different. Let's see. For ℓ_1 we recognize that what we've been given is a two-point form equation. (We can tell because of the $(1-t)$ multiplier.) From it we can see that $P(2,1,-1)$ and $Q(0,1,2)$ are two points on line ℓ_1. This also tells us that the vector

$$\vec{v} = \overrightarrow{PQ} = \vec{q} - \vec{p} = (0,1,2) - (2,1,-1) = (-2,0,3)$$

is parallel to line ℓ_1.

For ℓ_2 we're given parametric equations. It may be helpful to write these equations all the same way, with "constant + multiple of t" on the right hand side. We have:

$$\begin{aligned} x &= 2t-2 \\ y &= 1 \\ z &= 5-3t \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} x &= -2+2t \\ y &= 1+0t \\ y &= 5+(-3)t \end{aligned}$$

From the rearranged set of equations, using our knowledge of the form of parametric equations, we see that the point on ℓ_2 used to write these parametric equations is $R(-2,1,5)$. Also, the direction vector used for these equations is $\vec{u} = (2,0,-3)$.

Since $\vec{u} = (2,0,-3) = -(-2,0,3) = -\vec{v}$, we see that these vectors are scalar multiples of one another, so they are collinear. That is, the direction vector $\vec{u}$ used to write the equation of ℓ_2 is parallel to the vector which we know is parallel to ℓ_1. Therefore $\vec{u}$ is also parallel to ℓ_1, and thus lines ℓ_1 and ℓ_2 are parallel to one another. It's possible that they could be the same line. How can we tell whether they are?

Since ℓ_1 and ℓ_2 are parallel, then either they have no points in common or else they are the same line and have all points in common. So all we need to do is determine whether any point which is known to be on one line is also on the other. If it is, then they are actually the same line. But if it isn't, then they must be different, but parallel, lines.

We know that the point $P(0,1,2)$ is on line ℓ_1. Is it also on line ℓ_2? If it is, then $(x,y,z) = (0,1,2)$ must satisfy the parametric equations for ℓ_2, using the same value of t for each component (equation). Since the second coordinate of P is 1, the equation $y = 1$ is satisfied. For the first coordinate, we see that we need to have $x = 2t - 2$ satisfied for $x = 0$. This gives

$$0 = 2t - 2 \qquad \Rightarrow \qquad 0 + 2 = 2t \qquad \Rightarrow \qquad 2t = 2 \qquad \Rightarrow \qquad t = 1$$

Now, if we substitute $t = 1$ into the third of the parametric equations, we get

$$z = 5 - 3(1) = 5 - 3 = 2$$

Since $z = 2$ is the third coordinate of point P, we see that the point $(x,y,z) = (0,1,2)$ *does* satisfy the parametric equations of ℓ_2. That is, we have

$$(0,1,2) = (-2,1,5) + 1(2,0,-3)$$

so $(x, y, z) = (0, 1, 2)$ satisfies $x = -2 + 2t$, $y = 1 + 0t$ and $z = 5 - 3t$ with $t = 1$. Because the point $(0, 1, 2)$ does satisfy the equations for ℓ_2, it is a point on line ℓ_2.

So now we know that ℓ_1 and ℓ_2 are parallel lines, with a point in common, which means that they must have all points in common and be the same line. That is, since ℓ_1 and ℓ_2 are parallel and intersect at point $P(0, 1, 2)$, they must intersect at all other points as well, and actually be the same line.

Planes in $\Re^3$

We know that in $\Re^2$, something of the form $ax + by = c$ is the standard form of an equation of a line. What about the 3-dimensional equivalent, $ax + by + cz = d$. Is that an equation of a line? Well, let's see.

Let's think about a specific, uncomplicated, example. Consider the equation $x + y + z = 0$. This equation is satisfied by the point $P(2, -2, 0)$ (because $2 + (-2) + 0 = 0$) and also by the point $Q(3, -3, 0)$. And we know that there's a unique line that passes through those 2 points. Let's call that line ℓ_1. Using $\vec{v} = \vec{q} - \vec{p} = (3, -3, 0) - (2, -2, 0) = (1, -1, 0)$ as a vector which is parallel to ℓ_1, we can write an equation of ℓ_1 as

$$\vec{x}(t) = (2, -2, 0) + t(1, -1, 0)$$

Notice that for any point (x, y, z) on line ℓ_1 we have

$$\begin{aligned} x &= 2 + t \\ y &= -2 - t \\ z &= 0 \end{aligned}$$

and so $x + y + z = (2 + t) + (-2 - t) + 0 = 2 - 2 + t - t = 0$. That is, every point on ℓ_1 satisfies the equation $x + y + z = 0$.

But those aren't the only points which satisfy that equation. For instance, the point $R(1, 0, -1)$ also satisfies this equation. And this point is not on ℓ_1. The easiest way to tell is because from the parametric equations of ℓ_1 we can see that every point on ℓ_1 has $z = 0$, but the third coordinate of point R isn't 0, so it is not a point on ℓ_1.

Hmm. Every point on line ℓ_1 satisfies $x + y + z = 0$, but it's not true that every point that satisfies $x + y + z = 0$ is on line ℓ_1. So $x + y + z = 0$ cannot be an equation of line ℓ_1. Then what is it? Well, actually, it's the equation of a plane. $\Re^2$ (i.e. 2-space) is just a single plane. But $\Re^3$, which is to say 3-space, contains infinitely many planes. (For instance, think of the walls, ceilings and floors of the building you're in. And also every other building you ever have been or ever could be in. And all the ramps you've ever seen. And what those ramps would look like if they were knocked off kilter. And ... Each of those things lies in some particular plane in $\Re^3$, and there are many other planes besides those.)

So $x + y + z = 0$ is an equation of a plane. Let's call it the plane Π. (Planes are often named Π, which is just the Greek letter P, just like lines are named ℓ. ℓ for line, Π for plane. Same idea.) Any plane contains infinitely many lines. One of the lines that lies in the particular plane Π we've been talking about is the line ℓ_1. But there are many others. For instance, we saw that $P(2, -2, 0)$ and $R(1, 0, -1)$ both lie on this plane, so the line on which those points lie is another line in plane Π. We can call that one ℓ_2. And the vector $\vec{r} - \vec{p} = (1, 0, -1) - (2, -2, 0) = (-1, 2, -1)$ is parallel to ℓ_2 so we can express ℓ_2 as $\vec{x}(t) = (1, 0, -1) + t(-1, 2, -1)$.

Notice that $x+y+z=(1,1,1)\bullet(x,y,z)$. Let's think about the vector $(1,1,1)$ whose components are the coefficients in the equation of the plane Π. We know that $(1,-1,0)$ is parallel to line ℓ_1, which lies in plane Π. Notice that $(1,1,1)\bullet(1,-1,0) = 1(1)+1(-1)+1(0) = 1-1+0 = 0$, so the vector $(1,1,1)$ is a normal for (i.e. is perpendicular to) line ℓ_1. Likewise, we know that $(-1,2,-1)$ is parallel to line ℓ_2, which also lies in plane Π. Notice that $(1,1,1)\bullet(-1,2,-1) = 1(-1)+1(2)+1(-1) = -1+2-1 = 0$, so the vector $(1,1,1)$ is also a normal for (perpendicular to) line ℓ_2. However $(1,-1,0)$ is not a scalar multiple of $(-1,2,-1)$, so those vectors aren't orthogonal (i.e. parallel) and therefore lines ℓ_1 and ℓ_2 aren't parallel to one another. How can the same vector be perpendicular to both? Well, by being perpendicular to *the whole plane in which both lines lie.* This vector $(1,1,1)$ is actually perpendicular to, i.e. a normal for, the plane Π.

> **Definition:** A vector which is perpendicular to a particular plane in $\Re^3$ is said to be **normal** to the plane, and is called a **normal** for that plane, or a **normal vector** for the plane.

Point-Normal Form of an Equation of a Plane

At this point, it may not be too surprising to you to learn that if we write an equation using a normal vector and a point in $\Re^3$, what we get is an equation of a plane. That is, if we extend to $\Re^3$ the idea of the point-normal form of an equation of a line in $\Re^2$, the result is **not** an equation of a line in $\Re^3$, but rather an equation of a plane in $\Re^3$. This point-normal form equation of a plane in 3-space looks just like the point-normal form equation of a line in 2-space, except that the vectors have 3 components instead of 2. So it's the presence of that third component that distinguishes a point-normal form equation of a plane (with 3 components in the vectors) from a point-normal form equation of a line (with only 2 components in the vectors).

> **Definition:** The **point-normal form** of an equation of a plane Π in $\Re^3$, where $\vec{n} = (n_1,n_2,n_3)$ is any normal vector to the plane and $P(p_1,p_2,p_3)$ is any point on the plane, is given by
>
> $$\vec{n}\bullet(\vec{x}-\vec{p}) = 0 \qquad \textit{i.e.} \qquad (n_1,n_2,n_3)\bullet(\vec{x}-(p_1,p_2,p_3)) = 0$$

Example 3.13. Write an equation of the plane containing $P(1,2,3)$ with normal vector $\vec{n} = (1,0,-1)$ in point-normal form.

Solution:
The form of the equation is $\vec{n}\bullet(\vec{x}-\vec{p}) = 0$, so we get

$$(1,0,-1)\bullet(\vec{x}-(1,2,3)) = 0$$

Example 3.14. Write a point-normal form equation of the plane Π which contains the lines ℓ_1 represented by $\vec{x}(t) = (2,-2,0)+t(1,-1,0)$ and ℓ_2 represented by $\vec{x}(s) = (1,0,-1)+s(-1,2,-1)$.

Solution:
In order to write a point-normal form equation of the plane Π, we need a normal vector for the plane and a point that lies in the plane. Of course, the point-parallel form equations of ℓ_1 and ℓ_2 each give us a point that lies in the plane. That is, we know that $(2,-2,0)$ is a point in this plane, because this is a point on ℓ_1 which lies in plane Π, and likewise that $(1,0,-1)$ is another point on plane Π, because it is a point on line ℓ_2, which also lies in this plane. So we can use either one of these points in our equation of Π.

How do we find a normal for the plane? Well, we know from the equation of ℓ_1 that the vector $\vec{u} = (1, -1, 0)$ is parallel to ℓ_1, and likewise from the equation of ℓ_2 that the vector $\vec{v} = (-1, 2, -1)$ is parallel to ℓ_2. Of course any vector $\vec{n}$ which is a normal for Π (i.e. is perpendicular to this plane) must be perpendicular to any line that lies within Π. So if $\vec{n}$ is a normal for Π, then $\vec{n}$ is perpendicular to both ℓ_1 and ℓ_2 and therefore must be orthogonal to both $\vec{u}$ and $\vec{v}$. (That is, any vector which is perpendicular to ℓ_1 is also perpendicular to (orthogonal to) every vector that is parallel to ℓ_1. And similarly for ℓ_2.)

So how do we find a vector which is perpendicular to both $\vec{u}$ and $\vec{v}$? Well that's easy. We know that the vector $\vec{u} \times \vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$. So we can use

$$\begin{aligned} \vec{n} &= \vec{u} \times \vec{v} = (1, -1, 0) \times (-1, 2, -1) \\ &= ((-1)(-1), (0)(-1), (1)(2)) - ((2)(0), (-1)(1), (-1)(-1)) \\ &= (1, 0, 2) - (0, -1, 1) = (1, 1, 1) \end{aligned}$$

(Recall that we discussed previously that the vector $(1, 1, 1)$ was a normal for the plane containing these lines ℓ_1 and ℓ_2.)

Now we know both a normal vector for Π and a point in plane Π so we can write the point-normal form equation. We get

$$(1, 1, 1) \bullet (\vec{x} - (2, -2, 0)) = 0$$

<u>Standard Form Equation of a Plane</u>

Consider any point-normal form equation of a plane. For instance, let's work with the one we found in that last example. If we express the vector $\vec{x}$ as $\vec{x} = (x, y, z)$ and carry through the vector arithmetic, what do we get? Well let's see:

$$\begin{aligned} & (1, 1, 1) \bullet (\vec{x} - (2, -2, 0)) = 0 \\ \Rightarrow\ & (1, 1, 1) \bullet ((x, y, z) - (2, -2, 0)) = 0 \\ \Rightarrow\ & [(1, 1, 1) \bullet (x, y, z)] - [(1, 1, 1) \bullet (2, -2, 0)] = 0 \\ \Rightarrow\ & (1, 1, 1) \bullet (x, y, z) = (1, 1, 1) \bullet (2, -2, 0) \\ \Rightarrow\ & (1)(x) + (1)(y) + (1)(z) = (1)(2) + (1)(-2) + (1)(0) \\ \Rightarrow\ & x + y + z = 2 - 2 + 0 \\ \Rightarrow\ & x + y + z = 0 \end{aligned}$$

Ah, yes. We've seen that before. As an equation of the plane in which we found ℓ_1 and ℓ_2 to lie. That is, we started out our discussion of planes by wondering what $x + y + z = 0$ was the equation of, and we realized it was a plane. The equation looks just like the standard form equation of a line in $\Re^2$, except it has a z in it. So this form is called the standard form equation of a plane.

That is, as we said earlier, if we have an equation in the same form as a standard form equation of a line, but with z in it as well as x and y, then this equation isn't describing a single line. It is describing a whole plane in $\Re^3$. And now we see that the coefficients of x, y and z in this equation are the components of a normal vector for that plane.

Definition: The equation $ax + by + cz = d$ is a **standard form equation of a plane** and the vector (a, b, c) is a normal vector to this plane.

Example 3.15. Write an equation in standard form for the plane with normal vector $\vec{n} = (1, 2, 3)$ which contains the point $P(0, -1, 2)$.

Solution:
Since $\vec{n} = (1, 2, 3)$ is a normal vector for the plane, then the standard form equation must have the form $1x + 2y + 3z = d$ for some scalar d. But of course we would write that as $x + 2y + 3z = d$. How can we find the value of d? Well, we know that the point $P(0, -1, 2)$ lies on the plane, so $(x, y, z) = (0, -1, 2)$ must satisfy this equation. That is, we plug in $x = 0$, $y = -2$ and $z = 2$ to find the value of d. We get:

$$x + 2y + 3z = d \Rightarrow 0 + 2(-1) + 3(2) = d \Rightarrow -2 + 6 = d \Rightarrow d = 6 - 2 = 4$$

So a standard form equation of the plane is $x + 2y + 3z = 4$.

Notice: We could have used $\vec{n} \bullet \vec{p} = d$, from rearranging the point-normal equation for the plane. What we did here is just another explanation of the exact same arithmetic. (Look back at the examples in which we found point-normal equations of lines. We could have described the arithmetic we did there as "let $x = p_1$ and $y = p_2$" instead of "find $\vec{x} \bullet \vec{p}$".)

The Plane Determined by Three Points

In Example 3.14, we used the fact that if we know 2 vectors which are direction vectors for 2 lines contained in a plane, their cross product gives a normal for the plane. However that only works if the 2 vectors are non-collinear. That is, if the 2 vectors lie on the same line, then there are other vectors *in the same plane* that are orthogonal to both vectors. It's only if the 2 vectors are not parallel to one another that we can be sure that any (non-zero) vector which is perpendicular to both must be perpendicular to the whole plane.

Theorem 3.2. *If $\vec{u}$ is a direction vector for a line in some plane Π, and $\vec{v}$ is a direction vector for another line in plane Π, where $\vec{u}$ and $\vec{v}$ are not collinear, then $\vec{n} = \vec{u} \times \vec{v}$ is a normal vector for plane Π.*

We know that for any two points (whether in $\Re^2$ or in $\Re^3$), there is exactly one line which contains those two points. Consider 3 points in $\Re^3$. If the 3 points are all collinear, i.e. if they all lie on the same line, then there are many planes which contain those 3 points. (The infinitely many different planes that intersect along that line.) But if the 3 points are not all collinear, then there is only one plane that contains all three points.

Suppose we know three non-collinear points, P, Q and R. What do we need to do to find an equation of the plane containing those points? Well, the plane containing these points must contain the line on which P and Q both lie, and must also contain the line on which P and R both lie. (It must also contain the line on which Q and R both lie, but that's more lines than we need.) If the points are non-collinear, then these are different lines. So if we find direction vectors for those lines, then those vectors are not parallel, and we can use them to find a normal vector for the plane. Then we just use any one of the points, along with that normal vector, to write an equation of the plane. So finding an equation of the plane containing three specified points is a simple procedure. We refer to this as the plane **determined** by the three points.

Example 3.16. Find both a point-normal form equation and a standard form equation of the plane determined by the points $P(-1,0,1)$, $Q(1,2,3)$ and $R(2,-1,5)$.

Solution:
The line passing through points P and Q lies in this plane, and so any vector parallel to that line is also parallel to the plane. And if we let $\vec{u} = \vec{q} - \vec{p}$, then $\vec{u}$ is such a vector. Similarly, the vector $\vec{v} = \vec{r} - \vec{p}$ is parallel to the line which passes through both P and R, and since that line also lies in the plane, $\vec{v}$ is another vector which is parallel to the plane we need to describe. Also, we have

$$\begin{aligned} \vec{u} &= \vec{q} - \vec{p} = (1,2,3) - (-1,0,1) = (1-(-1), 2-0, 3-1) = (2,2,2) \\ \vec{v} &= \vec{r} - \vec{p} = (2,-1,5) - (-1,0,1) = (2-(-1), -1-0, 5-1) = (3,-1,4) \end{aligned}$$

and we can see that since $\vec{u}$ and $\vec{v}$ are not scalar multiples of one another then they are not collinear.

We use these two non-collinear vectors which are both parallel to the plane to find a normal for the plane:

$$\vec{n} = \vec{u} \times \vec{v} = (8-(-2), 6-8, -2-6) = (10,-2,-8)$$

Now we use this normal vector and any one of the three points to write a point-normal equation of the plane. For instance, using point P, the form $\vec{n} \bullet (\vec{x} - \vec{p}) = 0$ gives:

$$(10,-2,-8) \bullet (\vec{x} - (-1,0,1)) = 0$$

Finally, we can also rearrange this equation to standard form. Letting $\vec{x} = (x,y,z)$, we get:

$$\begin{aligned} (10,-2,-8) \bullet ((x,y,z) - (-1,0,1)) = 0 \quad &\Rightarrow \quad (10,-2,-8) \bullet (x,y,z) - (10,-2,-8) \bullet (-1,0,1) = 0 \\ &\Rightarrow \quad 10x - 2y - 8z = (10,-2,-8) \bullet (-1,0,1) \\ &\Rightarrow \quad 10x - 2y - 8z = -10 + 0 - 8 \\ &\Rightarrow \quad 10x - 2y - 8z = -18 \end{aligned}$$

(*Note:* We might prefer to divide through the equation by 2. That is, this plane would often be expressed as $5x - y - 4z = -9$.)

Determining the Distance between a Point and a Plane

Suppose we have some particular plane Π. Consider any point P which *does not* lie on this plane. How far is this point from the plane? That is, what is the shortest distance from the point to the plane?

The shortest way to get from the plane to point P is to start from the point on the plane which is nearest to point P. This will be the point P' on the plane such that the directed line segment from P' to P is normal (i.e. perpendicular) to Π. And the (shortest) distance from P to the plane will just be the length of that directed line segment, i.e. $\left|\left|\overrightarrow{P'P}\right|\right|$. However we don't want to have to actually find the point P', which is the point on the plane that is nearest to P, in order to find the distance from P to the plane. And in fact we don't need to.

Let plane Π have normal vector $\vec{n}$ and let Q be any known point on plane Π. Then it can be shown that the distance from P to P' is given by:

$$\left|\left|\overrightarrow{P'P}\right|\right| = \left|\left|\vec{p} - \vec{p'}\right|\right| = \frac{|\vec{n} \bullet (\vec{q} - \vec{p})|}{||\vec{n}||}$$

That is, we simply need to find the dot product of any normal vector to the plane with the vector equivalent to the directed line segment between the point P and *any* known point on the plane, discard the negative sign (if there is one), and divide by the magnitude of the normal vector used.

(*Notice:* We have not explained *why* this gives $\left|\left|\overrightarrow{P'P}\right|\right|$, so you should not be trying to understand that from the above. If you're interested, look at the explanation given in the text. All we've done here is to *assert* that *it can be shown that* this is true.)

Theorem 3.3. *Consider any plane* Π. *Let* $\vec{n}$ *be any normal vector for plane* Π *and let* Q *be any point on plane* Π. *Consider any other point* P *which is not on the plane* Π. *Then the distance between point* P *and plane* Π *is given by:*

$$\textit{distance} = \frac{|\vec{n} \bullet (\vec{q} - \vec{p})|}{||\vec{n}||}$$

Example 3.17. Find the distance between the point $P(1,2,3)$ and the plane with point-normal form equation $(1,2,1) \bullet (\vec{x} - (3,-1,0)) = 0$.

Solution:
We simply plug $\vec{n} = (1,2,1)$, $\vec{p} = (1,2,3)$ and $\vec{q} = (3,-1,0)$ into the formula. We have:

$$\begin{aligned} \vec{q} - \vec{p} &= (3,-1,0) - (1,2,3) = (2,-3,-3) \\ \text{so that } \vec{n} \bullet (\vec{q} - \vec{p}) &= (1,2,1) \bullet (2,-3,-3) = 1(2) + 2(-3) + 1(-3) = 2 - 6 - 3 = -7 \\ \text{and } ||\vec{n}|| &= \sqrt{1^2 + 2^2 + 1^2} = \sqrt{1+4+1} = \sqrt{6} \end{aligned}$$

and so the distance from P to the plane is

$$\text{distance} = \frac{|\vec{n} \bullet (\vec{q} - \vec{p})|}{||\vec{n}||} = \frac{|-7|}{\sqrt{6}} = \frac{7}{\sqrt{6}}$$

Example 3.18. Find the distance from the origin to the plane $x + y - z = 5$.

Solution:
The origin is the point $P(0,0,0)$. Notice that we know that this point is not on the plane because $(x,y,z) = (0,0,0)$ does not satisfy the equation of the plane. Also, we recognize from the standard form equation that $\vec{n} = (1,1,-1)$ is a normal for the plane. So now we just need to find any point that's on the plane. Simply pick any values of x, y and z that, when taken together, *do* satisfy the equation of the plane. For instance, for the point $(x,y,z) = (5,0,0)$ we get $x + y - z = 5 + 0 - 0 = 5$, so $Q(5,0,0)$ is a point on the plane.

Now we just use the formula. We see that the distance from the origin to the plane $x + y - z = 5$ is:

$$\frac{|\vec{n} \bullet (\vec{q} - \vec{p})|}{||\vec{n}||} = \frac{|(1,1,-1) \bullet ((5,0,0) - (0,0,0))|}{||(1,1,-1)||} = \frac{|(1,1,-1) \bullet (5,0,0)|}{\sqrt{1^2 + 1^2 + (-1)^2}} = \frac{5}{\sqrt{3}}$$

Notice: If we accidentally try to find the distance from a point P to a plane Π for a point P which actually *lies on the plane* Π, we will simply get the answer 0. In that case the directed line segment from P to another point Q known to be on the plane lies on the plane, so any normal for the plane is orthogonal to $\vec{q} - \vec{p}$. And we know that if two vectors are orthogonal then their dot product is 0, so the numerator of the formula will be 0, giving the answer 0 as the distance.

Finding the Distance Between a Point and a Line

We have already seen some ways in which planes in $\Re^3$ behave like lines in $\Re^2$. That is, we know that in $\Re^2$, a point-normal form equation corresponds to a line, but the similar form in $\Re^3$ corresponds to a plane. And the same is true for a standard form equation. In $\Re^2$ it represents a line, but in $\Re^3$ it represents a plane. And both of these forms use a normal vector. In fact, in $\Re^2$ we talk about a normal vector for a line, but in $\Re^3$ we only talk of a normal vector for a plane.

So it may not surprise you to learn that the same distance formula given in Theorem 3.3, which refers to the distance between a point and a plane in $\Re^3$, can also be used in $\Re^2$, but there it gives the distance between a point and a *line*.

Theorem 3.4. *Consider any line ℓ. Let $\vec{n}$ be any normal vector for line ℓ and let Q be any point on line ℓ. Consider any other point P which is not on line ℓ. Then the distance between point P and line ℓ is given by:*

$$distance = \frac{|\vec{n} \bullet (\vec{q} - \vec{p})|}{||\vec{n}||}$$

Example 3.19. Find the distance between the point $P(1, 2)$ and the line ℓ described by $2x + y = 1$.

Solution:
Line ℓ has normal $\vec{n} = (2, 1)$. We need to find some point Q on line ℓ. Letting $x = 0$ we get $2(0) + y = 1$, so $y = 1$. That is, the point on line ℓ which has x-coordinate 0 has y-coordinate 1, so the point $Q(0, 1)$ is a point on line ℓ. (Notice that for $(x, y) = (1, 2)$ we have $2x+y = 2(1)+2 = 4 \neq 1$, so $P(1, 2)$ is not on line ℓ.)

The distance between P and ℓ is

$$\frac{|\vec{n} \bullet (\vec{q} - \vec{p})|}{||\vec{n}||} = \frac{|(2,1) \bullet ((0,1) - (1,2))|}{||(2,1)||} = \frac{|(2,1) \bullet (-1,-1)|}{\sqrt{2^2 + 1^2}} = \frac{|-2-1|}{\sqrt{4+1}} = \frac{3}{\sqrt{5}}$$

Finding the Intersection of Two Lines

Any 2 lines which lie in the same plane and are not parallel intersect at exactly 1 point. To find this point of intersection, we simply need to find a point which is on both lines. We use parametric equations of at least one of the lines when we do this. We can either: (1) Use parametric equations of *both* lines to equate corresponding components of a point, and solve for the values of the parameters which satisfy all of those equations; or (2) Use parametric equations of one line and a standard form equation for the other line (if they are lines in $\Re^2$), and substitute for x and y in terms of the parameter, then solve for the value of the parameter.

Example 3.20. Find the point of intersection of the line ℓ_1: $\vec{x}(t) = (1,0) + t(2,1)$ with the line ℓ_2: $\vec{x}(s) = (1,1) + s(-1,0)$.

Solution:

For ℓ_1 we have parametric equations $\begin{array}{rcl} x & = & 1+2t \\ y & = & t \end{array}$ and for ℓ_2 we have $\begin{array}{rcl} x & = & 1-s \\ y & = & 1 \end{array}$.

If some point $P(x, y)$ is on both these lines, then it must be true that there are some values of t and s which give the same values of x and y. So we must have $1 + 2t = 1 - s$ and $t = 1$. Since $t = 1$, then $1 + 2t = 3$, so $1 - s = 3$ and we see that $s = 1 - 3 = -2$.

Notice that we've found values of the parameters, s and t, but we have not yet found the point on the line which corresponds to these values. That is, we know the value of t that gives the point on line ℓ_1 at which the two lines intersect, and likewise we know the value of s that gives that same point on line ℓ_2. But we were asked to find the actual point at which the two lines intersect. We're not finished until we've done that. And we have more information than we need to find the point, since we know two ways to get it. So we can use the value of t we found, in the equation for ℓ_1, to get the point P. And then we can use the value of s we found, in the equation of ℓ_2, to *check* our work. We get:

$$t = 1 \qquad \Rightarrow \qquad (x, y) = (1,0) + t(2,1) = (1,0) + 1(2,1) = (3,1)$$

as the point on ℓ_1 which we were looking for. We check that the point on ℓ_2 is the same point:

$$s = -2 \qquad \Rightarrow \qquad (x, y) = (1,1) + s(-1,0) = (1,1) + (-2)(-1,0) = (1,1) + (2,0) = (3,1)$$

Since we did find the same point on each line, this is the point we were looking for. We see that ℓ_1 and ℓ_2 intersect at the point $P(3,1)$.

Note: As we observed above, we found values of both parameters, but really we only need one. As we have seen, the other allows us to *check* our work. We're just checking that we didn't make an arithmetic error. If we got a different point on ℓ_2 than the one on ℓ_1 that would tell us that somewhere in our calculations we made an arithmetic mistake. Either in finding the points, or (more likely) in finding the values of the parameters. We would need to re-do our calculations until we find the mistake, and then finish the problem (including the check) again.

Example 3.21. Find the point of intersection of the line ℓ_1: $\vec{x}(t) = (1,1,2) + t(2,1,-1)$ with the line ℓ_2: $\vec{x}(s) = (0,1,2) + s(1,-1,1)$.

Solution:

For ℓ_1 we have $\begin{array}{rcl} x & = & 1+2t \\ y & = & 1+t \\ z & = & 2-t \end{array}$ and for ℓ_2 we have $\begin{array}{rcl} x & = & s \\ y & = & 1-s \\ z & = & 2+s \end{array}$.

The point of intersection of ℓ_1 and ℓ_2 is a point $P(x, y, z)$ which satisfies both sets of equations at the same time, so we must have:

$$\begin{array}{rcl} 1+2t & = & s \qquad (1) \\ 1+t & = & 1-s \qquad (2) \\ 2-t & = & 2+s \qquad (3) \end{array}$$

Equation (1) says that $s = 1 + 2t$, so that $1 - s = 1 - (1 + 2t) = 0 - 2t = -2t$. Therefore equation (2) gives $1 + t = -2t$, so $1 = -3t$ and thus $t = -\frac{1}{3}$. And then substituting $t = -\frac{1}{3}$ into $s = 1 + 2t$

gives $s = 1 + 2\left(-\frac{1}{3}\right) = 1 - \frac{2}{3} = \frac{1}{3}$. Checking these values in (3) we get

$$\begin{aligned} 2 - t &= 2 - \left(-\frac{1}{3}\right) = 2 + \frac{1}{3} = \frac{6}{3} + \frac{1}{3} = \frac{7}{3} \\ \text{and } 2 + s &= 2 + \frac{1}{3} = \frac{7}{3} \\ \text{so it's true that } 2 - t &= 2 + s \end{aligned}$$

Now, using $s = \frac{1}{3}$ in the equation for ℓ_2 we get the point

$$(x, y, z) = (0, 1, 2) + s(1, -1, 1) = (0, 1, 2) + \left(\frac{1}{3}\right)(1, -1, 1) = \left(\frac{0}{3}, \frac{3}{3}, \frac{6}{3}\right) + \left(\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}\right) = \left(\frac{1}{3}, \frac{2}{3}, \frac{7}{3}\right)$$

As before, we can use $t = -\frac{1}{3}$ to check that we haven't made any mistakes:

$$(x, y, z) = (1, 1, 2) + t(2, 1, -1) = (1, 1, 2) + \left(-\frac{1}{3}\right)(2, 1, -1) = \left(\frac{3}{3}, \frac{3}{3}, \frac{6}{3}\right) + \left(-\frac{2}{3}, -\frac{1}{3}, \frac{1}{3}\right) = \left(\frac{1}{3}, \frac{2}{3}, \frac{7}{3}\right)$$

We see that ℓ_1 and ℓ_2 intersect at this common point, $P\left(\frac{1}{3}, \frac{2}{3}, \frac{7}{3}\right)$.

Note: Any 2 non-parallel lines in $\Re^2$ always intersect at a single point. For 2 non-parallel lines in $\Re^3$, there are 2 possibilities. If they lie in the same plane, then they intersect at a single point. But they may lie in parallel planes and not intersect at all. In that case, it will be impossible to find values of s and t that satisfy all 3 equations at the same time.

As stated earlier, if we wish to find the point of intersection of 2 lines in $\Re^2$ and one of the lines is in point-normal form or standard form, we don't need to find a direction vector and write parametric equations of that line. Instead we can just use the standard form directly, as shown in the next example.

Example 3.22. Find the point of intersection of the lines ℓ_1: $\vec{x}(t) = (1, 0) + t(3, -1)$ and ℓ_2: $2x - y = 5$.

Solution:
For ℓ_1 we have $\begin{aligned} x &= 1 + 3t \\ y &= -t \end{aligned}$. Substituting for x and y in the standard form equation for ℓ_2 we have:

$$2x - y = 5 \quad \Rightarrow \quad 2(1 + 3t) - (-t) = 5 \quad \Rightarrow \quad 2 + 6t + t = 5 \quad \Rightarrow \quad 7t = 3 \quad \Rightarrow \quad t = \frac{3}{7}$$

Now we find the point on ℓ_1 corresponding to $t = \frac{3}{7}$:

$$(x, y) = (1, 0) + t(3, -1) = (1, 0) + \left(\frac{3}{7}\right)(3, -1) = \left(\frac{7}{7}, \frac{0}{7}\right) + \left(\frac{9}{7}, -\frac{3}{7}\right) = \left(\frac{16}{7}, -\frac{3}{7}\right)$$

Of course, we should check that this really is a point on ℓ_2 (i.e. check for arithmetic mistakes). For $(x, y) = \left(\frac{16}{7}, -\frac{3}{7}\right)$ we get:

$$2x - y = 2\left(\frac{16}{7}\right) - \left(-\frac{3}{7}\right) = \frac{32}{7} + \frac{3}{7} = \frac{35}{7} = 5$$

Since $2x - y = 5$, we see that $(x, y) = \left(\frac{16}{7}, -\frac{3}{7}\right)$ *is* a point on ℓ_2. So ℓ_1 and ℓ_2 intersect at the point $\left(\frac{16}{7}, -\frac{3}{7}\right)$.

The Intersection of a Line with a Plane

If line ℓ lies in plane Π, then all points on line ℓ are on plane Π. If line ℓ lies on a plane parallel to plane Π, then no point on line ℓ lies on plane Π. But if line ℓ does not lie on plane Π or on any plane parallel to Π, then ℓ intersects Π at a single point. That is, there is only a single point on line ℓ which lies on plane Π.

To find the point of intersection of line ℓ with plane Π, we use parametric equations of ℓ to express the coordinates of the point in terms of the parameter, then use the standard form equation of Π to solve for the value of the parameter. (This is exactly what we did in Example 3.22. But now there's z as well as x and y.)

Example 3.23. Find the point at which the line ℓ described by $\vec{x}(t) = (1,0,1) + t(3,2,1)$ intersects the plane Π described by $x + y - 2z = 3$.

Solution:
Parametric equations of line ℓ give:

$$\begin{aligned} x &= 1 + 3t \\ y &= 2t \\ z &= 1 + t \end{aligned}$$

We substitute these expressions into the equation for Π:

$$\begin{array}{llll} x + y - 2z = 3 & \Rightarrow \quad (1+3t) + 2t - 2(1+t) = 3 & \Rightarrow & 1 + 3t + 2t - 2 - 2t = 3 \\ & \Rightarrow \quad 3t = 3 - (-1) = 4 & \Rightarrow & t = \frac{4}{3} \end{array}$$

For this value of t we find the point on line ℓ to be

$$(x,y,z) = (1,0,1) + t(3,2,1) = (1,0,1) + \left(\frac{4}{3}\right)(3,2,1) = (1,0,1) + \left(4, \frac{8}{3}, \frac{4}{3}\right) = \left(5, \frac{8}{3}, \frac{7}{3}\right)$$

Therefore ℓ and Π intersect at the point $\left(5, \frac{8}{3}, \frac{7}{3}\right)$.

Check: We check that this point really is on plane Π:

$$x + y - 2z = 5 + \frac{8}{3} - 2\left(\frac{7}{3}\right) = \frac{15}{3} + \frac{8}{3} - \frac{14}{3} = \frac{15 + 8 - 14}{3} = \frac{9}{3} = 3$$

Since $(x,y,z) = \left(5, \frac{8}{3}, \frac{7}{3}\right)$ does give $x + y - 2z = 3$, this point is on plane Π.

Notice: This procedure only works when the line ℓ does not lie in the plane Π or in any plane parallel to Π. If line ℓ lies in plane Π, then when we substitute into the equation for Π and try to solve for t, the t's will all disappear and we'll be left with something like $3 = 3$. This equation is satisfied for *all* values of t, telling us that all points on ℓ satisfy the equation for Π. But if line ℓ lies on a plane parallel to Π, then again all the $t's$ will disappear, but we'll be left with something like $2 = 3$. This equation isn't satisfied for any value of t, telling us that there is *no* value of t for which $\vec{x}(t)$ is a point on plane Π.

Math 1229A/B

Unit 4:

Vectors in $\Re^m$

(text reference: Section 2.1)

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4 Vectors in $\Re^m$

We've learnt about $\Re^2$ and $\Re^3$, which you've seen before. Now, we're going to extend some of the same ideas to spaces with more than 3 dimensions. We refer to m-dimensional space as $\Re^m$ or m-space. "But", you're saying to yourself, "how can you have more than 3 dimensions?". Well, for starters, *time* is often referred to as the fourth dimension of physical space. The world we live in has depth, breadth and height, and also time. Where you're sitting right this minute is a particular location that could be described by x-, y- and z-coordinates. But you're in that location at this particular instant. At other times you weren't. However the location still existed. So time is another axis along which space can be measured.

And after that? Well, for higher dimensions, it's just much easier if you don't try to relate it to physical space. Because it tends to make your brain hurt if you try to actually picture $\Re^5$ or $\Re^6$ or $\Re^{20}$, and so forth. But to mathematicians, that's no reason not to talk about them. Theoretically, there could be more dimensions. And even if there aren't, there are still situations in which it is useful to use constructs which correspond to things we've already seen (points, vectors, etc.) but with more parts to them, as if they were from a higher-dimensional space. For instance, a company which produces 10 different products might have a use for considering a "point" with 10 coordinates, where each coordinate corresponds to a different one of the 10 products and indicates, for instance, how many of that product are in stock at the moment. There are many uses of the kind of mathematical constructs we've been working with, other than literally as points in space or directed line segments, in which the kind of arithmetic we've been doing still makes sense. And if these constructs aren't being used to represent physical space, then there's no reason that they would need to be limited to 3 dimensions.

We start our study of $\Re^m$ with a number of definitions, which will all look familiar to you from $\Re^2$ and $\Re^3$. These definitions just extend the construct we've been using, vectors, to higher dimensional space, and then extend (most of) the things we were doing with them, i.e. tell us how to do arithmetic etc. with these higher-dimensional vectors. And along the way we'll also see some theoretical results. But the arithmetic works just the same in higher dimensions as it did in $\Re^2$ and $\Re^3$, and the theorems will look familiar too, so it will be really easy for you to learn this. There's not much that's really new in this unit. It's just a matter of getting used to the idea of $\Re^m$ for $m > 3$.

Definition: For any positive integer $m \geq 2$, we use $\Re^m$, also called **m-space**, to denote the set of all ordered m-tuples $\vec{u} = (u_1, u_2, ..., u_m)$, where each value u_i may be any real number (i.e. for all $u_i \in \Re$). For any $\vec{u} \in \Re^m$, we refer to $\vec{u}$ as a **vector**, or an **m-vector**. The numbers u_1, u_2, ..., u_m are called the **components** of the m-vector $\vec{u}$.

Note: As we have already learnt, the terminology is sometimes a bit different for $\Re^2$ and $\Re^3$. For instance, we generally refer to an *ordered pair* or (rarely) *ordered duple* in $\Re^2$ and an *ordered triple* in $\Re^3$, rather than saying ordered 2-tuple or ordered 3-tuple. Also note that, as we have been doing with vectors in $\Re^2$ and $\Re^3$, when a vector is given a certain name, such as $\vec{u}$, the components (when not specific numbers) are assumed to be named with the same letter, with subscripts. For instance referring to the vector $\vec{u}$ we would understand that the first component is called $\vec{u_1}$, the second component is called u_2, and so forth. The only exception to this is, again as we've already seen, that often in $\Re^2$ we use $\vec{x} = (x, y)$ and likewise in $\Re^3$ we use $\vec{x} = (x, y, z)$.

Examples: $(1, 2, 3, 4)$ and $(-0.2, 61.3, 0.04, 0)$ are both vectors in $\Re^4$. $(5, 3, 6, 3, -1, -10)$ is a 6-vector. If $\vec{v} \in \Re^5$, then $\vec{v} = (v_1, v_2, v_3, v_4, v_5)$.

Definition: The **zero vector in** $\Re^m$ is the m-vector whose components are all 0. Also, two m-vectors are **equal** if and only if their corresponding components are identical. That is, for any $\vec{u}, \vec{v} \in \Re^m$, $\vec{u} = \vec{v}$ if and only if $u_1 = v_1$, $u_2 = v_2$, ..., and $u_m = v_m$.

Note: Only vectors which have the same number of components can be equal. That is, $\vec{u}$ can only be equal to $\vec{v}$ if they are both m-vectors, for the *same* value of m. (And of course will only actually be equal if their corresponding components are identical.)

Examples: $(0, 0, 0, 0)$ is the zero vector in $\Re^4$, and for $\vec{0} \in \Re^{12}$ we have $\vec{0} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$. If we have $\vec{u} = (1, 2, c, 3, -1)$ and $\vec{v} = (a, 2, 8, d, -1)$ and we know that $\vec{u} = \vec{v}$, then we must have $a = 1$, $c = 8$ and $d = 3$.

Definition: The **distance between** two m-vectors $\vec{u}$ and $\vec{v}$ is given by the **distance formula**:

$$d(\vec{u}, \vec{v}) = \sqrt{(v_1 - u_1)^2 + (v_2 - u_2)^2 + ... + (v_m - u_m)^2}$$

Also, the **magnitude** or **length** of an m-vector, sometimes referred to as the **norm** of the vector, is

$$||\vec{u}|| = \sqrt{(u_1)^2 + (u_2)^2 + ...(u_m)^2}$$

An m-vector whose magnitude is 1 is called a **unit vector**.

Note: These are precisely analogous to the notation, terminology and formulas we had for the distance between 2 vectors, and the magnitude of a vector, in $\Re^2$ and in $\Re^3$.

Example 4.1.
(a) Show that $(0, 0, 0, 0, -1, 0, 0, 0)$ is a unit vector.
(b) Show that there are no real numbers a and b for which $\vec{u} = (1, 1, a, b)$ is a unit vector.

Solution:
(a) We have

$$\begin{aligned} ||(0, 0, 0, 0, -1, 0, 0, 0)|| &= \sqrt{0^2 + 0^2 + 0^2 + 0^2 + (-1)^2 + 0^2 + 0^2 + 0^2} \\ &= \sqrt{0 + 0 + 0 + 0 + 1 + 0 + 0 + 0} \\ &= \sqrt{1} = 1 \end{aligned}$$

so this vector is a unit vector.

(b) We find the magnitude of $\vec{u}$:

$$||\vec{u}|| = ||(1, 1, a, b)|| = \sqrt{1^2 + 1^2 + a^2 + b^2} = \sqrt{1 + 1 + a^2 + b^2} = \sqrt{2 + a^2 + b^2}$$

Since $a^2 \geq 0$ and $b^2 \geq 0$ for all real numbers a and b, then $2 + a^2 + b^2 \geq 2$ and so $\sqrt{2 + a^2 + b^2} \geq \sqrt{2}$. But then we see that $||\vec{u}|| \geq \sqrt{2} > 1$, so $||\vec{u}|| \neq 1$ and $\vec{u}$ cannot be a unit vector.

Example 4.2. If $\vec{u} = (1, 2, 3, 4)$ and $\vec{v} = (0, -1, 12, 7)$, find the distance between $\vec{u}$ and $\vec{v}$ and also find the magnitude of each.

Solution:
The distance between $\vec{u}$ and $\vec{v}$ is

$$\begin{aligned} d(\vec{u}, \vec{v}) &= \sqrt{(0-1)^2 + (-1-2)^2 + (12-3)^2 + (7-4)^2} = \sqrt{(-1)^2 + (-3)^2 + (9)^2 + (3)^2} \\ &= \sqrt{1 + 9 + 81 + 9} = \sqrt{100} = 10 \end{aligned}$$

For the magnitudes of $\vec{u}$ and $\vec{v}$ we get

$$\begin{aligned} ||\vec{u}|| &= ||(1, 2, 3, 4)|| = \sqrt{1^2 + 2^2 + 3^2 + 4^2} = \sqrt{1 + 4 + 9 + 16} = \sqrt{30} \\ \text{and } ||\vec{v}|| &= ||(0, -1, 12, 7)|| = \sqrt{0^2 + (-1)^2 + (12)^2 + 7^2} = \sqrt{0 + 1 + 144 + 49} = \sqrt{194} \end{aligned}$$

Example 4.3. How long is the 5-vector $(1, 1, -1, 0, -1)$?

Solution:
We are being asked to find the length, i.e. the magnitude, of this vector. We get:

$$||(1, 1, -1, 0, -1)|| = \sqrt{1^2 + 1^2 + (-1)^2 + 0^2 + (-1)^2} = \sqrt{1 + 1 + 1 + 0 + 1} = \sqrt{4} = 2$$

We see that the vector $(1, 1, -1, 0, -1)$ is 2 units long.

Definition: For any scalar c and any m-vector $\vec{u}$, the **scalar multiple** of $\vec{u}$ by c is

$$c\vec{u} = (cu_1, cu_2, ..., cu_m)$$

For the scalar -1, the scalar multiple of any $\vec{u} \in \Re^m$ by -1 is called the **negative** of $\vec{u}$, denoted $-\vec{u}$, so that

$$-\vec{u} = (-u_1, -u_2, ..., -u_m)$$

If two m-vectors are scalar multiples of one another, we say that they are **parallel**.

Note: Again, this is the same terminology and notation, and similar calculations, as we had in $\Re^2$ and $\Re^3$. To find a scalar multiple of an m-vector, we multiply each component of that vector by the scalar. And to find the negative of an m-vector, we multiply each component by -1, i.e. switch the sign of each component. Also, we use the term parallel to describe two m-vectors which have the same property that would lead us to conclude that they were parallel if they were vectors with fewer components.

Example 4.4. For $\vec{u} = (1, 2, -1, 2)$, $\vec{v} = (2, 0, 3, 0, 1)$ and $\vec{w} = (1, 0, 1, 0, 0, 3, 2)$, find $-\vec{u}$, $5\vec{v}$ and $-2\vec{w}$.

Solution:
We simply need to multiply the components of each vector by the corresponding scalar:

$$\begin{aligned} -\vec{u} &= -(1, 2, -1, 2) = (-1, -2, -(-1), -2) = (-1, -2, 1, -2) \\ 5\vec{v} &= 5(2, 0, 3, 0, 1) = (5 \times 2, 5 \times 0, 5 \times 3, 5 \times 0, 5 \times 1) = (10, 0, 15, 0, 5) \\ -2\vec{w} &= -2(1, 0, 1, 0, 0, 3, 2) \\ &= (-2 \times 1, -2 \times 0, -2 \times 1, -2 \times 0, -2 \times 0, -2 \times 3, -2 \times 2) \\ &= (-2, 0, -2, 0, 0, -6, -4) \end{aligned}$$

Of course, we could have found $-\vec{u}$ more quickly by realizing that we just needed to switch the sign of each component:

$$-\vec{u} = -(1, 2, -1, -2) = (-1, -2, 1, 2)$$

Definition: Two m-vectors are said to be **collinear** if and only if each is a scalar multiple of the other. That is, $\vec{u}$ and $\vec{v} \in \Re^m$ are collinear if and only if there is some scalar c such that $\vec{u} = c\vec{v}$. If two non-zero m-vectors $\vec{u}$ and $\vec{v}$ are collinear, so that $\vec{u} = c\vec{v}$, then they are said to **have the same direction** if $c > 0$ and are said to **have opposite directions** if $c < 0$.

Examples: The vectors $(1,2,3,4)$ and $(2,4,6,8)$ are collinear because $(2,4,6,8) = 2(1,2,3,4)$. These vectors have the same direction. Also, for $\vec{u} = (1,0,-1,0,1)$ and $\vec{v} = (-10,0,10,0,-10)$, $\vec{v} = -10\vec{u}$ and so $\vec{u}$ and $\vec{v}$ are collinear, with directions opposite to one another.

Theorem 4.1. *For any m-vector $\vec{u}$ and any scalar c,*

$$||c\vec{u}|| = |c|||\vec{u}||$$

That is, the magnitude of the scalar multiple $c\vec{u}$ is simply the magnitude of $\vec{u}$ times the absolute value of the scalar multiplier.

Example 4.5. Find the magnitude of $(4,8,-20,12)$.

Solution:
The arithmetic is easier if we realize that $(4,8,-20,12) = 4(1,2,-5,3)$. We get

$$||(4,8,-20,12)|| = ||4(1,2,-5,3)|| = |4|||(1,2,-5,3)|| = 4\sqrt{1^2+2^2+(-5)^2+3^2} = 4\sqrt{1+4+25+9} = 4\sqrt{39}$$

Example 4.6. If $\vec{u} = (1,-1,2,-2,3,-3)$ and $\vec{v} = -5\vec{u}$, find $||\vec{v}||$.

Solution:
It will be easier to find the magnitude of $\vec{u}$ and use that to find the magnitude of $\vec{v}$, rather than to find the magnitude of $\vec{v}$ directly. We get:

$$\begin{aligned} ||\vec{v}|| &= ||-5\vec{u}|| = |-5|||\vec{u}|| = 5||(1,-1,2,-2,3,-3)|| \\ &= 5\sqrt{1^2+(-1)^2+2^2+(-2)^2+3^2+(-3)^2} \\ &= 5\sqrt{1+1+4+4+9+9} \\ &= 5\sqrt{28} = 5\sqrt{4\times 7} = 5\sqrt{4}\sqrt{7} = 5(2)\sqrt{7} = 10\sqrt{7} \end{aligned}$$

Example 4.7. If $\vec{u} = (1,-1,1,0)$, find a unit vector in the opposite direction to $\vec{u}$.

Solution:
Let $\vec{v}$ be a unit vector in the opposite direction to $\vec{u}$. Then $\vec{v}$ is a scalar multiple of $\vec{u}$, i.e. $\vec{v} = c\vec{u}$, for some scalar value $c < 0$, such that $||\vec{v}|| = 1$. Since $\vec{v} = c\vec{u}$, then we get

$$||\vec{v}|| = ||c\vec{u}|| = |c|||\vec{u}||$$

and so $||\vec{v}|| = 1$ gives $|c|||\vec{u}|| = 1$, which means that we must have $|c| = \frac{1}{||\vec{u}||}$. (Notice that we have previously observed this in $\Re^2$ and $\Re^3$.) For $\vec{u} = (1,-1,1,0)$, we have

$$||\vec{u}|| = \sqrt{1^2+(-1)^2+1^2+0^2} = \sqrt{1+1+1+0} = \sqrt{3}$$

so we see that we need $|c| = \frac{1}{\sqrt{3}}$ in order for $\vec{v} = c\vec{u}$ to be a unit vector, and we need $c < 0$ in order for $\vec{v} = c\vec{u}$ to have the opposite direction to $\vec{u}$. Therefore we need $c = -\frac{1}{\sqrt{3}}$. So we see that a unit vector in the opposite direction to $\vec{u}$ is

$$\vec{v} = c\vec{u} = \left(-\frac{1}{\sqrt{3}}\right)(1,-1,1,0) = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0\right)$$

Definition: Consider any $\vec{u}$ and $\vec{v}$ in $\Re^m$. The **vector sum** of the vectors is

$$\vec{u}+\vec{v}=(u_1,u_2,...,u_m)+(v_1,v_2,...,v_m)=(u_1+v_1,u_2+v_2,...,u_m+v_m)$$

and the **vector difference** of $\vec{u}$ and $\vec{v}$ is

$$\vec{u}-\vec{v}=\vec{u}+(-\vec{v})=(u_1,u_2,...,u_m)+(-v_1,-v_2,...,-v_m)=(u_1-v_1,u_2-v_2,...,u_m-v_m)$$

Note: This says that, as we do in $\Re^2$ and $\Re^3$, we find the sum of 2 vectors as the vector whose components are the sums of the corresponding components of the 2 vectors, and we find the difference of 2 vectors, which can be considered to be the sum of the first vector and the negative of the second vector, as the vector whose components are the differences of the corresponding components.

Example 4.8. If $\vec{u}=(1,2,3,4)$ and $\vec{v}=(0,-1,3,-2)$, find $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$.

Solution:

$$\begin{aligned} \text{We get } \vec{u}+\vec{v} &= (1,2,3,4)+(0,-1,3,-2)=(1+0,2+(-1),3+3,4+(-2))=(1,1,6,2) \\ \text{and } \vec{u}-\vec{v} &= (1,2,3,4)-(0,-1,3,-2)=(1-0,2-(-1),3-3,4-(-2))=(1,3,0,6) \end{aligned}$$

Example 4.9. If $\vec{u}=(1,0,1,0,-1,0)$ and $\vec{v}=(0,1,1,0,-1,-1)$, find $\vec{w}$ where it is known that $\vec{w}=3\vec{u}-2\vec{v}$.

Solution:
We simply carry out the specified arithmetic operations. We get:

$$\begin{aligned} \vec{w} &= 3\vec{u}-2\vec{v} \\ &= 3(1,0,1,0,-1,0)-2(0,1,1,0,-1,-1) \\ &= (3,0,3,0,-3,0)-(0,2,2,0,-2,-2) \\ &= (3-0,0-2,3-2,0-0,-3-(-2),0-(-2)) \\ &= (3,-2,1,0,-1,2) \end{aligned}$$

Since all of the arithmetic operations we have defined for $\Re^m$ are exactly analogous to the arithmetic operations we defined for $\Re^2$ and $\Re^3$, then the same properties hold. That is, in Theorem 1.4, on page 13 we stated some properties which hold for vectors in $\Re^2$ and $\Re^3$, and these same properties hold for vectors in $\Re^m$. You should review Theorem 1.4, and think about those same properties when applied to m-vectors for $m>3$. Experiment a bit to convince yourself that those properties still hold when we have vectors from a higher dimensional space.

Dot Product in $\Re^m$

Definition: Let $\vec{u}$ and $\vec{v}$ be any 2 m-vectors. Then the **dot product of $\vec{u}$ and $\vec{v}$**, written $\vec{u}\bullet\vec{v}$, is the scalar value given by

$$\vec{u}\bullet\vec{v}=(u_1,u_2,...,u_m)\bullet(v_1,v_2,...,v_m)=u_1v_1+u_2v_2+...+u_mv_m$$

Note: This says that just as before, we find the dot product of 2 vectors by taking the sum of the products of corresponding components.

Example 4.10. Find $(1, 2, 3, 4) \bullet (1, 0, -1, 0)$.

Solution:

$$(1, 2, 3, 4) \bullet (1, 0, -1, 0) = (1)(1) + (2)(0) + (3)(-1) + (4)(0) = 1 + 0 + (-3) + 0 = 1 - 3 = -2$$

Example 4.11. If $\vec{u} = (1, 1, -1, 2, -2)$ and $\vec{v} = (-1, 2, -1, 1, -2)$, find $\vec{u} \bullet \vec{v}$ and $\vec{v} \bullet (-\vec{u})$.

Solution:

$$\begin{aligned}
\vec{u} \bullet \vec{v} &= (1, 1, -1, 2, -2) \bullet (-1, 2, -1, 1, -2) \\
&= (1)(-1) + (1)(2) + (-1)(-1) + (2)(1) + (-2)(-2) \\
&= -1 + 2 + 1 + 2 + 4 = 8 \\
\text{and } \vec{v} \bullet (-\vec{u}) &= (-1, 2, -1, 1, -2) \bullet (-1, -1, 1, -2, 2) \\
&= (-1)(-1) + (2)(-1) + (-1)(1) + (1)(-2) + (-2)(2) \\
&= 1 - 2 - 1 - 2 - 4 = -8
\end{aligned}$$

Notice that in that example, we found that $\vec{v} \bullet (-\vec{u}) = -8 = -(\vec{u} \bullet \vec{v})$. That's because once again, having defined the dot product for m-vectors to be exactly analogous to the dot product for vectors in $\Re^2$ and $\Re^3$, the same properties hold. That is, all of the properties of the dot product that we observed in Theorem 2.1, on page 16, hold for m-vectors just like they do for vectors in $\Re^2$ and $\Re^3$. As before, you should review the properties stated in that theorem, think about what they mean when applied to vectors with more than 3 components, and experiment a bit to convince yourself that these properties do still hold.

For instance, the theorem tells us that $\vec{v} \bullet \vec{u} = \vec{u} \bullet \vec{v}$, and also that $\vec{v} \bullet (-\vec{u}) = \vec{v} \bullet (-1)\vec{u} = (-1)(\vec{v} \bullet \vec{u}) = -(\vec{v} \bullet \vec{u})$, so in Example 4.11 it is not surprising that we found that $\vec{v} \bullet (-\vec{u}) = -(\vec{u} \bullet \vec{v})$, because we have

$$\vec{v} \bullet (-\vec{u}) = -(\vec{v} \bullet \vec{u}) = -(\vec{u} \bullet \vec{v})$$

In $\Re^2$ and $\Re^3$, the word *orthogonal* is used to describe 2 vectors which form a right angle at the origin, i.e. two vectors which are perpendicular to one another. And we know an easy way to determine when two vectors are orthogonal, because the value of the dot product of those vectors is 0. We extend this sense of orthogonal to vectors in $\Re^m$.

> **Definition:** For vectors $\vec{u}$ and $\vec{v}$ in $\Re^m$, we say that the vectors are **orthogonal** if and only if $\vec{u} \bullet \vec{v} = 0$.

Example 4.12. For what value(s) of k are the vectors $\vec{u} = (1, -2, -4, k, 2)$ and $\vec{v} = (2, 1, -1, 3, 1)$ orthogonal?

Solution:
In order for $\vec{u}$ and $\vec{v}$ to be orthogonal, it must be true that $\vec{u} \bullet \vec{v} = 0$. We calculate $\vec{u} \bullet \vec{v}$, in terms of the unknown constant k, and then solve for the value(s) of k which make that expression have the value 0. We have

$$\vec{u} \bullet \vec{v} = (1, -2, -4, k, 2) \bullet (2, 1, -1, 3, 1) = 2 - 2 + 4 + 3k + 2 = 6 + 3k$$

Therefore we need $6 + 3k = 0$ which gives $3k = -6$, so $k = -2$.

That is, $\vec{u}$ and $\vec{v}$ are orthogonal only when $k = -2$.

Notice: We have extended the definitions of vector sums and differences, scalar multiplication, and the dot product of vector to include vectors in $\Re^m$ for any m. But we *have not* extended the idea of cross product. As stated in Unit 2, the cross product of 2 vectors is defined *only* for vectors in $\Re^3$. (Or at least for our purposes it is.)

Lines in $\Re^m$

We understand physically (i.e. geometrically) what a line is in $\Re^2$ or $\Re^3$. That geometric concept of a line does not, perhaps, easily extend to "higher dimensions" in which we cannot attach the same physical interpretation. However the algebraic constructs which we associate with lines can easily be extended to $\Re^m$. Since we are already using terms like "vector" and even "point" in $\Re^m$, it makes sense that we would extend to $\Re^m$ the other terminology we use along with those. For instance, we call the thing represented by $\vec{x}(t) = \vec{p} + t\vec{v}$ a line if $\vec{p}$ and $\vec{v}$ are vectors in $\Re^2$ or in $\Re^3$ (where $\vec{p}$ is the vector from the origin to a point P on the line, and $\vec{v}$ is a direction vector for the line), so why not also refer to $\vec{x}(t) = \vec{p} + t\vec{v}$ as a line when $\vec{p}$ and $\vec{v}$ happen to be vectors with more than 3 components?

Note: In $\Re^2$, we use $\vec{x} = (x, y)$, and in $\Re^3$ we use $\vec{x} = (x, y, z)$. But now we've reached the end of the alphabet ... what do we do in $\Re^4$? Or in $\Re^5$? Generally, when $\vec{x} \in \Re^m$ for $m > 3$ we use subscripted x's. That is, we use $\vec{x} = (x_1, x_2, ..., x_m)$.

Definition: Borrowing from the terminology used for $\Re^2$ and $\Re^3$, we define equations of lines in $\Re^m$ as follows:

1. Consider any points (i.e. m-tuples) P and Q in R^m, and let $\vec{p}$ and $\vec{q}$ be the corresponding vectors in $\Re^m$. The **two-point form** of an **equation of the line through P and Q** is given by

$$\vec{x}(t) = (1 - t)\vec{p} + t\vec{q}$$

2. For any vector $\vec{v}$ which is parallel to the vector $\vec{q} - \vec{p}$, we say that $\vec{v}$ is a **direction vector** for the line through P and Q, and we can write a **point-parallel form equation** of the line as

$$\vec{x}(t) = \vec{p} + t\vec{v}$$

3. For any line $\vec{x}(t) = \vec{p} + t\vec{v}$ in $\Re^m$, **parametric equations** of the line can be obtained by equating corresponding components from the point-parallel form equation. For instance, for the line

$$\vec{x}(t) = (p_1, p_2, p_3, p_4) + t(v_1, v_2, v_3, v_4)$$

the parametric equations would be

$$\begin{aligned} x_1 &= p_1 + v_1 t \\ x_2 &= p_2 + v_2 t \\ x_3 &= p_3 + v_3 t \\ x_4 &= p_4 + v_4 t \end{aligned}$$

There will, of course, be one parametric equation for each component of the m-vector. That is, m parametric equations are required to describe a line in $\Re^m$.

Example 4.13. Write equations of the line in $\Re^4$ containing $P(1,2,3,4)$ and $Q(2,0,-1,1)$ in the following forms: two-point form, point-parallel form, parametric equations.

Solution:
We use the two points given to write a two-point form equation of the line:

$$\vec{x}(t) = (1-t)(1,2,3,4) + t(2,0,-1,1)$$

As in $\Re^2$ and $\Re^3$, we can find a direction vector, $\vec{v}$, for the line as the vector equivalent to $\overrightarrow{PQ}$, given by $\vec{q} - \vec{p}$. We get

$$\vec{v} = \vec{q} - \vec{p} = (2,0,-1,1) - (1,2,3,4) = (1,-2,-4,-3)$$

and so we can write a point-parallel form equation of the line using this direction vector and either point:

$$\vec{x}(t) = (1,2,3,4) + t(1,-2,-4,-3)$$

From this point-parallel for equation we get the parametric equations:

$$\begin{aligned} x_1 &= 1+t \\ x_2 &= 2-2t \\ x_3 &= 3-4t \\ x_4 &= 4-3t \end{aligned}$$

Example 4.14. Write parametric equations of the line through $P(1,0,2,1,-3,2)$ which is parallel to $\vec{v} = (0,1,0,1,-1,1)$.

Solution:
We see (by counting the coordinates of P and the components of $\vec{v}$) that this is a line in $\Re^6$, so there will be 6 parametric equations. Let $\vec{x}(t) = (x_1, x_2, x_3, x_4, x_5, x_6)$ be any point on this line. A point-parallel equation of the line is $\vec{x}(t) = (1,0,2,1,-3,2) + t(0,1,0,1,-1,1)$. Therefore the parametric equations for the line are:

$$\begin{aligned} x_1 &= 1+0t \\ x_2 &= 0+t \\ x_3 &= 2+0t \\ x_4 &= 1+1t \\ x_5 &= -3-t \\ x_6 &= 2+t \end{aligned}$$

Of course, we would normally rewrite these:

$$\begin{aligned} x_1 &= 1 \\ x_2 &= t \\ x_3 &= 2 \\ x_4 &= 1+t \\ x_5 &= -3-t \\ x_6 &= 2+t \end{aligned}$$

Hyperplanes

We know that in $\Re^2$, a point-normal form equation or standard form equation corresponds to a line, while in $\Re^3$, these forms correspond to a plane. The generic term for the higher-dimensional constructs which correspond to planes is **hyperplane**. That is, just as $\Re^2$ contains infinitely many different lines, and $\Re^3$ contains infinitely many different planes, it is likewise the case that $\Re^m$ contains infinitely many "hyperplanes", for any $m > 3$. And it is these hyperplanes which are described by point-normal form or standard form equations in $\Re^m$.

> **Definition:** In $\Re^m$, $\vec{n} \bullet (\vec{x} - \vec{p}) = 0$ is a **point-normal form equation of the hyperplane through P with normal vector $\vec{n}$**, and the **standard form equation** of this hyperplane is
>
> $$n_1x_1 + n_2x_2 + ... + n_mx_m = c$$
>
> where the coefficients n_i are the corresponding components of $\vec{n}$ and c is a constant whose value is given by $\vec{n} \bullet \vec{p}$.

Example 4.15. Write an equation of the hyperplane in $\Re^5$ which passes through $(1, 0, -2, 3, 1)$ with normal $(2, 3, -1, -2, 1)$ in:
(a) point-normal form (b) standard form

Solution:
(a) We have $\vec{n} = (2, 3, -1, -2, 1)$ and $\vec{p} = (1, 0, -2, 3, 1)$, so a point-normal form equation of this hyperplane is:

$$(2, 3, -1, -2, 1) \bullet (\vec{x} - (1, 0, -2, 3, 1)) = 0$$

(b) The normal vector gives the coefficients of the x_i's, and the value of the constant on the right hand side of the equation is found by evaluating the equation at the given point (i.e. by calculating $\vec{n} \bullet \vec{p}$). That is, the equation has the form $2x_1+3x_2-x_3-2x_4+x_5 = c$ where $c = 2(1)+3(0)-(-2)-2(3)+1 = 2+0+2-6+1 = -1$, so a standard form equation of the hyperplane is:

$$2x_1 + 3x_2 - x_3 - 2x_4 + x_5 = -1$$

Example 4.16. Find a normal vector for the hyperplane in $\Re^4$ corresponding to the standard form equation $5x_1 - 3x_2 + x_4 = 7$. Find any point on this hyperplane.

Solution:
The components of the normal vector are the coefficients of the x_i's in the standard form equation. Since the left hand side of the equation is $5x_1 - 3x_2 + x_4$, with no x_3 visible, the coefficient of x_3 must be 0. So the normal vector is $\vec{n} = (5, -3, 0, 1)$.

To find a point on this hyperplane, we simply need to find any point $P(x_1, x_2, x_3, x_4)$ which satisfies the standard form equation. For instance, if we set $x_1 = 0$ and $x_2 = 0$ then we get $0 - 0 + x_4 = 7$, and x_3 could have *any* value, so $(0, 0, 0, 7)$ is a point on the hyperplane. Likewise, $(0, 0, -43, 7)$ is another. Many other points could be found, with any combination of x_1, x_2 and x_4 values which makes $5x_1 - 3x_2 + x_4 = 7$. (For instance, if we chose $x_1 = 2$ and $x_2 = 1$, we would get $x_4 = 0$. And once again, x_3 could have any value, since it is multiplied by 0.)

Notice: As we saw in this example, some of the x_i's might be invisible in the standard form equation, because the corresponding component of the normal vector is 0. This can be true for one or more x_i's with i larger than the largest subscript showing, as well as for "missing" ones between those that show. In the example, we were told that the hyperplane was in $\Re^4$, so we knew that there weren't any more x_i's after the last one we could see. But if it hadn't said that, we wouldn't know how many components the normal vector should have. We might *assume* that there should be 4, but it could just as well have been 5 or 6 or ... That is, the hyperplane through $(0, 0, 0, 7, 0, 0)$ with normal vector $(5, -3, 0, 1, 0, 0)$, in $\Re^6$, also has standard form equation $5x_1 - 3x_2 + x_4 = 7$.

The same thing can happen with $\Re^2$ and $\Re^3$. The equation $x + y = 1$ is an equation of a line in $\Re^2$, with normal $(1, 1)$. But it's also the equation of a plane in $\Re^3$, with normal $(1, 1, 0)$. (Since the unknowns are called x and y, rather than x_1 and x_2, it's unlikely to correspond to a hyperplane in $\Re^m$ for some $m > 3$, although it could. The text sometimes uses $\vec{x} = (x, y, z, w)$ for a vector in $\Re^4$, and by that convention, $x + y = 1$ is also a hyperplane in $\Re^4$ with normal $(1, 1, 0, 0)$. And I suppose we could similarly use (x, y, z, w, v) for $\Re^5$ and so on ...)

Please Note: At the end of this section, the text contains some material that we aren't going to talk about, or at least not right now. The concepts of *linear equation* and *linear combination* we'll meet soon enough, and there's no need to talk about them here and now. And the concept of a plane determined by two vectors isn't covered in this course.

Math 1229A/B

Unit 5:
Systems of Linear Equations

(text reference: Section 2.2)

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5 Systems of Linear Equations

You know what the standard form of an equation of a line in $\Re^2$ looks like. It has the form $ax+by=c$, for some constants a, b and c. Any equation in 2 variables in which each variable is only multiplied by a constant is the standard form equation of a line in $\Re^2$. The variables are usually called x and y, but they could be called x_1 and x_2, or m and n, or anything.

Of course, we can have other kinds of equations with 2 variables, and you've probably seen some of those before. For instance, $x^2+y^2=1$ is the equation of the circle with radius 1. More complicated curves in 2-space have equations like $x^2+xy+y^2=4$ or $5x^4y-3x^2y^2+2xy^5=0$ or $\sqrt{x}-x\sqrt{y}+y=3$. And then there are equations like $x+\frac{y}{x}-\frac{2}{y^2}=6$, and $2^{x+y}=4$ and $x+\sin y=1$. But none of those are *linear* equations. They're not lines. They're curvy things.

You also know that in $\Re^3$, an equation like $x+y+z=1$ is *not* a line. It's a plane. But ... what is a plane? It's just a whole bunch of lines side-by-side. Well, okay, we wouldn't really think of a plane that way, but a plane does have some characteristics which make it similar to a line in some ways. It's flat. No curvy bits. And the same goes for a hyperplane in $\Re^m$. Any plane or hyperplane, in standard form, has an equation which is just the sum of a bunch of constant multiples of variables, set equal to some constant. There are never any more complicated things done to the variables, like squaring, or taking the square root, or multiplying two variables together, or dividing by a variable. Just a constant times a variable, plus a constant times another variable, plus ... and equal to some constant. The most interesting things that happen are that sometimes a constant is negative, so that we're actually subtracting, or sometimes a constant is 0, so that the variable isn't even there.

Equations like this are called *linear equations*, because the relationship between the variables is always *like* the relationship between the variables in an equation of a line in $\Re^2$. We're going to be working with linear equations a lot in this course. In fact, the textbook is called *Elementary Linear Algebra*, because the kind of algebra we're studying all relates to these linear equations. So we need a careful definition of what these are. But it's not any more complicated than what we've already said.

Definition: A **linear equation** in the variables x_1, x_2, ... x_n is an equation which can be put into the form

$$a_1x_1+a_2x_2+...a_nx_n=b$$

for some constants a_1, a_2, ..., a_n and b, where not all of the a_i values are 0.

Note: That last bit just means that the equation $0=0$ isn't considered to be a linear equation. There has to be at least one variable actually appearing in the equation. Also, as you already know, the variables don't necessarily have to have the names shown in the definition. They could be x, y, z, w, r and t, or they could be subscripted y's or s's instead of x's. The names of the variables don't matter. Any equation which says, or can be rearranged to say, that the sum of scalar multiples of variables is equal to a constant is a linear equation.

Definition:

- A collection of a finite number of linear equations involving the same variables is called a **system of linear equations**, often abbreviated as **SLE**.
- A system of m linear equations in n variables is called an $m\times n$ SLE (pronounced "m by n").
- An n-vector $(x_1, x_2, ..., x_n)$ is called a **solution** to a particular SLE with n variables if it satisfies all of the equations in the system.

- A SLE is in **standard form** if all of the equations have all of the variables appearing on the left hand side of the equation, in the same order for all equations, with spaces left for any variables missing in that equation, and the constant term appears on the right hand side of each equation.
- A SLE is said to be **consistent** if it has at least one solution.
- A SLE which has no solutions is called **inconsistent**.

For instance, the equations

$$\begin{array}{rrcrcrcr} & x & + & y & + & z & = & 4 \\ \text{and} & x & - & y & & & = & -1 \end{array}$$

form a system of linear equations, because each is a linear equation in the variables x, y and z. We can refer to this as a 2×3 SLE, which says that it has 2 equations and 3 variables. This SLE is in standard form, because each equation has all variables on the left hand side with the constant term on the right hand side, and the variables appear in the same order, lined up in columns. The vector $(1, 2, 1)$ is a solution to this SLE because $(x, y, z) = (1, 2, 1)$ has $x + y + z = 1 + 2 + 1 = 4$ so it satisfies the first equation, and also has $x - y = 1 - 2 = -1$, so the second equation is also satisfied. And since there is at least one solution to the SLE (we know one, and there are many others), the system is consistent.

Likewise, the set of equations

$$\begin{array}{rrcrcr} & 6x & + & 2y & = & 4 \\ \text{and} & 3x & + & y & = & 0 \end{array}$$

is another SLE, again in standard form. However this SLE is inconsistent, because if $6x + 2y = 4$, then $3x + y = \frac{1}{2}(6x + 2y) = \frac{1}{2}(4) = 2 \neq 0$ so any 2-vector satisfying the first equation cannot also satisfy the second equation and so the system has no solutions.

On the other hand, the equations

$$\begin{array}{rrcrcrcr} & x & + & y & + & z & = & 1 \\ \text{and} & 3^x & - & 2\cos y & + & z & = & 0 \end{array}$$

is *not* a system of linear equations, because the second equation is not linear. Neither 3^x nor $2\cos y$ are expressions which can appear in a linear equation.

Example 5.1. Consider the SLE $\begin{array}{rcl} x + 2z & = & \dfrac{y}{2} \\ y & = & 2z \end{array}$.

(a) Put this system into standard form.
(b) Show that any vector of the form $(x, y, z) = (-t, 2t, t)$ is a solution to this system.

Solution:
(a) To put the SLE into standard form, we collect all the variables on the left hand side and put any constant on the right hand side. For the first equation, we can start by getting rid of the fraction (although this isn't necessary). We have

$$x + 2z = \frac{y}{2} \quad \Rightarrow \quad 2x + 4z = y \quad \Rightarrow \quad 2x + 4z - y = 0 \quad \Rightarrow \quad 2x - y + 4z = 0$$

Also, for the second equation we have

$$y = 2z \quad \Rightarrow \quad y - 2z = 0$$

So we can write the SLE in standard form as

$$\begin{array}{rcrcrcr} 2x & - & y & + & 4z & = & 0 \\ & & y & - & 2z & = & 0 \end{array}$$

Notice that the last step in rearranging the first equation wasn't necessary. The variables don't necessarily have to be in any particular order, as long as they're in the same order in each equation. We could have left the first equation as $2x + 4z - y = 0$, but in that case we would have needed to state the second equation as $-2z + y = 0$ in our standard form. Usually, though, we do list the variables in their "natural" order. It's just easier to think of that way.

(b) From the original statement of the second equation, we know that any solution to this system must satisfy $y = 2z$. Substituting this into the original form of the first equation gives $x + 2z = \frac{y}{2} = \frac{2z}{2} = z$. And from $x + 2z = z$ we see that we need $x = z - 2z = -z$. So for any particular value of z, setting $x = -z$ and $y = 2z$ gives a solution to the SLE. That is, if we set z equal to some value t, then if we also set $x = -t$ and $y = 2t$ we get a solution to the system. So $(x, y, z) = (-t, 2t, t)$ is a solution for any value t. For instance, setting $t = 1$ we see that $(-1, 2, 1)$ is a solution to this SLE. Likewise, $t = -2$ gives $(2, -4, -2)$ as another solution.

Of course, to show that $(-t, 2t, t)$ is a solution, we didn't actually need to do that. What we did above actually found the form of solution. All that was really necessary here was to show that $(x, y, z) = (-t, 2t, t)$ does satisfy both equations. Substituting $x = -t$, $y = 2t$ and $z = t$ into the left hand side of each equation in the standard form of the SLE we get:

$$\begin{array}{rclcl} 2x - y + 4z & = & 2(-t) - (2t) + 4(t) & = & -2t - 2t + 4t \quad = \quad 0 \\ y - 2z & = & 2t - 2(t) & = & 0 \end{array}$$

Since for both equations what we got was the right hand side value of the corresonding equation in the standard form of the SLE, we see that this vector *does* satisfy both equations and therefore *is* a solution to the system.

Definition: When a solution to a system of equations can be stated with one or more parameters, such that any value of the parameter(s) gives a solution to the system, the system is said to have an r**-parameter family of solutions**, where r is the number of parameters in the solution.

For instance, we saw that the SLE in Example 5.1 has the 1-parameter family of solutions $(-t, 2t, t)$ for any $t \in \Re$.

Definition: Two systems of linear equations are said to be **equivalent** if they have exactly the same solutions.

For instance, consider the SLE in Example 5.1. We know that in any solution to this system, it must be true that $y = 2z$. But if $y = 2z$ then it must also be true that $2y = 4z$, i.e. that $2y - 4z = 0$. Likewise, we know that any solution to the system must satisfy $\dfrac{y}{2} = x + 2z$, and with $y = 2z$ we could express this as $\dfrac{y}{2} = x + y$ to get $x + y - \dfrac{y}{2} = 0$ which gives $x + \dfrac{y}{2} = 0$.

So the SLE's

$$\begin{array}{rcrcrcl} & & 2y & - & 4z & = & 0 \\ x & + & \dfrac{y}{2} & & & = & 0 \end{array} \qquad \text{and} \qquad \begin{array}{rcrcrcl} 2x & - & y & + & 4z & = & 0 \\ & & y & - & 2z & = & 0 \end{array}$$

are equivalent.

Definition: To **solve** a system of linear equations means to find all solutions to the system, or to determine that the system is inconsistent.

For instance, given the SLE $\begin{array}{rcrcl} x & + & y & = & 1 \\ & & y & = & 0 \end{array}$, we can solve the system as follows. We see that we must have $y = 0$. Substituting this into the first equation, we get $x + 0 = 1$ so we see that we must have $x = 1$. Therefore $(x, y) = (1, 0)$ is the only solution to this SLE.

You have most likely solved systems of equations before. In the next unit, we'll learn a way to organize the data from a SLE so that we can find all solutions in a systematic way which is less cumbersome than working with the actual equations. But for now we'll look at what's going on with the kind of manipulations we're going to be doing, while working with the equations. We'll be doing things like "add two equations together", so you need to understand that what we mean by that is to create a new equation, whose left hand side (LHS) is the sum of the LHS's of the equations being added, and whose right hand side (RHS) is the sum of the RHS's. For instance, if we want to add the equation $x + y = 1$ and the equation $x + 2y = -2$, we form the new equation

$$(x + y) + (x + 2y) = (1) + (-2) \qquad \Rightarrow \qquad 2x + 3y = -1$$

Similarly, when we talk about multiplying an equation by a scalar, what we mean is to multiply both sides of the equation by that scalar, so when we multiply the equation LHS = RHS by the scalar (i.e. the constant) c, we make a new equation that says that $c \times \text{LHS} = c \times \text{RHS}$. For instance, when we multiply $2x + 3y - z = 5$ by 2 we get $4x + 6y - 2z = 10$.

Our objective in manipulating equations in this way is to solve a SLE. In order to do this, any manipulation we do must transform the SLE into an *equivalent* SLE. If the SLE we end up with isn't equivalent to the one we started with, then the solutions to the system we now have are not necessarily solutions to, or are not necessarily all of the solutions to, the original system, and so finding them won't solve the SLE we were trying to solve.

There are only 3 kinds of things we're allowed to do when manipulating equations to solve a SLE. These are called *elementary operations.* It's important to remember what these operations are, and to do **only** these kinds of operations, because doing anything else will almost certainly result in a SLE which is *not* equivalent to the system we're trying to solve.

Definition: The following operations are the **elementary operations** which are allowed in solving a SLE:

I Multiply an equation by a non-zero scalar.

II Interchange the positions of two equations in the system.

III Replace one of the equations by the sum of that equation and a scalar multiple of another one of the equations in the system.

No other operations are allowed.

Theorem 5.1. *Performing one of the elementary operations to transform a system of linear equations always results in a SLE which is equivalent to the original system.*

We're not going to worry about *why* that's true. We'll just accept that it is. If you only perform elementary operations, and of course perform them properly, then each system you get is equivalent to the one before, so when you arrive at one for which the solution(s) are obvious, or for which it is clear that there is no solution, you have solved the system you started with.

Example 5.2. Solve the SLE $\begin{array}{rcrcr} x & + & 2y & = & 4 \\ x & - & y & = & 1 \end{array}$.

Solution:
If we add 2 times the second equation to the first equation, the y term in the resulting equation will have coefficient 0, because $2 + 2(-1) = 0$. So the first elementary operation we're going to perform is to replace the first equation by the first equation plus 2 times the second equation. This is an elementary operation of type III. The new first equation is

$$(x + 2y) + 2(x - y) = 4 + 2(1) \qquad \Rightarrow \qquad x + 2x + 2y - 2y = 6 \qquad \Rightarrow \qquad 3x = 6$$

Replacing the first equation by this new equation we get the transformed, equivalent, SLE

$$\begin{array}{rcrcr} 3x & & & = & 6 \\ x & - & y & = & 1 \end{array}$$

Now, we can see that the first equation is going to tell us the value of x. We multiply the first equation (of this transformed system) by $\frac{1}{3}$ (which is to say, we divide through by 3) to get just x on the LHS. This is an elementary operation of type I. The new first equation is

$$\left(\frac{1}{3}\right)(3x) = \left(\frac{1}{3}\right)(6) \qquad \Rightarrow \qquad x = 2$$

Re-writing the system with this new version of the first equation, we have the SLE

$$\begin{array}{rcrcr} x & & & = & 2 \\ x & - & y & = & 1 \end{array}$$

Next, we can eliminate x from the second equation by subtracting the first equation. That is, we perform another type III elementary operation, replacing the second equation by itself plus -1 times the first equation. The new second equation is

$$(x - y) + [-1(x)] = 1 + [-1(2)] \qquad \Rightarrow \qquad x - y - x = 1 - 2 \qquad \Rightarrow \qquad -y = -1$$

So now we have the system

$$\begin{array}{rcr} x & & = & 2 \\ & -y & = & -1 \end{array}$$

All we need to do now is to multiply the second equation by -1 to get rid of the negative on the y. Another type I operation. This gives the new second equation $y = 1$ and so we have the SLE

$$\begin{array}{rrcr} x & & = & 2 \\ & y & = & 1 \end{array}$$

Well, it's pretty easy to see what the solution to that SLE is. It's staring us right in the face. Clearly, $(x, y) = (2, 1)$ is the only solution to this system of equations. And since the only things we did were elementary operations, this system is equivalent to, i.e. has the same solutions as, the original system. And so $(x, y) = (2, 1)$ is also the only solution to the SLE we started out with.

Example 5.3. Find all solutions to the system of linear equations:

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 5 \\ 2x & + & y & & & = & 10 \\ & & y & + & 2z & = & 0 \end{array}$$

Solution:
Let's think about what our elementary operations accomplished in Example 5.2. First, we eliminated y from the first equation, so that it was telling us the value of x. Later, we eliminated x from

the second equation, so that it was telling us the value of y. This general approach of eliminating variables from the equations, so that each equation tells us about a different variable, is what we want to accomplish with elementary operations.

In this case, we want to end up with the first equation being the only one with an x in it, so that this is the equation telling us about the value of x. Therefore we'll start as before by eliminating x from the other equations. Conveniently, the third equation already doesn't have x in it, so we only need to eliminate x from the second equation. In that equation, we have $2x$, so in order to eliminate this, we want to add $-2x$. We can do that by adding -2 times the first equation to the second equation. Therefore we start with a type III elementary operation: replace the second equation by the sum of itself and -2 times the first equation. The new second equation is

$$(2x+y)+(-2)(x+y+z) = 10+(-2)(5) \quad \Rightarrow \quad 2x+y-2x-2y-2z = 10-10 \quad \Rightarrow \quad -y-2z = 0$$

The transformed SLE is

$$\begin{array}{rcrcrcl} x & + & y & + & z & = & 5 \\ & & -y & - & 2z & = & 0 \\ & & y & + & 2z & = & 0 \end{array}$$

Since we want the second equation to be telling us about the value of y, it will be useful to have the coefficient of y in the second equation being 1. That is, we want the second equation to start with just y, rather than the $-y$ it currently starts with. There are at least a couple of different ways to accomplish this, but the easiest is to recognize that we already have an equation that starts with just y, so we can simply move that equation, to make it the second equation. That is, we can use a type II elementary operation to interchange the positions of the second and third equations. (In this case, this elementary operation isn't really very much easier than the alternatives, but generally speaking type II operations are the easiest — and least prone to error — so we prefer those to either of the others.) This transforms the SLE to

$$\begin{array}{rcrcrcl} x & + & y & + & z & = & 5 \\ & & y & + & 2z & = & 0 \\ & & -y & - & 2z & = & 0 \end{array}$$

Next, in keeping with our approach of eliminating variables from every equation except the one that's going to tell us about the value of that variable, we want to eliminate the y terms from the first and third equations. For the first equation, we need a type III elementary operation. (There isn't another equation we can interchange with the first, that doesn't have a y in it, so we can't use a type II operation. Also, there's no non-zero scalar that we can multiply the first equation by to transform $+y$ to 0, so we also can't use a type I operation.) We see that since the coefficient of y in the first equation is 1, the same as in the second equation, we can get rid of the y in the first equation by simply subtracting the second equation. That is, we will replace the first equation by the sum of itself and -1 times the second equation. This gives the new first equation as

$$(x+y+z)+(-1)(y+2z) = 5+(-1)(0) \quad \Rightarrow \quad x+y+z-y-2z = 5-0 \quad \Rightarrow \quad x-z = 5$$

The newly transformed SLE is

$$\begin{array}{rcrcrcl} x & & & - & z & = & 5 \\ & & y & + & 2z & = & 0 \\ & & -y & - & 2z & = & 0 \end{array}$$

And since the coefficient on y in the third equation is -1, we can eliminate the y term by adding y, i.e. by simply adding the second equation. So we replace the third equation by the sum of itself and 1 times the second equation, which gives the new third equation as

$$(-y-2z)+(y+2z) = 0+0 \quad \Rightarrow \quad 0 = 0$$

Hmm. Well, okay. The equivalent SLE is now

$$\begin{array}{rcrcrcl} x & & & - & z & = & 5 \\ & & y & + & 2z & = & 0 \\ & & & & 0 & = & 0 \end{array}$$

At this point, we have the first equation starting with just x, and with no y term in it, to tell us about the value of x, and the second equation starting with just y, with no x term in it, to tell us about the value of y. We want to next make the third equation start with just z, and eliminate the z term from the other two equations, so that the third equation tells us about the value of z, and the others involve *only* the variable that each is telling us about the value of. But we have a problem. The third equation has gone away! It doesn't have a z in it anymore. All it says is the seemingly unhelpful fact that zero is equal to zero. *Note:* An equation that tells us that $0 = 0$ isn't actually a problem. It simply says "everything's okay". In the next example, we'll see that sometimes we get an equation that tells us that 0 equals something other than 0. *That* does indicate a problem, of sorts. But the assertion that 0 equals 0 is just a statement of fact that doesn't bother us.

But then, how do we proceed if there's no equation to tell us about the value of z? Well, we don't. That is, we're pretty much done. The fact that there's no equation telling us about the value of z tells us that z could have *any value*. That is, we can have $z = t$ for any value $t \in \Re$. And the first equation, telling us about the value of x, tells us about the value of x *relative to* z. Likewise, the second equation is telling us about the value of y *relative to* z. Let's look at those two equations, and re-express them so that the LHS has only the variable we want the equation to be telling us about:

$$\begin{aligned} x - z = 5 \quad &\Rightarrow \quad x = z + 5 \\ y + 2z = 0 \quad &\Rightarrow \quad y = -2z \end{aligned}$$

So if z is equal to some number t, we need $x = t + 5$ and $y = -2t$. That is, $(x, y, z) = (t + 5, -2t, t)$ is a solution for any value $t \in \Re$. That is, there are infinitely many different solutions to this last SLE, and they all have this same form (for different values of t). Therefore the SLE we started out with,

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 5 \\ 2x & + & y & & & = & 10 \\ & & y & + & 2z & = & 0 \end{array}$$

(which is of course equivalent to the one we ended up with, since we only transformed the SLE using elementary operations), has the one-parameter family of solutions $(x, y, z) = (t + 5, -2t, t)$ for any $t \in \Re$.

Example 5.4. Solve the SLE $2x = 2 - 2y$ and $3y - 6 = -3x$.

Solution:
This time, the SLE we're given is not in standard form. Whenever we're given an SLE that isn't in standard form, the first thing we do, *always*, is to put the system into standard form. So we start by rearranging each of the equations into (in this case, since we have 2 variables, x and y) the form $ax + by = c$. For the first equation, we need to add $2y$ to each side of the equation, i.e. move the $2y$ from the RHS to the LHS. We get the equation $2x + 2y = 2$. For the second equation, we add $3x$ to both sides, to move the $3x$ from the RHS to the LHS, and we also need to add 6 to both sides, i.e. move the constant to the RHS. We get $3x + 3y = 6$. This gives the SLE as

$$\begin{array}{rcrcr} 2x & + & 2y & = & 2 \\ 3x & + & 3y & = & 6 \end{array}$$

Notice: What we did here, so far, didn't involve elementary operations. But we weren't transforming the SLE into an equivalent SLE, we were simply re-writing the given equations to get to standard form. Moving things from one side of an equation to another, by adding or subtracting the same thing on both sides of the equation, doesn't change the equation in any fundamental way. It's still the same equation, it just looks a bit different.

You may be able to tell already, from the standard form SLE, what the conclusion is going to be. But let's see what we can do with elementary operations to make it more obvious. (Besides, it's good practice.) We'll follow the same basic procedure as before, making the first equation tell us about the value of x, and elimating x from the other equation. First, in order to have the first equation tell us about x, we want it to start with just x, rather than $2x$. The easiest way to accomplish that this time is to divide through by 2, i.e. to multiply the first equation by the non-zero constant $\frac{1}{2}$. This is a type I elementary operation. The new version of the first equation is $\left(\frac{1}{2}\right)(2x+2y) = \left(\frac{1}{2}\right)(2)$, i.e. $x + y = 1$. We get the equivalent SLE

$$\begin{array}{rcrcr} x & + & y & = & 1 \\ 3x & + & 3y & = & 6 \end{array}$$

Next, we want to eliminate x from the second equation. Since the coefficient of x in the second equation is 3, we want to subtract $3x$. We can accomplish that by subtracting 3 times the first equation. That is, we perform a type III elementary operation, replacing the second equation by itself plus -3 times the first equation. When we do this we get:

$$(3x+3y) + (-3)(x+y) = 6 + (-3)(1) \quad\Rightarrow\quad 3x+3y-3x-3y = 6-3 \quad\Rightarrow\quad 0 = 3$$

This gives the transformed system

$$\begin{array}{rcrcr} x & + & y & = & 1 \\ & & 0 & = & 3 \end{array}$$

Because all we did was perform elementary operations, this system is equivalent to the system we started with. But this system is nonsense! Why does it say that $0 = 3$? What does that mean? Obviously that equation can *never* be satisfied. It's always false. ... But that's a very useful thing to know. If this transformed system is never satisfied, and is equivalent to the original system, then that means that the original system can never be satisfied either. If the transformed system is nonsense, then the original SLE must also have been nonsense. That is, since there are no values of x and y which make both equations in the transformed system true, then there are no values of x and y that make both the equations in the original SLE true, i.e. the system has no solutions. So in this case, our conclusion is that the given SLE is inconsistent.

Notice: Whenever we obtain an equation that says that $0 = c$ where c is any non-zero constant, that contradictory equation is telling us that the system is inconsistent. Not just that this particular equation has no solution, but more, that the whole system has no solution.

In these last 3 examples, we had examples of the only 3 things that can happen when we solve a system of linear equations. The system can have a unique solution, like in Example 5.2. Or the system can have no solution, as we saw in Example 5.4. The *only* other possibility is that the system has infinitely many solutions, expressed as a parametric family of solutions. This was the situation in Example 5.3, which we had previously encountered in Example 5.1, as well.

Math 1229A/B

Unit 6:

Row Reduction

(text reference: Section 2.3)

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6 Row Reduction

Next, we learn a method known as row-reduction for solving SLE's. This method works in basically the same way as the method we used in the previous unit, but we will use a structure called a matrix to eliminate the repetitive writing down of symbols which never change from one step to the next. This helps us to organize what we're doing and develop a systematic procedure for getting from the system which needs to be solved to an equivalent system in which the solution(s), or the fact that there is no solution, is obvious. First, we must define what this mathematical structure called a matrix is.

> **Definition:** A **matrix** is a rectangular array of numbers:
>
> $$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$
>
> The horizontal lines of numbers are called **rows** and the vertical lines of numbers are called **columns**. The number, a_{ij}, in row i and column j is called the **(i,j)-entry** of the matrix. A matrix with m rows and n columns is called an $\mathbf{m \times n}$ **matrix** (pronounced "m by n").

Note: The plural of *matrix* is *matrices*.
Also Note: Matrices are sometimes written using parentheses (i.e. round brackets) instead of the square brackets shown here. But in this course, we'll use square brackets.

Here are some examples of matrices:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \qquad \begin{bmatrix} -1 & 0 & 5 \\ 1 & 2 & 4 \\ -3 & 6 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & -5 & 0 \end{bmatrix}$$

We will learn more about these things called matrices later in the course. For now, we want to learn about one particular use of a matrix, to use it as shorthand to represent a system of linear equations.

> **Definition:** Consider any system of m linear equations in n variables, written in standard form. The **coefficient matrix** of the SLE is the $m \times n$ matrix in which the (i, j)-entry is the coefficient, in the i^{th} equation, of the j^{th} variable. And the **augmented matrix** for a SLE is obtained by appending the m right hand side values of the equations to the coefficient matrix, as an extra column in the matrix. We always delineate this extra column in an augmented matrix by placing a vertical line before it.

That is, row i of the coefficient matrix contains, in order, the coefficients from equation i. Likewise, column j of the coeffient matrix contains the coefficients of the j^{th} variable in the order in which they appear in the equations. When we form the augmented matrix, by also including the column of RHS values, the matrix looks just like the SLE, with all of the variables omitted, and with a vertical line instead of all the equal signs.

Example 6.1. Write the coefficient matrix and the augmented matrix for the following SLE:

$$\begin{array}{rcrcrcr} x & - & 4y & + & 3z & = & 5 \\ -x & + & 3y & - & z & = & -3 \\ 2x & & & - & 4z & = & 6 \end{array}$$

Solution:
Since the system is already in standard form, we can obtain the coefficient matrix by simply writing the coefficients from the system in the same order in which they appear in the SLE. For instance, we get the first row of the coefficient matrix by extracting the coefficients of x, y and z, i.e. 1, -4 and 3, from the first equation. *Note:* The $(3,2)$-entry is 0, recognizing that in the third equation, there is a 0 coefficient making the y-term invisible. We get the Coefficient Matrix:

$$\begin{bmatrix} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{bmatrix}$$

For the augmented matrix, we write the columns from the coefficient matrix, then write a vertical line, and add a new column containing the right hand side values of the equations. In this case, the Augmented Matrix is:

$$\left[\begin{array}{rrr|r} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 2 & 0 & -4 & 6 \end{array}\right]$$

Notice: The augmented matrix for the SLE contains all of the numbers which appear in the system; we're simply omitting all the non-numeric objects (i.e. variables and equal signs).

Example 6.2. Write the augmented matrix for the system of linear equations: $\begin{array}{rcl} x_1 & = & 1 + x_2 - 3x_3 \\ x_2 & = & 2 - x_1 + x_4 \\ x_3 & = & x_2 + x_4 - x_1 \end{array}$.

Solution:
Remember: Our definition of the coefficient matrix and the augmented matrix require that the system be in standard form. So the first thing we have to do is rearrange each equation to get the standard form SLE. We get:

$$\begin{array}{rcrcrcrcr} x_1 & - & x_2 & + & 3x_3 & & & = & 1 \\ x_1 & + & x_2 & & & - & x_4 & = & 2 \\ x_1 & - & x_2 & + & x_3 & - & x_4 & = & 0 \end{array}$$

Now, we just have to leave out all the x's, $+$'s, $-$'s and $=$'s, while filling in the invisible 0's and ± 1's. That is, we write the coefficients as we see (and don't see) them, and write the RHS's, with a vertical line separating the column of RHS values from the coefficients. We get the augmented matrix:

$$\left[\begin{array}{rrrr|r} 1 & -1 & 3 & 0 & 1 \\ 1 & 1 & 0 & -1 & 2 \\ 1 & -1 & 1 & -1 & 0 \end{array}\right]$$

Example 6.3. If the augmented matrix for a particular system of linear equations is $\left[\begin{array}{rr|r} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right]$, write the SLE.

Solution:
The augmented matrix tells us everything we need to know about the SLE *except* what the variables are called. Then again, it doesn't matter much what names we use for the variables. We can tell from the number of columns in the coefficient matrix part of the augmented matrix that there are

2 variables in the SLE, so let's use x and y. And since there are 2 rows in the augmented matrix, the SLE must have 2 equations. We just have to attach the coefficients to the variables for each equation, replace the vertical line with equal signs, and use the RHS values from the extra column. We get:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \text{ corresponds to the SLE } \begin{array}{rcrcr} x & + & 2y & = & 3 \\ 4x & + & 5y & = & 6 \end{array}$$

Looking at these examples we see that it is easy to translate a standard form system of linear equations to its augmented matrix, and to translate from the augmented matrix back to the system. We are going to use these augmented matrices to transform from one SLE to an equivalent SLE, using operations corresponding to the elementary operations we have already learnt, until we get to an augmented matrix which corresponds to a SLE in which it is easy to find the solution(s) or to recognize that the system is inconsistent. We will learn new operations to perform on the augmented matrices, as well as a procedure for deciding which operations to apply, in what order. But before we do that, there is another very important consideration ... How do we know when to *stop* performing these operations? That is, how do we recognize that we have obtained "an augmented matrix which corresponds to a SLE in which it is easy to find the solution(s) or to recognize that the system is inconsistent"? We look for something called *row-reduced echelon form*.

Definition: A matrix A is said to be in **row-reduced echelon form**, abbreviated as **RREF**, if each of the following four conditions is met:

(i) Every row of A which contains at least one non-zero entry has a 1 as its first (from left to right) non-zero entry. By convention, we refer to this 1 as the *leading 1* of the row in which it occurs.

(ii) Each column of A which contains a leading 1 for some row contains no other non-zero entries (i.e. all other entries are 0's).

(iii) In any two rows of A which each contain some non-zero entries, the leading 1 from the lower row must occur farther to the right than the leading 1 from the upper row. That is, the leading ones in the matrix move from left to right as we read down the matrix.

(iv) All rows of A which consist entirely of zeros are placed at the "bottom" of the matrix, i.e. are lower in the matrix than all rows which contain some non-zero entries.

For instance, the matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

satisfies each condition above. That is, looking at row 1, we see that the first entry is a 1 — the leading 1 for row 1. And looking at the rest of column 1 (the column which contains row 1's leading 1), we see that the other entries in this column are all 0's. That is, this column contains no other non-zero entries. Now, looking at row 2, the first non-zero entry is in the second column, and it is a 1, so this is row 2's leading 1. Again, looking at the rest of this column, there are no other non-zero entries. Similarly, in row 3 we see that the first two entries are 0's, so the "leading" entry is the third, and as before it is a 1. And the rest of the column which contains this 1 is all 0's. That is, the column which contains the leading 1 for row 3 contains no other non-zero entries. So conditions (i) and (ii) of the requirements of RREF are satisfied. Now, looking at the bigger picture, we do see the leading 1's moving from left to right as we read down the matrix. That is, row 1 has the left-most leading 1, and in the rows below that, each leading 1 is further to the right than the leading 1 in the preceding row. Therefore condition (iii) is satisfied. Finally, looking at the matrix we see that *there are no rows which consist entirely of zeros*, which means that where such rows should come in the matrix is irrelevant. We only need to be concerned with condition (iv) when there are some rows like that. If there aren't any, then the condition is not violated, so it is satisfied. Therefore all of the conditions required for row-reduced echelon form are satisfied by this matrix, so the matrix *is* in RREF. Notice that column 4 does not contain any row's leading 1, so the requirements of RREF say nothing about what the column 4 entries may or may not be.

Example 6.4. Which of the following matrices are in RREF?

$$(a)\left[\begin{array}{cccc|c} 1 & 0 & 2 & 2 & 3 \\ 0 & 1 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad (b)\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \qquad (c)\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right] \qquad (d)\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

Solution:
(a) We have the matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 2 & 3 \\ 0 & 1 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Looking at this matrix, we see that in each row that has any non-zero entries, the first non-zero entry is a 1. That is, row 1 has a leading 1 (in column 1) and row 2 also has a leading 1 (in column 2). Row 3 doesn't have any non-zero entries, so it doesn't need (and cannot have) a leading 1. Therefore condition (i) of the requirements for RREF is satisfied. Also, looking at columns 1 and 2, which are the only columns which contain any row's leading 1, we see that all of the other entries are 0. That is, the column which contains the leading 1 for row 1 does not contain any other non-zero entries, and the same is true for the column which contains the leading 1 for row 2. Since these are the only columns which contain leading 1's for some row, these are the only columns to which condition (ii) of the requirements for RREF pertain. Therefore this condition is satisfied. And we see that row 2's leading 1 is further to the right than row 1's, above it, so condition (iii) of the requirements for RREF is satisfied, too. Finally, row 3 contains only 0's and is at the bottom of the matrix, further down than all the rows which contain some non-zero entries. Thus condition (iv) of the requirements for RREF is also satisfied. Since all of the conditions are satisfied, this matrix does fulfill the requirements and therefore *is* in row-reduced echelon form.

(b) Now we look at the matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Here we see that each row does contain a leading 1, so condition (i) is satisfied. But this matrix fails condition (ii), because column 2 contains the leading 1 for row 2, but also contains a non-zero entry in row 1. In order to satisfy condition (ii) there would have to be a 0 in the $(1, 2)$-entry of the matrix, since the leading 1 for some row is in that column. Therefore this matrix is *not* in RREF.

(c) Next we look at the matrix

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right]$$

In this case, each row does contain a leading 1, and each column that contains the leading 1 for some row has no other non-zero entries, so both condition (i) and condition (ii) are satisfied. However, this matrix does not satisfy condition (iii) because the leading 1 for row 3 is not strictly to the right of the leading 1 for row 2. That is, the leading 1's do not move from left to right as we read down the matrix. The row whose leading 1 is in column 3 would have to be lower down in the matrix than the row whose leading 1 is in column 2 in order for codition (iii) to be satisfied. (Unless, of course, there wasn't any row whose leading 1 was in column 2. But that's not the case here.) Therefore this matrix also is *not* in RREF.

(d) Finally, we look at the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This one fails condition (i). The first non-zero entry in row 2 is not a 1. That is, row 2 does have some non-zero entries, but it does not have a leading 1. Therefore condition (i) is violated, so the matrix is *not* in RREF.

Suppose that we have a system of linear equations, and that we also have a RREF augmented matrix that we know corresponds to a system which is equivalent to that SLE. How do we use it to find the solution(s) to the SLE? We simply use the RREF matrix to write the corresponding SLE so that we can see the solution(s) to that SLE, and since this SLE is equivalent to the original SLE, we have also found the solution(s) to the original SLE. For instance, suppose that we have the system

$$\begin{array}{ccccc} x & + & y & = & 3 \\ x & - & y & = & 1 \end{array}$$

for which the augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & -1 & 1 \end{array}\right]$$

And suppose that (by some means we haven't yet learnt) it has been determined that the RREF matrix

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$$

corresponds to a SLE which is equivalent to the original system. Then we write the SLE corresponding to this RREF augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right] \quad \text{corresponds to} \quad \begin{array}{ccccc} 1x & + & 0y & = & 2 \\ 0x & + & 1y & = & 1 \end{array} \quad \text{i.e. to} \quad \begin{array}{cccc} x & & = & 2 \\ & y & = & 1 \end{array}$$

so, because we know that this system is equivalent to, and thus has the same solutions as, the original system, we see that $(x, y) = (2, 1)$ is the only solution to $\begin{array}{ccccc} x & + & y & = & 3 \\ x & - & y & = & 1 \end{array}$.

Example 6.5. For each of the following, find all solutions to a system of linear equations which is equivalent to the SLE corresponding to the given RREF augmented matrix, where the variables are as stated.

(a) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right]$ with variables x, y and z.

(b) $\left[\begin{array}{cccc|c} 1 & 0 & 2 & 2 & 3 \\ 0 & 1 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ with variables x_1, x_2, x_3 and x_4.

(c) $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 5 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$ with variables w, x, y and z.

Solution:
For each of the given RREF matrices, we write the corresponding SLE, using the variables given, and state all solutions to that SLE.

(a) We can easily confirm that this matrix is in RREF. We see that

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right] \quad \text{corresponds to} \quad \begin{array}{ccccc} x & & & = & 1 \\ & y & & = & 2 \\ & & z & = & 0 \end{array}$$

so the only solution to any SLE which is equivalent to this SLE is $(x, y, z) = (1, 2, 0)$.

(b) Notice that this is the augmented matrix which we have already confirmed, in Example 6.4(a) *is* in RREF. This time we have

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 2 & 3 \\ 0 & 1 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \text{which corresponds to} \quad \begin{array}{ccccccccc} x_1 & & & + & 2x_3 & + & 2x_4 & = & 3 \\ & & x_2 & + & 3x_3 & + & x_4 & = & 2 \\ 0x_1 & + & 0x_2 & + & 0x_3 & + & 0x_4 & = & 0 \end{array}$$

We see that the first equation is telling us about the value of x_1 (relative to x_3 and x_4) and the second equation is telling us about the value of x_2 (relative to x_3 and x_4). The third equation simply states that $0 = 0$, which tells us nothing (except that there isn't a problem). That is, the original system must have had 3 equations, but only the first 2 ended up giving us useful information, and the third was consistent with those 2, but contained no new information. There is no equation telling us about the value of x_3, so it must be free to have any real value. Likewise, with no equation telling us about the value of x_4, this variable is also free to have any real value. Since *each* of these variables could have any value, we use a *different* parameter for each. That is, we can have $x_3 = s$ for any $s \in \Re$ and also $x_4 = t$ for any $t \in \Re$. Using these values in the first and second equations, and rearranging to state the corresponding values of x_1 and x_2 we get

$$\begin{array}{ccccccccc} x_1 & & + & 2s & + & 2t & = & 3 \\ & x_2 & + & 3s & + & t & = & 2 \end{array} \quad \begin{array}{c} \Rightarrow \\ \Rightarrow \end{array} \quad \begin{array}{ccccccc} x_1 & = & 3 & - & 2s & - & 2t \\ x_2 & = & 2 & - & 3s & - & t \end{array}$$

Therefore any SLE which is equivalent to this SLE has the *two-parameter family of solutions* $(x_1, x_2, x_3, x_4) = (3 - 2s - 2t, 2 - 3s - t, s, t)$.

(c) Checking for leading 1's, and that their columns are otherwise empty, and seeing that the leading 1's move from left to right as we read down the matrix (and observing that there is no row which contains only 0's), we see that this matrix is, as stated, in RREF. We write the corresponding SLE:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 5 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \quad \text{corresponds to} \quad \begin{array}{ccccccccc} w & & & & & + & 2z & = & 5 \\ & & x & & & & & = & 3 \\ & & & & y & - & 3z & = & 2 \\ 0w & + & 0x & + & 0y & + & 0z & = & 1 \end{array}$$

In this system, it doesn't matter what the first 3 equations say, because the fourth equation says that $0 = 1$. The existence of 4 equations tells us that the original SLE, whatever it was, had 4 equations. But the fourth equation has been transformed into an equation which is nonsense, i.e. cannot be true, no matter what the values of w, x, y and z are. So the SLE corresponding to the RREF augmented matrix has no solution, and therefore any SLE equivalent to this SLE must be inconsistent.

At this point, we have learnt how to write the augmented matrix for any SLE in standard form, and how to recognize a matrix that is in RREF, as well as how to find the solution(s), if any, to the SLE corresponding to an augmented matrix which is in RREF. But we don't know yet how to transform the augmented matrix for a SLE into a matrix in RREF for an equivalent system. That's what we learn next.

Manipulating the augmented matrix for a system of linear equations corresponds to, i.e. is equivalent to, manipulating the equations in the system. In the previous unit, we learnt how to manipulate equations using the three elementary operations, in order to solve a SLE. What we need to learn now is how to perform operations on an augmented matrix which correspond to those elementary operations we perform on equations. But what we are going to learn applies more broadly than just to augmented matrices corresponding to SLE's. The operations we are going to learn can be applied to any kind of matrix.

In an augmented matrix representing a SLE, each row of the matrix corresponds to a different equation in the system. Therefore the kind of operations we perform on equations in a system are performed on *rows* of a matrix. There are 3 elementary operations which we are allowed to perform on equations, and so there are three corresponding *elementary row operations* which we can perform on a matrix.

Definition: The following operations are the **elementary row operations** (abbreviated **ero's**) which can be performed to transform a matrix into RREF:

I Multiply any row of the matrix by any non-zero scalar.

II Interchange the positions of any two rows in the matrix.

III Replace any row in the matrix by the sum of that row and a scalar multiple of any other row of the matrix.

No other operations are allowed.

Also, we say that two matrices are **row equivalent** if one matrix can be transformed into the other by applying a sequence of elementary row operations.

Notice: When we 'multiply a row by a scalar', or 'sum one row and a scalar multiple of another row', we perform these arithmetic operations by treating a row of a matrix like a vector, using the vector operations we have learnt previously. That is, we multiply a row of a matrix by a scalar by multiplying each entry in the row by that scalar. Likewise, adding 2 rows of a matrix means summing corresponding entries (components) in the rows.

Also Notice: Compare the definitions of the elementary row operations, stated here, to the definition of the elementary operations on equations used to transform an SLE into an equivalent SLE, stated on page 64. You will see that they correspond exactly, and so when we perform these elementary row operations on an augmented matrix, it is just like performing the corresponding elementary operations on the equations in the SLE represented by that augmented matrix. And therefore we have a theorem for augmented matrices similar to Theorem 5.1(see page 64) which told us that when we perform elementary operations to transform a SLE, the new SLE is always equivalent to the one we started with.

Definition: The process of applying elementary row operations to transform a matrix into RREF is called **row-reduction**, or simply **reduction**. We can refer to this as **row-reducing** or **reducing** the matrix.

Theorem 6.1. *Applying any sequence of elementary row operations to the augmented matrix for a SLE produces a new augmented matrix whose corresponding SLE has exactly the same solution(s) as (i.e. is equivalent to) the original SLE.*

Using the definition of *row equivalent*, another way to state this theorem is:

> *If 2 augmented matrices are row equivalent, then their corresponding SLE's have the same solution(s).*

Before we think about applying elementary row operations to an augmented matrix, to find the solution(s) to a SLE, let's look at some examples of row equivalent matrices, and practice using these ero's more generally to row-reduce any matrix (to RREF). For instance, the matrices

$$\begin{bmatrix} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -2 & 8 & -6 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{bmatrix}$$

are row equivalent since multiplying row 1 of the first matrix by -2 (a type I ero) transforms the first matrix into the second matrix.

Similarly, the matrices

$$\begin{bmatrix} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & 3 & -1 \\ 1 & -4 & 3 \\ 2 & 0 & -4 \end{bmatrix}$$

are row equivalent because the second matrix can be obtained from the first by interchanging rows 1 and 2 (a type II ero).

Also, the matrices

$$\begin{bmatrix} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & 2 & 1 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{bmatrix}$$

are row equivalent since performing the type III ero *add twice the second row to the first row* in the first matrix transforms it to the second matrix.

There is shorthand which we can use to indicate what ero we are performing. If we transform a matrix by multiplying a row of the matrix by a scalar, i.e. if we replace Row i of the matrix by c times that row, for some non-zero scalar c, we indicate this by writing $R_i \leftarrow cR_i$. (This says calculate c times Row i and put it where Row i was before. That is, Row i is replaced by c times Row i.) Likewise, when we interchange two rows of a matrix, i.e. put Row i where Row j was, and vice versa, we write $R_i \leftrightarrow R_j$. And for a type III ero, in which we add a scalar multiple of another row to a row, i.e. Row i is replaced by Row i plus c times Row j, we write $R_i \leftarrow R_i + cR_j$. So for the three transformations we observed above, we can write

$$\begin{bmatrix} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{bmatrix} \xrightarrow{R_1 \leftarrow (-2)R_1} \begin{bmatrix} -2 & 8 & -6 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} -1 & 3 & -1 \\ 1 & -4 & 3 \\ 2 & 0 & -4 \end{bmatrix}$$

$$\text{and} \quad \begin{bmatrix} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 + 2R_2} \begin{bmatrix} -1 & 2 & 1 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{bmatrix}$$

The long arrow says "transform the matrix to a row equivalent matrix by performing the ero indicated".

Example 6.6. For each of the matrices in Example 6.4 which is not already in RREF, perform elementary row operations to transform the matrix into a row equivalent matrix which is in RREF.

Solution:

(a) As we observed in Example 6.4(a), the matrix $\left[\begin{array}{cccc|c} 1 & 0 & 2 & 2 & 3 \\ 0 & 1 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ is already in RREF.

(b) In Example 6.4(b), we observed that the augmented matrix $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$ is not in RREF because column 2, which contains the leading 1 for row 2, has a non-zero entry in row 1. That is, the problem here is that the $(1,2)$-entry of the matrix must be 0 in order to have RREF. Notice that row 2's leading 1 in column 2 means that we can eliminate the non-zero entry in this column, in row 1, by adding the negative of the existing non-zero $(1,2)$-entry times row 2 to row 1. That is, we can use a type III ero in which we add a scalar multiple of row 2 to row 1, to get rid of the non-zero $(1,2)$-entry. And the scalar we need is just the negative of the entry which we are trying to transform into 0. Also notice that row 2 has a 0 in column 1, so adding a multiple of row 2 to row 1 will not have any effect on the $(1,1)$-entry, which means that after we do this, row 1 will still have a leading 1. (It's important that when we perform ero's we don't "mess up" the things we already have which *do* comply with RREF, so that we don't create more work for ourselves.) When we perform the ero $R_1 \leftarrow R_1 + (-1)R_2$, i.e. $R_1 \leftarrow R_1 - R_2$, we get the new version of Row 1 by performing this arithmetic on the *vectors* which look like those rows of the matrix. (*Note:* It doesn't matter that the matrix happens to be an augmented matrix. We just ignore the | when we do the arithmetic.) So the new row 1 corresponds to

$$\text{Row 1} - \text{Row 2} = (1,1,0,0) - (0,1,0,0) = (1,0,0,0)$$

Therefore transform the matrix by

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 + (-1)R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

We see that the new matrix, which is row equivalent to the old one, is in RREF, so we're done.

(c) In Example 6.4(c), we found that the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is not in RREF because although each row does have a leading 1, and the columns containing these leading 1's each have no other non-zero entries, the leading 1's do not follow the required pattern of moving from left to right as we read down the matrix. The leading 1 in row 3 is not farther to the right than the leading 1 in row 2. We can easily fix this by simply switching the positions of those two rows. That is, we need to perform a type II ero to interchange the positions of Rows 2 and 3. We get:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We see that the transformed matrix is in RREF, so we're done.

(d) Finally, in part (d) of Example 6.4, we observed that the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is not in RREF because row 2 does not have a leading 1. The first non-zero entry in row 2 is a 2, not a 1. We can always transform a "leading c", for some value c which is neither 0 nor 1, into a leading 1 by multiplying it (i.e. by multiplying the row) by $\frac{1}{c}$. So all we need to do here is to perform the type I ero *multiply Row 2 by the non-zero scalar* $\frac{1}{2}$. We get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftarrow \left(\frac{1}{2}\right) R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We see that the transformed matrix is now in RREF.

In this example, for each of the matrices which was not already in row-reduced echelon form, we were able to, quite easily, transform the matrix into RREF using elementary row operations. But those matrices were very close to RREF to start with. In each case, we needed only a single ero to get to RREF. Usually, several or perhaps even many ero's will be required to obtain RREF. One natural question which may occur to you is *Is this always possible?*, that is *Is it* **always** *possible to transform a matrix into RREF using ero's?*. And when more than one ero is needed, assuming that it *is* possible to obtain RREF, you might also wonder *Does it matter* **which** *ero's we do, and does the* **order** *we do them in matter?*. Well, clearly it does matter *which* ero's we do, because some will take us to RREF and others won't. But still, there will often be choices about which problem to tackle next, or which type of ero to use to accomplish a particular goal in moving towards RREF. So the question is, *Does it matter what choices we make?*. The next theorem addresses these questions.

Theorem 6.2. Facts about Row-Reduced Echelon Form

(a) Every matrix can be transformed into RREF by applying a finite sequence of elementary row operations.

(b) The row-reduced echelon form of a matrix is unique. That is, if the same matrix is reduced by two different sequences of ero's, the RREF obtained in both cases will be identical.

Although part (a) of Theorem 6.2 tells us that it is *always possible* to transform a matrix into RREF by a finite sequence of ERO's, and part (b) tells us that any sequence of ERO's that results in RREF will produce *the same* RREF, it is still possible to perform many ERO's on a matrix without ever getting any closer to having the matrix in RREF. So it is **very important** to approach the reduction of a matrix in a systematic way. We want to follow a procedure which ensures that each ero performed is moving the matrix closer to being in RREF, so that we arrive at RREF quickly.

In order for a matrix to be in row-reduced echelon form, we need each row, if it has any non-zero entries in it, to have a leading 1. Also, we need the column which has a row's leading 1 in it to contain 0's in all the other rows. When we want to transform a matrix to RREF, we tackle the rows of the matrix from the top down, and deal with those 2 requirements, in order, for the row we're currently dealing with. That is, starting with the top row, we make sure that we have a leading 1 in that row, and then make the rest of the entries in the column that contains that leading 1 be 0's. Once we've accomplished that, we move on to the next row. As we go along, we move any rows which contain only 0's down to the bottom of the matrix, and also move rows around if necessary to ensure that the leading 1 we're currently getting, or working with, is the left-most of the entries which don't yet conform.

And we want to be sure that we never "undo" any of the work done in previous steps, which can happen if we're not careful. For instance, if we've obtained a leading 1 in a particular row, we don't want that 1 to be transformed into some other number later on. A leading 1 should always stay a leading 1 as we continue to reduce the matrix. Likewise, if we've "zeroed out" a column, so that the leading 1 it contains is the only non-zero entry in that column, we want to be sure that we don't re-introduce any non-zero entries into that column later on in the reduction process.

With these considerations in mind, we use the procedure shown below whenever we need to row-reduce a matrix. It is **strongly recommended** that you become familiar with, and comfortable with using, this procedure/algorithm, in order to save time and work in reaching RREF. While it may, on rare occasions, be true that the reduction could have been completed slightly more quickly by some other sequence of operations, far more often this procedure gives a highly efficient route to obtaining the row-reduced echelon form of the matrix.

Procedure for Transforming a Matrix into RREF:

1. Always work from the top down.
2. Always work from left to right.
   So if there is a non-zero entry lower down in the matrix that is farther to the left than the left-most non-zero entry in the highest row (among rows that do not yet have a leading 1), interchange rows so that you can simultaneously be working with the highest, and the left-most, such entry. For instance, for $\begin{bmatrix} 0 & 0 & 3 & 4 \\ -1 & 1 & 2 & 1 \end{bmatrix}$ we would start with $R_1 \leftrightarrow R_2$.
3. In the highest up row which has not yet been "dealt with", obtain a leading 1. That is, if the first non-zero entry in the row is not a 1, perform one or more ero's to transform that non-zero entry into a 1. (See Note 1.)
4. As soon as a leading 1 has been obtained, use it to obtain 0's everywhere else in the column which contains that leading 1. (See Note 2.)
5. As soon as a row containing only 0's is obtained, move it to the bottom of the matrix.
6. Continue in this manner until the matrix is in RREF.

Notes:

1. It is always possible to obtain a leading 1 in a particular row (which has at least one non-zero entry) using a type I ero, by multiplying that row by $\frac{1}{c}$, where c is the number which is to be transformed into a 1. However, if there is another row further down in the matrix which already has a 1 in the column in which the leading 1 is to be obtained, use a type II ero and interchange those rows. If no other row has a 1 in this column, but there is a row further down whose entry in this column will lead to less complicated fractions, interchange the rows, to simplify the arithmetic. For instance, if the top-most row which does not have a leading 1 in it has a 12 as its left-most non-zero entry, and further down in the matrix some row has a 2 in that column, the arithmetic is usually easier if you interchange the rows and multiply by $\frac{1}{2}$ to get a leading 1 than if you introduce a leading 1 by multiplying the current row by $\frac{1}{12}$. So for $\begin{bmatrix} 12 & 10 & 3 & 7 \\ 2 & -1 & 5 & 2 \end{bmatrix}$, it's easier to use $R_1 \leftrightarrow R_2$ to get $\begin{bmatrix} 2 & -1 & 5 & 2 \\ 12 & 10 & 3 & 7 \end{bmatrix}$ and then use $R_1 \leftarrow \frac{1}{2}R_1$ than to apply $R_1 \leftarrow \frac{1}{12}R_1$ to the original matrix, since halves are arithmetically easier to work with than twelfths.
2. Once a leading 1 has been obtained in a particular row, it is always possible to "zero out" the rest of the column which contains that leading 1 using type III ero's, adding scalar multiples of the row with the (newly obtained) leading 1 to the other rows (those which have non-zero entries in that column). And the particular scalar

needed is $-c$, where c is the entry which is to become 0. For instance, if we have just obtained a leading one in row i, and some row farther down in the matrix, row j, has a 2 in the column containing row i's leading one, perform the type III ero: replace row j by row j plus (-2) times row i. Or for the matrix obtained at the end of Note 1, the next step would be

$$\begin{bmatrix} 1 & -\frac{1}{2} & \frac{5}{2} & 1 \\ 12 & 10 & 3 & 7 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 + (-12)R_1} \begin{bmatrix} 1 & -\frac{1}{2} & \frac{5}{2} & 1 \\ 0 & 16 & -27 & -5 \end{bmatrix}$$

3. Because we always "zero out" the column containing one leading 1 before moving on to obtain the next leading 1, then when we perform any subsequent type III ero's, it is always 0 which is being added to a previously-obtained leading 1, leaving it as a 1. For instance, look at the first ero being done in the line marked (*) in Example 6.7 below (on page 81). Prior to this ero, we have obtained a leading 1 in row 1, and "zeroed out" its column. At this point, having obtained a leading 1 in row 2, we need to perform a type III ero to obtain a 0 in the $(1, 2)$-entry. When we do this, because of the 0 in the first entry of row 2, the leading 1 in row 1 remains 1. (If we had not already obtained a 0 in the $(1, 2)$-entry, this next ero would end up with something other than a 1 as the leading entry in row 1.)
4. Similarly, since we always use *replace this row by this row plus a scalar multiple of the row we're currently working with*, we never multiply "this row" by a constant, and so we never replace a leading 1 with anything other than 1. To see this, look at the (**) line in Example 6.7 below (on page 82). For row 1, we need to eliminate the 3 in column 3, using the 1 in row 3. We *don't* replace row 1 by $\left(-\frac{1}{3}\right) R_1 + R_3$, because row 1's leading 1 would become $-\frac{1}{3}$, and we'd need to start over to obtain a leading 1 in row 1. What we use is $R_1 \leftarrow R_1 + (-3)R_3$. That way, the leading 1 in Row 1 isn't changed.
5. Likewise, when we replace a row using a type III ero, since it is always the row containing the leading 1 whose column we need to "zero out" that we are adding scalar multiples of to other rows, and that row has 0's in all of the columns previously "zeroed out", we never add anything other than 0 to a previously-obtained 0, and so the columns we have previously "zeroed out" remain that way. Look again at what's being done to Row 1 in the (**) line in Example 6.7 – when we do the ero $R_1 \leftarrow R_1 + (-3)R_3$, it's $-3 \times 0 = 0$ that we add to the 0 in Row 1.
6. Although a leading 1 exists in a column, and we work with that column (to "zero it out"), it's important to remember that a leading 1 *belongs to* a particular row, not a column. That is, it is rows which must have leading 1's, not columns. If some column doesn't contain any row's leading 1, but does have non-zero entries, that doesn't matter.
7. Because this procedure moves us toward RREF in a systematic fashion, with each ero taking us closer to RREF and carefully not undoing the work already accomplished by previous ero's, using this procedure will never "take you around in circles", which is definitely a danger when row-reduction is approached less systematically.

Let's look at a couple of examples of using this procedure.

Example 6.7. Find the RREF matrix which is row equivalent to the matrix $\begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 0 & -3 \end{bmatrix}$.

Solution:
Since there *are* some non-zero entries in column 1, i.e. there *are* rows which have a non-zero first

entry, then we will certainly have a row whose leading 1 is in the first column. And that will have to be row 1, since any leading 1 in a row further down would have to be farther to the right than row 1's leading 1. So we will want the $(1,1)$-entry of the matrix to be the leading 1 for row 1. We currently have a 0 in the $(1,1)$-entry. The easiest way to get a 1 there is to put a row which already has a 1 as its first entry into the top position. Since both rows 2 and 3 have 1's as their first entries, either would do. Let's choose row 2. Then our first ero is the type II ero: interchange the positions of rows 1 and 2. So we have:

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 0 & -3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & -3 \end{bmatrix}$$

Now we have a leading 1 in row 1. The next thing we want to do is eliminate the other non-zero entries in the column containing that leading 1, i.e. column 1. Conveniently, row 2 already has a 0 in column 1, so the only non-zero we need to eliminate is the first entry in row 3. It is currently a 1, and we need it to be a 0. We can accomplish that by subtracting the 1 that's in the same position in row 1. That is, we can obtain a 0 in the $(3,1)$-entry of the matrix by subtracting row 1 from row 3. Therefore our next step is to perform the type III ero: replace row 3 by row 3 plus (-1) times row 1. We get:

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & -3 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 + (-1)R_1} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & -5 \end{bmatrix}$$

Now that we have a leading 1 in row 1 and its column contains no other non-zero entries, we turn our attention to row 2. We need the first non-zero entry in row 2 to be a 1. Aha! It already is! That's convenient. So, no ero needed to accomplish that. We've (already) got our next leading 1. And now we need to transform the matrix so that there are no other non-zero entries in the column which contains row 2's leading 1, i.e. in column 2. Currently both row 1 and row 3 have non-zero entries in column 2, so we need to eliminate both of these. That will require 2 separate ero's, but both will have the form: replace row i by row i plus some multiple of row 2. What are the multiples we need? The ones which will give the value 0 for the $(i,2)$-entry after this operation. For row 1, we want to be adding a 1 to the -1 that's there now, so we can just add row 2. That is, the multiplier is just 1 (the negative of the -1 which we need to transform to 0). And for row 3, we have a 1 there (i.e. in the second column) now, so we need to be subtracting 1. Which means adding (-1) times the row 2 entry. So the multiplier we need this time is -1 (again, the negative of the entry which is to be transformed to 0). We perform these 2 ero's, one after the other:

$$(*) \qquad \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & -5 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 + R_2} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & -5 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 + (-1)R_2} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -6 \end{bmatrix}$$

So we have "dealt with" rows 1 and 2. That leaves row 3. We need the first non-zero entry in row 3 to be a 1, instead of the -6 that's there now. We can obtain this by simply dividing row 3 by -6. That is, we perform the type I ero: multiply row 3 by $-\dfrac{1}{6}$. We get:

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -6 \end{bmatrix} \xrightarrow{R_3 \leftarrow \left(-\frac{1}{6}\right) R_3} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

And now that we've obtained another leading 1, we need to "zero-out" it's column. That is, we need all the other entries in column 3 (which contains row 3's leading 1) to be 0. Currently in column 3 we have a 3 in row 1 and a 1 in row 2. To transform each of these to 0, we will perform two type III ero's adding a scalar multiple of row 3 to each row, separately. In each case, the scalar multiplier we need is just the negative of the number that's already there. That is, -3 is the multiplier for the ero we apply to row 1, and -1 is the multiplier for the ero we apply to row 2.

Notice that when row 1 is replaced by row 1 minus 3 times row 3, neither row 2 nor row 3 is affected by this change. Likewise, when row 2 is replaced by row 2 minus one times row 3, neither row 1 nor row 3 is affected by the change. Whenever we do these "clear out the column which contains the leading 1 we just found" operations, only the row whose entry we're "clearing out" changes. This means that we can actually perform more than one of these operations at the same time. Well, we do them one at a time, but we don't need to re-write the whole matrix with each one. We can just re-write the matrix once, incorporating both of the changes we need, at the same time. Or if there were 3 lines that all needed their non-zero entries to become 0, we could write this as a single step in which all 3 changes appear to take place simultaneously. We still want to say what ero's we're doing, so if there are 2 of them, we write one of them above the arrow and one below. When there are more than 2, we can write two (or even more) on the same line, separated by commas.

So we can write the transformations corresponding to the 2 ero's we need to do to eliminate the non-zeroes in the column containing row 3's leading 1 as:

$$(**) \qquad \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2 \leftarrow R_2 - R_3]{R_1 \leftarrow R_1 - 3R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Checking the properties we need, we see that this matrix is in RREF, so we're done. Since we only performed ero's, each matrix is row equivalent to the one preceeding it, and so this last matrix is row equivalent to the first. That is, the RREF matrix we found is row equivalent to the one we were given.

Notice: You should write down the transformed matrices showing only one change at a time for as long as you need to. If it's too confusing to you to re-write the matrix only once for two or more changes, don't do it until you're more comfortable with it. After some practise, you'll catch on and you'll get sick of re-writing the matrix.

Example 6.8. Use elementary row operations to bring the augmented matrix of the system of linear equations shown below to row-reduced echelon form, and state all solutions to the SLE.

$$\begin{array}{rcrcrcr} 2x & + & 2y & + & z & = & 4 \\ 3x & + & y & - & 2z & = & 5 \\ x & & & - & 3z & = & -2 \end{array}$$

Solution:
First, we need to form the augmented matrix for the system. Remember, we omit the variables and equal signs (and insert the invisible zeroes and ones) to get an augmented matrix which looks very much like the system, but only contains the numbers (with any needed negative signs). The RHS values are offset by a vertical line. We get:

$$\left[\begin{array}{rrr|r} 2 & 2 & 1 & 4 \\ 3 & 1 & -2 & 5 \\ 1 & 0 & -3 & -2 \end{array}\right]$$

We start at the top and as far left as possible. Row 1's left-most entry *is* non-zero, so we will want it to be a leading 1. Currently it's a 2. We could multiply row 1 by $\frac{1}{2}$ to change the 2 into a 1, but it's easier to notice that there's another row which already has a 1 as the first entry. We can just make that row be row 1, by switching the positions of that row (row 3) and row 1:

$$\left[\begin{array}{rrr|r} 2 & 2 & 1 & 4 \\ 3 & 1 & -2 & 5 \\ 1 & 0 & -3 & -2 \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_1} \left[\begin{array}{rrr|r} 1 & 0 & -3 & -2 \\ 3 & 1 & -2 & 5 \\ 2 & 2 & 1 & 4 \end{array}\right]$$

Having just obtained a leading 1, the next thing we do is to eliminate all other non-zeroes in the column that leading 1 is in, i.e. column 1. We can do this by subtracting 3 times row 1 from row 2 and subtracting 2 times row 1 from row 3. That is, for each of the other rows we replace that row by itself plus the negative of the current column 1 entry times row 1. We get:

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 3 & 1 & -2 & 5 \\ 2 & 2 & 1 & 4 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - 2R_1]{R_2 \leftarrow R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 2 & 7 & 8 \end{array}\right]$$

Now that we've "dealt with" row 1, we turn our attention to row 2. We see that row 2 already has a 1 as its first non-zero entry, and that this 1 is immediately to the right of row 1's leading 1, so this is our leading 1 for row 2. (That is, fortuitously we don't have to do any work to obtain a leading 1 in row 2, since it already has a 1 in the appropriate position.) Since row 2's leading 1 is in column 2, we next need to eliminate all other non-zero entries from column 2. We do this using the same kind of ero's as when we were doing this for column 1, except this time it is row 2 which a multiple of will be added to each other row as necessary. We see that row 1 already has a 0 in column 2, so we don't need to do anything to row 1. It is only row 3 which has a column 2 entry which must be transformed into a 0. Since this entry is a 2, we add -2 times row 2 to row 3:

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 2 & 7 & 8 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & -7 & -14 \end{array}\right]$$

Row 2 has now been "dealt with" so we move on to row 3. This time, we don't already have a leading 1, so we have to get one. We need to transform the -7 which is currently row 3's first non-zero entry into a 1. We do that with a type I ero: multiply row 3 by $\dfrac{1}{-7}$:

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & -7 & -14 \end{array}\right] \xrightarrow{R_3 \leftarrow \left(-\frac{1}{7}\right) R_3} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

And of course now that we have a new leading 1 we must eliminate all other non-zero entries from the column containing that leading 1, i.e. column 3. Again, we use type III ero's, this time with multiples of row 3 being added to the other rows. We need to eliminate a -3 in row 1, so to that row we add $-(-3)$, i.e. 3, times row 3. Likewise, to eliminate the 7 in row 2, we add -7 times row 3. We get:

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow[R_2 \leftarrow R_2 - 7R_3]{R_1 \leftarrow R_1 + 3R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

This final matrix is in RREF. (Since column 4 (the RHS column) does not contain the leading 1 for any row, it doesn't matter what its entries are for purposes of RREF.) This RREF augmented matrix corresponds to the linear system:

$$\begin{array}{cccc} x & & & = 4 \\ & y & & = -3 \\ & & z & = 2 \end{array}$$

Because we only performed elementary row operations in transforming the matrix, the final augmented matrix is row equivalent to the original augmented matrix and therefore this new SLE has the same solution(s) as the original system. We see that the only solution is $(x, y, z) = (4, -3, 2)$.

Next, we want to formalize what we did in this example into a systematic approach for solving systems of linear equations. It's quite straightforward. We simply do exactly what we did in the example: Write the augmented matrix for the given SLE, transform it to RREF, and use that RREF matrix to find the solutions to the underlying transformed SLE, which is equivalent to the SLE we started with. This procedure is called *Gauss-Jordan Elimination.*

Define: The method of **Gauss-Jordan Elimination**
Given any system of linear equations:

1. form the augmented matrix for the SLE;
2. use elementary row operations to transform the augmented matrix to one in which the coefficient matrix part is in RREF;
3. determine the set of all solutions to the system corresponding to this final augmented matrix.

The final SLE is equivalent to the original SLE, so these are also the solutions to the original system.

Note: When we row-reduce an augmented matrix, it is never necessary to bring the whole matrix to RREF, just the coefficient matrix part. That is, if there is a row whose leading 1 would be in the RHS-value column, there is actually no need to convert that entry into a 1. (We'll see why a bit later, when we do an example in which there is such a row.)

Example 6.9. Solve the following system:

$$\begin{array}{rcrcrcl} 3x & + & 3y & + & 12z & = & 6 \\ x & + & y & + & 4z & = & 2 \\ 2x & + & 5y & + & 20z & = & 10 \\ -x & + & 2y & + & 8z & = & 4 \end{array}$$

Solution:
We write the augmented matrix for the given system, and reduce it. Hopefully by this time you are able to understand why we do the ero's specified here, so the explanation will be omitted. If you have trouble seeing what the reasoning was, look again at the procedure we described for transforming a matrix to RREF. We get:

$$\left[\begin{array}{rrr|r} 3 & 3 & 12 & 6 \\ 1 & 1 & 4 & 2 \\ 2 & 5 & 20 & 10 \\ -1 & 2 & 8 & 4 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{rrr|r} 1 & 1 & 4 & 2 \\ 3 & 3 & 12 & 6 \\ 2 & 5 & 20 & 10 \\ -1 & 2 & 8 & 4 \end{array}\right]$$

$$\xrightarrow[(R_3 \leftarrow R_3+(-2)R_1),(R_4 \leftarrow R_4+R_1)]{R_2 \leftarrow R_2+(-3)R_1} \left[\begin{array}{rrr|r} 1 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 12 & 6 \\ 0 & 3 & 12 & 6 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_4} \left[\begin{array}{rrr|r} 1 & 1 & 4 & 2 \\ 0 & 3 & 12 & 6 \\ 0 & 3 & 12 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow \left(\frac{1}{3}\right) R_2} \left[\begin{array}{rrr|r} 1 & 1 & 4 & 2 \\ 0 & 1 & 4 & 2 \\ 0 & 3 & 12 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_3 \rightarrow R_3+(-3)R_2]{R_1 \rightarrow R_1+(-1)R_2} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This final matrix is in RREF and has as its underlying SLE:

$$\begin{array}{rcrcrcl} x & & & & & = & 0 \\ & & y & + & 4z & = & 2 \end{array}$$

$$\text{which gives} \qquad \begin{array}{rcl} x & = & 0 \\ y & = & 2-4z \end{array}$$

The first equation tells us about the value of x, and the second equation tells us about y, because the leading 1 for row 1 is in the first column (the x column) and the leading 1 in row 2 is in column

2 (the column corresponding to y). The last two rows of the RREF matrix just said that $0 = 0$, so we didn't bother writing those as part of the SLE. We do not have an equation telling us about the value of z, so z may have any value. We set z equal to a parameter, t, to indicate this, and substitute t for z everywhere else. We get

$$x = 0, y = 2 - 4t \text{ and } z = t,$$

where t may have any real value, so the solutions to the system are described by:

$$(x, y, z) = (0, 2 - 4t, t),\ t \in \Re.$$

That is, for any real value of t, $(x, y, z) = (0, 2 - 4t, t)$ gives a solution to the SLE.

Note: It is always a good idea to check that the solution(s) obtained *does* satisfy all of the equations in the original SLE. As we saw in the previous unit, we can do this by substituting the unique solution, or the parametric solution, into the original equations and verifying that left side equals right side for each. (This is left to the reader here.) If the supposed "solution" obtained does *not* satisfy the equations in the original SLE, this indicates that an arithmetic error has been made at some point in the calculations (or perhaps that some operation *other than* an ERO was performed at some point, so that the final matrix is not row equivalent to the original matrix).

Example 6.10. Solve the following system:

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 2 \\ & & 2y & - & z & = & 4 \\ x & + & 3y & & & = & 5 \end{array}$$

Solution:
We have:

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 2 \\ 0 & 2 & -1 & 4 \\ 1 & 3 & 0 & 5 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - R_1} \left[\begin{array}{rrr|r} 1 & 1 & 1 & 2 \\ 0 & 2 & -1 & 4 \\ 0 & 2 & -1 & 3 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow (\frac{1}{2}) R_2} \left[\begin{array}{rrr|r} 1 & 1 & 1 & 2 \\ 0 & 1 & -\frac{1}{2} & 2 \\ 0 & 2 & -1 & 3 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - 2R_2]{R_1 \leftarrow R_1 - R_2} \left[\begin{array}{rrr|r} 1 & 0 & \frac{3}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 2 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

The coefficient matrix is now in RREF, so we stop. The SLE corresponding to this last augmented matrix is:

$$\begin{array}{rcrcrcr} x & & & + & \frac{3z}{2} & = & 0 \\ & & y & - & \frac{z}{2} & = & 2 \\ 0x & + & 0y & + & 0z & = & -1 \end{array}$$

The last equation says that $0 = -1$, which of course is nonsense. As we have seen before, this sort of thing is telling us that the SLE is inconsistent, so our conclusion is that the given SLE has no solution.

Notice what the final augmented matrix looked like in this example. The last row of the matrix has only 0's in the coefficient matrix part, and a non-zero value in the RHS column. There are 3 things to note about this.

First of all, it doesn't matter *what* non-zero value is in the RHS column for a row in which the coefficient matrix part contains only zeroes. As long as the coefficient matrix part is all 0's and the RHS column isn't, the corresponding equation tells us that the system is inconsistent. And this is the only situation in which we would, if we insisted on putting the *whole* augmented matrix into RREF instead of just the coefficent matrix part, have a leading 1 in the RHS column. Of course, we could obtain an augmented matrix (wholly) in RREF by simply multiplying the last row by -1 (or in general, by 1 over whatever the non-zero RHS value is). But there's no need to do so. This is why we express the Gauss-Jordan Method as only requiring that the *coefficient matrix part* of the augmented matrix be in RREF.

Next, suppose we get a row like this in the augmented matrix before we've finished bringing the coefficient matrix to RREF. If we continue reducing the augmented matrix, this row is not going to change. The only ero's we might perform involving this row, no matter how much work remains to get to RREF, would be interchanging this row with other rows, moving it farther and farther down the matrix. (In the example above, it was already at the bottom of the matrix, and we were finished by the time we got it, so that didn't happen.) So when we finish reducing the augmented matrix, there will still be a row in which the coefficient matrix part contains only 0's and the RHS column contains a non-zero. And that row corresponds to an equation saying that zero equals some non-zero value, which tells us that the system is inconsistent. And in that case, any work we did after getting this row of the matrix was unnecessary. There's no need to have the rest of the matrix in RREF if we can already tell that the conclusion is going to be that the SLE has no solution. Therefore, you can stop row-reducing as soon as a row like this appears in the matrix, and simply conclude at that point that the system is inconsistent and has no solution.

Finally, if we know that a row which contains only 0's in the coefficient matrix part and a non-zero in the RHS column corresponds to the equation $0 = c$ for some $c \neq 0$ and thus tells us that the system is inconsistent, then we don't need to bother writing the SLE which corresponds to the reduced augmented matrix at all. We simply observe *from the final augmented matrix* that the SLE has no solution.

Similarly, though, for any other kind of final augmented matrix we can also just "read off" the solution from the final augmented matrix, without necessarily having to think about "okay, what is the SLE which corresponds to this augmented matrix?" and then "what is/are the solution(s) to that SLE?". We can describe a procedure for identifying the set of solutions to a SLE directly from the final augmented matrix. Consider what we found in the last 3 examples, in reverse order.

First of all, in Example 6.10 which we just did, we had the situation we've just been discussing – a row in which there are only 0's in the coefficient matrix part but with a non-zero entry in the RHS column. And as we have discussed, regardless of anything else that may be in the augmented matrix, this means that the system has no solution, because it is inconsistent. So that's the first thing we check.

Now, look at the final augmented matrix in Example 6.9. In that example we found that the system had a parametric family of solutions. What characteristics did the final augmented matrix have, to give a corresponding SLE which lead us to that conclusion? Well, let's think about that.

In an augmented matrix corresponding to a SLE, each row corresponds to a different equation in the SLE and each column in the coefficient matrix part corresponds to a different variable in the SLE. A non-zero row of an RREF augmented matrix corresponds to an equation which tells us about the value of the variable whose coefficient is the leading 1 in that row (i.e. tells us about the value of the variable corresponding to the column in which the leading 1 for that row occurs). Any column of the coefficient matrix part of the RREF augmented matrix which does not contain a leading 1 for any row corresponds to a variable for which there is no equation telling us the value of that variable.

For instance, in the example, we had the final augmented matrix $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$. We see that row 1's leading 1 is in column 1, and row 2's is in column 2. Rows 3 and 4 do not have leading 1's, so there is no row whose leading 1 is in column 3. The variables in the original SLE were x, y and z, so column 3 is the z-column. Row 1 is telling us about the value of x (it says $x = 0$), because the leading 1 for row 1 is in the x-column (i.e. the first column). Row 2 is telling us about the value of y, because the leading 1 in row 2 is in the y-column. But there is no row telling us about the value of z, because the z-column does not contain the leading 1 for any row.

As we have seen, this means that z is free to take on *any* value. And likewise, *any* variable whose column in the final augmented matrix does not contain the leading 1 for any row is free to take on any value. Thus, once we have obtained the row-reduced augmented matrix, and we have checked that the system is not inconsistent, if there is any variable whose column does not contain a leading 1 for some row in that matrix, then we set that variable equal to a *parameter* to obtain the set of solutions for the system, and this solution set is a "parametric family of solutions". If there is more than one such variable, then we must set each of them equal to a different parameter. In that case, the solution set to the SLE is a multiple parameter family of solutions.

Of course, if we do have a parametric family of solutions, then we need to write the SLE corresponding to the final augmented matrix in order to find those solutions. But we can skip the part where we literally write the underlying system, and introduce the parameter(s) immediately.

Notice that if a SLE is consistent, then having a column in the coefficient matrix part of the augmented matrix which does not contain the leading 1 for any row can only occur when the number of leading 1's in the final augmented matrix is smaller than the number of variables in the system.

If the system is not inconsistent, and does not have a parametric family of solutions, then the only other possibility is that there is a unique solution. Consider the final augmented matrix. If the system is not inconsistent, then there isn't a row which contains only 0's in the coefficient matrix part but with a non-zero in the RHS column. So either there are no rows of the matrix which contain only 0's in the coefficient matrix, or else each such row has 0 also in the RHS column, i.e. is a row containing *only* 0's. And if the system does not have a parametric family of solutions, then there is no column in the coefficient matrix part which does not contain a leading 1 for any row. That is, every column in the coefficient matrix part of the augmented matrix *does* contain a leading 1 for some row.

And if every column in the coefficient matrix part of the final augmented matrix does contain the leading 1 for some row, then the RREF of the coefficient matrix contains nothing but 0's and 1's. And when you write the SLE corresponding to the final augmented matrix, then (ignoring any equations which just say 0=0) each equation has only a different one variable on the LHS and a number on the RHS. So this kind of reduced augmented matrix is telling us about a unique solution, which we can find as follows: For each row, the variable whose column contains the leading 1 which is the only non-zero entry in the row is equal to the RHS value in that row.

For instance, in Example 6.8 we had the final augmented matrix $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$. We see that in the coefficient matrix part, each column does contain the leading 1 for some row, so this part of the matrix only contains these leading 1's and the 0's which the rest of each of these columns must be filled with. Row 1 has its leading 1 in column 1, the x-column, so it's telling us that $x = 4$. Row 2 has its leading 1 in column 2, the y-column, so this row says that $y = -3$. And row 3 has its leading 1 in column 3, which is the z-column, so we have $z = 2$.

We can summarize all of this as shown in the following theorem.

Theorem 6.3. Recognizing the solution(s) for a SLE from the final augmented matrix *Given an augmented matrix whose coefficient matrix part is in RREF:*

1. *If any row of the matrix contains only 0's in the coefficient matrix part, with a non-zero in the RHS column, then the system is inconsistent and has no solution.*
2. *Otherwise, if there are any columns in the coefficient matrix part which do not contain the leading 1 for any row, the system has a parametric family of solutions (i.e. infinitely many solutions). For each such column, set the corresponding variable equal to a different parameter. Use the final matrix to write the underlying SLE and re-arrange to express each of the other variables in terms of the parameter(s).*
3. *And if neither of the conditions above is true, then the system has a unique solution. Simply set each variable equal to the RHS value for the row whose leading 1 is in that variable's column.*

Example 6.11. Solve the following system:

$$\begin{array}{rcrcrcl} 2x & - & y & + & 3z & = & 24 \\ & + & 2y & - & z & = & 14 \\ 7x & - & 5y & & & = & 6 \end{array}$$

Solution:
We write the augmented matrix, row-reduce it, and then observe from the final matrix what the solution(s), if any, to this SLE is/are.

$$\left[\begin{array}{rrr|r} 2 & -1 & 3 & 24 \\ 0 & 2 & -1 & 14 \\ 7 & -5 & 0 & 6 \end{array}\right] \xrightarrow{R_1 \leftarrow \frac{1}{2}R_1} \left[\begin{array}{rrr|r} 1 & -\frac{1}{2} & \frac{3}{2} & 12 \\ 0 & 2 & -1 & 14 \\ 7 & -5 & 0 & 6 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 + (-7)R_1} \left[\begin{array}{rrr|r} 1 & -\frac{1}{2} & \frac{3}{2} & 12 \\ 0 & 2 & -1 & 14 \\ 0 & -\frac{3}{2} & -\frac{21}{2} & -78 \end{array}\right] \xrightarrow{R_2 \leftarrow (\frac{1}{2})R_2} \left[\begin{array}{rrr|r} 1 & -\frac{1}{2} & \frac{3}{2} & 12 \\ 0 & 1 & -\frac{1}{2} & 7 \\ 0 & -\frac{3}{2} & -\frac{21}{2} & -78 \end{array}\right]$$

$$\xrightarrow[R_3 \leftarrow (R_3 + (\frac{3}{2})R_2)]{R_1 \leftarrow (R_1 + (\frac{1}{2})R_2)} \left[\begin{array}{rrr|r} 1 & 0 & \frac{5}{4} & \frac{31}{2} \\ 0 & 1 & -\frac{1}{2} & 7 \\ 0 & 0 & -\frac{45}{4} & -\frac{135}{2} \end{array}\right] \xrightarrow{R_3 \leftarrow (-\frac{4}{45}R_3)} \left[\begin{array}{rrr|r} 1 & 0 & \frac{5}{4} & \frac{31}{2} \\ 0 & 1 & -\frac{1}{2} & 7 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

$$\xrightarrow[R_2 \leftarrow (R_2 + (\frac{1}{2})R_3)]{R_1 \leftarrow (R_1 + (-\frac{5}{4})R_3)} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

Looking at the final RREF augmented matrix, we see that (1) there is no row which contains only 0's in the coefficient matrix part with a non-zero in the RHS column, so the system is consistent; (2) every column of the coefficient matrix part does contain the leading 1 for some row, so no parameters are needed, i.e. each variable has a unique value; and (3) from row 1, x has the value 8, from row 2, y has the value 10, and from row 3, z has the value 6. Therefore the unique solution to the given SLE is $(x, y, z) = (8, 10, 6)$.

Example 6.12. Solve the following system:

$$\begin{array}{rcrcrcrcrcr} x_1 & + & x_2 & - & x_3 & + & x_4 & & & = & 0 \\ & & & & x_3 & + & x_4 & + & x_5 & = & 0 \\ 2x_1 & + & 2x_2 & - & x_3 & & & + & x_5 & = & 0 \\ x_1 & + & x_2 & - & 2x_3 & & & - & x_5 & = & 0 \\ & & & & 2x_3 & - & 4x_4 & + & 2x_5 & = & 0 \\ -x_1 & - & x_2 & + & 2x_3 & - & 3x_4 & + & x_5 & = & 0 \end{array}$$

Solution:
When we write the augmented matrix, we see that row 1 already has a leading 1. We "zero out" its column (column 1):

$$\left[\begin{array}{rrrrr|r} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 2 & 2 & -1 & 0 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 2 & -4 & 2 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[(R_4 \leftarrow R_4+(-1)R_1),(R_6 \leftarrow R_6+R_1)]{R_3 \leftarrow R_3+(-2)R_1} \left[\begin{array}{rrrrr|r} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & 2 & -4 & 2 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{array}\right]$$

Notice that, below row 1, the left-most non-zero entry is in column 3, and is a leading 1 for row 2. That is, column 2 (like column 1) contains only 0's below row 1. Thus, as we proceed, column 2 will not contain a leading 1 for any row. (Although this is a bit unusual, it does sometimes happen. It means, of course, that we will (at the end) need to introduce a parameter for this column.) Since row 2 already has a leading 1, we now need to clear out column 3.

$$\xrightarrow[(R_5 \leftarrow R_5+(-2)R_2),(R_6 \leftarrow R_6-R_2)]{(R_1 \leftarrow R_1+R_2),(R_3 \leftarrow R_3-R_2),(R_4 \leftarrow R_4+R_2)} \left[\begin{array}{rrrrr|r} 1 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -6 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \end{array}\right]$$

After moving the zero row to the bottom of the matrix ($R_4 \leftrightarrow R_6$), we proceed by obtaining our next leading 1 (in row 3, column 4) and then clear out column 4.

$$\xrightarrow{R_3 \leftarrow -\frac{1}{3}R_3} \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[(R_4 \leftarrow R_4 + 3R_3),(R_5 \leftarrow R_5 + 6R3)]{R_1 \leftarrow (R_1 - 2R_3),(R_2 \leftarrow R_2 - R_3)} \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The last matrix is in RREF. We see that there are no rows in which the coefficient matrix part contains only 0's while the RHS contains a non-zero. Therefore the system is consistent. We see that as expected the x_2 column does not contain the leading 1 for any row, and neither does the x_5 column, so we need to set each of these equal to a parameter. Setting $x_2 = s$ and $x_5 = t$ we see that row 1 says that $x_1 + s + t = 0$, so $x_1 = -s - t$; row 2 says that $x_3 + t = 0$, so $x_3 = -t$; and row 3 says that $x_4 = 0$. Therefore the system has the 2-parameter family of solutions

$$(x_1, x_2, x_3, x_4, x_5) = (-s - t, s, -t, 0, t)$$

Math 1229A/B

Unit 7:

Matrix Operations

(text reference: Section 3.1)

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7 Matrix Operations

At the beginning of Unit 6, we defined the mathematical construct called a matrix. Next we learn more about matrices, especially how to do matrix arithmetic. First, we should review the definition of a matrix. And define one more term, that we didn't need in what we have done so far.

> **Definition:** A **matrix** is a rectangular array of numbers:
>
> $$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$
>
> The horizontal lines of numbers are called **rows** and the vertical lines of numbers are called **columns**. The number, a_{ij}, in row i and column j is called the **(i,j)-entry** of the matrix. A matrix with m rows and n columns is called an $\mathbf{m \times n}$ **matrix** (pronounced "m by n"). The numbers m and n are called the **dimensions** of the matrix.

When one of the dimensions of a matrix is 1, so that the matrix has only one row or one column, the matrix is very similar to a vector. Because of that, we sometimes use the word vector in describing the matrix. But we need be clear about *which* dimension is 1, so we qualify the term vector. This also helps to remind us that we're really talking about a matrix, rather than an actual vector.

> **Definition:** For any value $n > 1$, a $1 \times n$ matrix can be referred to as a **row vector**. Similarly, for any value $m > 1$, an $m \times 1$ matrix can be referred to as a **column vector**.

Example 7.1. Describe each of the following matrices, and identify both the $(1, 2)$-entry and the $(2, 1)$-entry, if the matrix has one.

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \\ 9 & 10 \end{bmatrix}$ (c) $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ (d) $\begin{bmatrix} -3 & 0 & \frac{3}{7} & 2 \end{bmatrix}$ (e) $\begin{bmatrix} -32 \end{bmatrix}$

Solution:

(a) Since the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 5 \end{bmatrix}$ has 2 rows and 3 columns, it is a 2×3 matrix. The $(1, 2)$-entry of this matrix is 2 (i.e. the number in row 1, column 2), and the $(2, 1)$-entry is 4 (i.e. the number in row 2, column 1). (Notice that both in stating the dimensions of the matrix and in referring to a particular entry, the first number always refers to row(s).)

(b) The matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \\ 9 & 10 \end{bmatrix}$ has 5 rows and 2 columns, so it is a 5×2 matrix. The $(1, 2)$-entry is 2 and the $(2, 1)$-entry is 3.

(c) We have the matrix $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, which has 3 rows and only 1 column, so this is a 3×1 column vector. Since it doesn't have any column 2, there is no $(1, 2)$-entry. The $(2, 1)$-entry is 1.

(d) This matrix, $\begin{bmatrix} -3 & 0 & \frac{3}{7} & 2 \end{bmatrix}$, has only 1 row, with 4 columns, so it is a 1×4 row vector. The $(1, 2)$-entry is 0, and of course there is no $(2, 1)$-entry.

(e) Is that a matrix? Just $\begin{bmatrix} -32 \end{bmatrix}$? Sure it is. We can tell by the square brackets around it (as well as by the fact that the question said that each of the question parts involved a matrix). This matrix has only 1 row and 1 column. It is a 1×1 matrix, and therefore has neither a $(1, 2)$-entry nor a $(2, 1)$-entry. Its only entry is the $(1, 1)$-entry, which is -32. *Note:* According to our definitions of row vector and column vector, the "other" dimension must be bigger than 1, so a 1×1 matrix is not considered to be either of these things. We just call it a 1×1 matrix (which helps us to remember that it *is* a matrix, rather than just a scalar which happens to be written in square brackets.)

There is some more terminology and notation for matrices that we should talk about. In vectors, we talk about *corresponding components*, meaning the numbers in the same position in 2 vectors in the same space. Similarly, when we're talking about two matrices which have the same dimensions, we use the term **corresponding entries** to refer to the numbers in the same positions in the 2 matrices. So for instance if we have two $m \times n$ matrices, i.e. with the same values of m and of n for each, the $(1, 1)$-entries of the 2 matrices are corresponding entries. And the $(3, 2)$-entries of the matrices, if there are any, are also corresponding entries. In general, the (i, j)-entry of one matrix and the (i, j)-entry of the other matrix, for the same values of i and j, are corresponding entries.

Matrices are named with capital letters. And when a matrix is named with a particular capital letter, we often use the lower case version of the same letter, subscripted with row and column indices, to denote entries in the matrix. For instance, if we have a matrix called A, we can use a_{ij} to denote the (i, j)-entry of A. And then sometimes we want to define a matrix as the matrix A whose (i, j)-entry is called a_{ij}. We do this by saying "Let $A = [a_{ij}]$", or "Consider the matrix $A = [a_{ij}]$". So $[a_{ij}]$ simply denotes the matrix containing entries which are referred to as a_{ij}. For instance, we could say "Consider the 2×3 matrix $A = [a_{ij}]$ with $a_{ij} = i - j$", which defines A to be the 2×3 matrix in which each entry is its row number minus its column number. So we would have

$$A = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \end{bmatrix}$$

Here is some more terminology that we use:

Definition:

- Any matrix in which every entry is zero is called a **zero matrix**. So for any positive integers m and n, there is an $m \times n$ zero matrix.
  Notice: This is similar to the idea of the *zero vector* in $\Re^n$.
- For any $n > 1$, any $n \times n$ matrix is called a **square matrix of order** n.
  That is, a square matrix is just a matrix which has the same number of rows and columns. And the order of the matrix is the number of rows (or the number of columns).
- In a square matrix of order n, the entries a_{ii} for $i = 1, ..., n$ are called the **main diagonal** of the matrix.
  That is, the main diagonal of a square matrix runs diagonally, from the top left corner to the bottom right corner of the matrix.

- Any matrix in which the only non-zero entries appear on the main diagonal is called a **diagonal matrix**.
  So in a diagonal matrix, all the entries a_{ij} for $i \neq j$ are 0. Of course, there may also be some zeroes along the main diagonal.
- The **identity matrix of order** n is the $n \times n$ diagonal matrix in which $a_{ii} = 1$ for all $i = 1, ..., n$. The identity matrix of order n is often denoted I_n, or just I.
  That is, an identity matrix is a square matrix which has 1's all along the main diagonal, and 0's everywhere else.

Consider the matrices shown here:

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -5 \end{bmatrix} \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Here, A is the 2×2 zero matrix. It is also a square matrix of order 2. And since it is a square matrix, and all *off-diagonal* entries are 0, we *could* also say that it is a diagonal matrix. (Any square zero matrix can be said to be a diagonal matrix. But diagonal matrices usually do have *some* non-zero entries.) B is another square matrix of order 2. And C is another zero matrix — the 2×3 zero matrix. Matrix D is a square matrix of order 3, and since all the non-zero entries are along the main diagonal, with zeroes everywhere else, it is a diagonal matrix. And I_4, of course, is the identity matrix of order 4. Which means it's also a square matrix, and a diagonal matrix. (*Notice:* We've seen identity matrices before. A square matrix in RREF which doesn't have any rows of only zeroes is always an identity matrix.)

Some matrix concepts, definitions, and/or arithmetic operations are just like the corresponding concepts, definitions and/or arithmetic operations for vectors in $\Re^n$. We've already seen some, like the zero matrix. Next we learn some more.

Definition:

- **Matrix Equality:** Two matrices are said to be **equal** if and only if they have the same dimensions, and their corresponding entries are equal.
  That is, $A = B$ if and only if A and B are both $m \times n$ matrices (for the same m and n) and it is true that $a_{ij} = b_{ij}$ for all values of i and j.
- **Matrix Addition:** If A and B have the same dimensions, then the **sum** of matrices A and B is obtained by summing the corresponding entries.
  So if A and B are both $m \times n$ matrices, the matrix $C = A + B$ has $c_{ij} = a_{ij} + b_{ij}$ for all i and j. Notice that if A and B do not have the same dimensions, then $A + B$ is not defined. We can *only* add matrices which have the same dimensions.
- **Scalar Multiplication:** For any matrix A and any scalar c, the **scalar multiple** cA is obtained by multiplying every element of A by c.
  So the matrix $B = cA$ has $b_{ij} = c(a_{ij})$ for all i and j.
- **Negation:** For any matrix A, the **negative of** A, denoted $-A$, is the matrix $(-1)A$. That is, each entry of $-A$ is the negative of the corresponding entry of A, so if $B = -A$, then $b_{ij} = -a_{ij}$ for all i and j.

- **Matrix Subtraction:** For any matrices A and B which have the same dimensions, the **matrix difference** $A - B$ is defined to be the sum of A and $-B$.
  That is, if $C = A - B$, then $C = A + (-B)$, so $c_{ij} = a_{ij} - b_{ij}$ for all i and j.

Notice that each of these works in exactly the same way as the analogous operation for vectors. Vectors can only be equal, or be added or subtracted, if they're from the same space. For matrices, they must have the same dimensions. That is, in both cases, they must have the same number of entries (components), in the same configuration. And the scalar multiplication operation multiplies every element by the scalar, both for vectors and for matrices. Likewise, just as $-\vec{v} = (-1)\vec{v}$, we have $-A = (-1)A$ for any matrix A.

Example 7.2. State whether matrices A and B are equal.

(a) $A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 0 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 0 & 6 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 0 & 3 \\ 5 & 1 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 3 \\ 5 & 1 & 2 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

(d) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

Solution:

(a) $A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 0 & 6 \end{bmatrix} = B$

Since A and B are both 2×3 matrices and $a_{ij} = b_{ij}$ for each pair (i, j), they are equal matrices.

(b) $A = \begin{bmatrix} 1 & 0 & 3 \\ 5 & 1 & -2 \end{bmatrix} \neq B = \begin{bmatrix} 1 & 0 & 3 \\ 5 & 1 & 2 \end{bmatrix}$

Although A and B are both 2×3 matrices, with many of their entries identical, there is a combination ij for which $a_{ij} \neq b_{ij}$ (*i.e.* $a_{23} = -2$ whereas $b_{23} = 2$). Therefore, A and B are not equal matrices.

(c) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \neq B = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

Here, A has dimension 2×3, whereas B has dimension 3×2, so they cannot be equal matrices, no matter how similar their entries may be.

(d) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \neq B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

Again, A and B do not have the same dimension (A is 2×2 while B is 3×2), so they are not equal.

Before we look at more examples, there is one more matrix operation we should define. This one is not like any operation on vectors, because it involves changing the dimensions of the matrix, by interchanging the rows and columns, which has no counterpart in the context of vectors, since a vector has only one dimension. Effectively, we turn the matrix sideways, so that the rows become columns and the columns become rows. We refer to this as *transposing*, i.e. finding the *transpose* of, the matrix.

> **Definition:** For any $m \times n$ matrix A, the **transpose of** A, denoted A^T, is the $n \times m$ matrix whose (i,j)-entry is the (j,i)-entry of A. That is, if $B = A^T$, then $b_{ij} = a_{ji}$ for all values of i and j.

For instance, to find the transpose of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, the entries in the first row of A become the entries in the first column of A^T and the entries of the second row of A become the entries of the second column of A^T. Or, looked at the other way, the first column of A is the first row of A^T, the second column of A is the second row of A^T and the third column of A is the third row of A^T. It doesn't matter whether we think of switching rows to columns or switching columns to rows — the effect is the same. In terms of entries, if $B = [b_{ij}]$ where $B = A^T$, since A is a 2×3 matrix then B is a 3×2 matrix, with $b_{11} = a_{11}$, $b_{12} = a_{21}$, $b_{21} = a_{12}$, $b_{22} = a_{22}$, $b_{31} = a_{13}$ and $b_{32} = a_{23}$. We get

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Example 7.3. If $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \\ 0 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & a & b \\ c & 2 & -1 \end{bmatrix}$, are there any values of a, b and c for which $A = 2B^T$?

Solution:

We see that A is a 3×2 matrix and B is a 2×3 matrix, so that B^T is a 3×2 matrix. Recall that taking a scalar multiple of a matrix does not change the dimensions of the matrix, so $2B^T$ will also be a 3×2 matrix. Therefore it may be possible to find values of a, b and c for which $A = 2B^T$. (If the dimensions of B^T were not the same as the dimensions as A then it would not be possible for $2B^T$ to be equal to A.) We need to find B^T and then $2B^T$. For B^T we simply interchange the rows and columns of B. And then for $2B^T$ we multiply each entry of B^T by 2. We get:

$$B = \begin{bmatrix} 1 & a & b \\ c & 2 & -1 \end{bmatrix} \Rightarrow B^T = \begin{bmatrix} 1 & c \\ a & 2 \\ b & -1 \end{bmatrix} \Rightarrow 2B^T = 2\begin{bmatrix} 1 & c \\ a & 2 \\ b & -1 \end{bmatrix} = \begin{bmatrix} 2(1) & 2(c) \\ 2(a) & 2(2) \\ 2(b) & 2(-1) \end{bmatrix} = \begin{bmatrix} 2 & 2c \\ 2a & 4 \\ 2b & -2 \end{bmatrix}$$

Comparing this last matrix to matrix A, we see that all of the known values match. That is, both matrices have 2 has their $(1,1)$-entry, 4 as their $(2,2)$-entry and -2 as their $(3,2)$-entry, so it *will* be possible to find values of a, b and c which make these matrices equal. (If there was any entry for which known values in the 2 matrices were not identical, then it would not be possible for the matrices to be equal.)

We need to have $\begin{bmatrix} 2 & 3 \\ -1 & 4 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 2c \\ 2a & 4 \\ 2b & -2 \end{bmatrix}$, so it must be true that $2c = 3$, $2a = -1$ and $2b = 0$. We see that we need $c = \frac{3}{2}$, $a = -\frac{1}{2}$ and $b = 0$.

Example 7.4. Find the sum of matrices A and B, if possible, in each of the following.

(a) $A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 2 & 5 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 5 & 0 \\ -2 & 4 & -6 \end{bmatrix}$

(b) $A = \begin{bmatrix} -2 & 3 \\ 1 & 4 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 3 & -2 \\ -2 & 1 & -3 \end{bmatrix}$

(c) $A = \begin{bmatrix} 5 \\ 0 \\ -3 \end{bmatrix}$ $\quad -B^T = \begin{bmatrix} 2 & 7 & -4 \end{bmatrix}$

Solution:

(a) Recall that in order to add two matrices they must have the same dimensions. Since A is a 2×3 matrix and B is also a 2×3 matrix, the sum $A + B$ is defined. Also recall that the sum of two matrices is the matrix whose entries are the sums of the corresponding entries of the two matrices. We get

$$A + B = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 2 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 5 & 0 \\ -2 & 4 & -6 \end{bmatrix} = \begin{bmatrix} (2+1) & (-1+5) & (3+0) \\ (0-2) & (2+4) & (5-6) \end{bmatrix} = \begin{bmatrix} 3 & 4 & 3 \\ -2 & 6 & -1 \end{bmatrix}$$

(b) This time, A is a 2×2 matrix while B is a 2×3 matrix. Matrix addition can only be performed when the matrices to be added have the same dimensions, so in this case $A+B$ is not defined.

(c) We see that A is a 3×1 column vector. And since $-B^T$ is a 1×3 row vector, then so is B^T and therefore B is a 3×1 column vector. (*Notice:* The transpose of the transpose of a matrix is just the original matrix. So $B = (B^T)^T$.) Therefore it *will* be possible to add A and B. But of course, we need to find B. Recall that the negative of a matrix can be obtained by changing the sign of each entry in the matrix. And of course the negative of the negative of a matrix is just the original matrix (i.e. $-(-M) = M$ for any matrix M). So here, $B^T = -(-B^T)$.

$$-B^T = \begin{bmatrix} 2 & 7 & -4 \end{bmatrix} \Rightarrow B^T = -\begin{bmatrix} 2 & 7 & -4 \end{bmatrix} = \begin{bmatrix} -2 & -7 & 4 \end{bmatrix} \Rightarrow B = \begin{bmatrix} -2 \\ -7 \\ 4 \end{bmatrix}$$

Now that we have found B (which is, as we knew it would be, a (3×1) matrix, so that it can be added to A), we find $A + B$:

$$A + B = \begin{bmatrix} 5 \\ 0 \\ -3 \end{bmatrix} + \begin{bmatrix} -2 \\ -7 \\ 4 \end{bmatrix} = \begin{bmatrix} 5-2 \\ 0-7 \\ -3+4 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \\ 1 \end{bmatrix}$$

Notice: We could have done this more directly as follows, using the fact that the transpose of the transpose of a matrix is the matrix itself. (That is, if we switch the rows and columns, and then switch them again, we have just put them back where they were in the first place.) So we can consider B as $(B^T)^T$, and of course adding can be considered as subtracting the negative of the matrix, and to subtract one matrix from another we just subtract the corresponding components. Furthermore, whether we change the signs of a matrix before or after transposing it clearly makes no difference. That is, $(-B)^T = -(B^T)$, so we have

$$A + B = A - (-B) = A - [-(B^T)^T] = A - (-B^T)^T$$

Therefore to add A and B we can subtract the transpose of $-B^T$ from A:

$$A+B = A-(-B^T)^T = \begin{bmatrix} 5 \\ 0 \\ -3 \end{bmatrix} - \begin{bmatrix} 2 & 7 & -4 \end{bmatrix}^T = \begin{bmatrix} 5 \\ 0 \\ -3 \end{bmatrix} - \begin{bmatrix} 2 \\ 7 \\ -4 \end{bmatrix} = \begin{bmatrix} 5-2 \\ 0-7 \\ -3-(-4) \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \\ 1 \end{bmatrix}$$

Example 7.5. Given A and B as follows, find (a) $3A - B$ and (b) $(2A - 3I + B^T)^T$.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ -2 & 0 \end{bmatrix}$$

Solution:

(a) Notice that since A and B are both 2×2 matrices, the stated operations are all defined. We find $3A$ by multiplying each element of A by 3, and then subtract B by subtracting corresponding components:

$$\begin{aligned} 3A - B &= 3\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 3(1) & 3(2) \\ 3(3) & 3(4) \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ -2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 3-1 & 6-2 \\ 9-(-2) & 12-0 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 11 & 12 \end{bmatrix} \end{aligned}$$

(b) Recall that I is an identity matrix, i.e. a square matrix whose main diagonal elements are all 1's and whose off-diagonal elements are all 0's. Since I here appears in a sum/difference of 2×2 matrices, clearly it is I_2, i.e. the identity matrix of order 2, which is meant. (That is, we assume that I here means the particular identity matrix for which the required calculation *is defined*.) We start by finding the matrix whose transpose will be the final answer. That is, we can find $2A - 3I + B^T$, and then take the transpose of that matrix to get $(2A - 3I + B^T)^T$. We have:

$$\begin{aligned} 2A - 3I + B^T &= 2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ -2 & 0 \end{bmatrix}^T \\ &= \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2-3+1 & 4-0+(-2) \\ 6-0+2 & 8-3+0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 8 & 5 \end{bmatrix} \\ \text{Therefore } (2A - 3I + B^T)^T &= \begin{bmatrix} 0 & 8 \\ 2 & 5 \end{bmatrix} \end{aligned}$$

So far we have mostly been dealing with arithmetic operations for matrices which are very similar to the corresponding arithmetic operations for vectors. But with matrices, there is also a *multiplication* operation defined. Recall that with vectors, we don't have a *multiplication* operation. We do have two different kinds of products, the *dot product* and the *cross product*, but neither of these is considered to be multiplication, so for vectors there is nothing that directly corresponds to multiplication. And the dot product operation for vectors is not one which could easily and directly be extended to the context of matrices. Also, the cross product is only defined for vectors in $\Re^3$, so it is very exclusive and cannot be extended to matrices. However, part of the multiplication operation for matrices will look very familiar, because it does involve what is effectively the dot product.

Matrix Multiplication

Matrix multiplication is more complicated than the other matrix operations that we've looked at so far. It's not hard, just somewhat more complicated. Once you get the hang of it, it's easy. But let's work up to it gradually, to make sure you remember the steps. First, we'll look at the rules about *which matrices can be multiplied together?*.

Definition: Consider 2 matrices A and B. The **matrix product** AB is **only** defined if A is an $m \times n$ matrix and B is an $n \times p$ matrix. That is, two matrices can be multiplied together only when the number of *columns* in the first matrix of the product is the same as the number of *rows* in the second matrix of the product.

Well, that's probably not what you were expecting! It seems a little quirky, but it's not that hard a rule. And there's a good reason for it. Once we learn *how* to calculate a matrix product, you'll see *why* we need those dimensions to match. And then it will be easy to remember, because if they don't match, you won't be able to calculate the entries of the product matrix. You can think of it as the "inner" dimensions of the product. That is, if we multiply an $m \times n$ matrix times an $n \times p$ matrix, we're doing $(m \times n) \times (n \times p)$ and it's those two inside dimensions, that are right next to each other but from different matrices, that have to be the same. (Of course, when we say $(m \times n) \times (n \times p)$, the middle $\times$ doesn't mean the same thing that the other two do. But the fact that it looks the same is kind of helpful. Or we could write it as $(m \times n) \cdot (n \times p)$, because sometimes we use $\cdot$ to represent multiplication. And using the $\cdot$ might be even better, to help you remember what to do ... but we're not there yet.)

So for instance, if A is a 2×3 matrix, and B is a 3×2 matrix, then we can form the matrix product AB, because $(2 \times 3) \times (3 \times 2)$ has the 2 inner dimensions matching. We can always mutliply a "something" by 3 times a 3 by "anything". Likewise, we can multiply a 3×2 times a 2×3, so the matrix product BA is also defined. However, the products $A(B^T)$ and $(A^T)B$ are not defined, because for $A(B^T)$ we're trying to multiply a 2×3 times a 2×3, and for $(A^T)B$ we're trying to multiply a 3×2 times a 3×2. In both of those, the number of rows in the second matrix is *not* the same as the number of columns in the first matrix.

And now, suppose that we also have C, which is a 2×2 matrix. Then we can use C in a matrix product as the first matrix if it's multiplying a matrix that has 2 rows, or as the second matrix in the product if it's being multiplied by a matrix that has 2 columns. So the matrix product CA is defined (i.e. $(2 \times 2) \times (2 \times 3)$ works) and the matrix product BC is defined (i.e. $(3 \times 2) \times (2 \times 2)$ works). But the matrix product AC is not defined (because $(2 \times 3) \times (2 \times 2)$ doesn't match) and neither is the matrix product CB (because $(2 \times 2) \times (3 \times 2)$ doesn't match either). On the other hand, the products $(A^T)C$ and $C(B^T)$ *are* defined.

A couple of notes about notation

1. We always just write the names of the matrices beside each other to express a matrix product. We don't use a $\times$ or a $\cdot$ to indicate that we're multiplying. Just the same as with unknowns. We never write $x \times y$, we just write xy to say x times y. With numbers, we need a multiplication symbol between them, or brackets, because two number written beside each other means something else ... another number. (e.g. if we write 62, that *doesn't* mean 6 times 2, it means sixty-two.) But if A is a matrix and B is a matrix, then AB never means anything but A times B, so we don't need a symbol to say "times".
2. When we write a ^{T} to indicate the transpose of a matrix, it always means just the matrix it's attached to, i.e. right beside. So we don't usually write something like $A(B^T)$. There's no need for the brackets. We just write AB^T and we know that it means A times the transpose of B, because the ^{T} is on the B. If we wanted to say "the transpose of the product matrix AB", then we would have to write it as $(AB)^T$. We need the brackets, so that the transpose can be "attached" to the brackets to show that it's the whole thing inside the brackets that is being transposed.

Example 7.6. Consider the matrices shown here:

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 2 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} -2 & 3 \\ 1 & 4 \end{bmatrix} \qquad C = \begin{bmatrix} 5 \\ 0 \\ -3 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & -2 & 5 \\ 0 & 3 & 4 & -1 \\ 5 & -1 & 2 & 4 \end{bmatrix}$$

How many different matrix products of the form M_1M_2 are defined, where each of M_1 and M_2 is either one of the given matrices or the transpose of one of the given matrices?

Solution:
A is a 2×3 matrix, so A^T is a 3×2 matrix. Both B and B^T are 2×2 matrices. C is a 3×1 matrix and D is a 3×4 matrix, so C^T is a 1×3 matrix and D^T is a 4×3 matrix.

Let's consider the matrices, one by one, as the first matrix in the product, and see which of the 8 matrices could be the second matrix in the product. We can form a matrix product of the form AM for any matrix M which has 3 rows, i.e. as long as M is a $3\times n$ matrix for any value n. Therefore the products AA^T, AC and AD are all defined. For the matrix product BM, M must be a $2\times n$ matrix. A satisfies this requirement, as do B itself, and its transpose, so the products BA, BB and BB^T are all defined. For CM we would need M to be a $1\times n$ matrix. Only C^T meets this requirement, so the only product of this form which is defined is CC^T. And DM requires that M be a $4\times n$ matrix, which only describes D^T, so DD^T is the only product with D as the first matrix.

Of course, we could also have a transposed matrix as the first matrix in the product. For A^TM we need M to be a $2\times n$ matrix, and that means that A^TA, A^TB and A^TB^T are all defined. Since B^T is a 2×2 matrix, we can again have any of those same matrices as the second matrix in a product B^TM, so B^TA, B^TB and B^TB^T are defined. C^TM needs M to be a $3\times n$ matrix, so C^TC, C^TD and C^TA^T are all defined. Similarly, since D^T also has 3 columns, it can multiply any of those same matrices, that all have 3 rows, so D^TC, D^TD and D^TA^T are all defined.

Using only these 4 matrices and their transposes, any of 20 different matrix products can be formed.

Notice: For any matrix M, if M is an $m\times n$ matrix, then M^T is an $n\times m$ matrix, so both MM^T and M^TM are defined. And if M is a square matrix, then both MM and M^TM^T are also defined.

Okay, so we know which matrix products are defined. But what do we get when we multiply one matrix by another? That is, if the matrix product AB is defined, what does it produce? Well, it gives a new matrix. And the 2 inner dimensions, that are the same, collapse in on themselves and disappear, as we see in the following.

> **Definition:** If A is an $m\times n$ matrix and B is an $n\times p$ matrix, then the **product matrix** AB has **dimensions** $m\times p$. That is, multiplying and $m\times n$ matrix times an $n\times p$ matrix produces an $m\times p$ matrix.

For instance, if we multiply a 3×4 matrix times a 4×2 matrix, we get a 3×2 matrix. If we multiply a 2×1 matrix times a 1×2 matrix we get a 2×2 matrix, but if we reverse the order of the matrices in the product, so that we're multiplying a 1×2 matrix times a 2×1 matrix, we get a 1×1 matrix, i.e. a matrix containing only a single number.

Example 7.7. Recall the matrices defined in Example 7.6. Which of the matrix products identified in that example as being defined give a matrix which is either a row vector or a column vector?

Solution:
We had the following: A is a 2×3 matrix, B is a 2×2 matrix, C is a 3×1 matrix (a column vector) and D is a 3×4 matrix, so that A^T is a 3×2 matrix, B^T is another 2×2 matrix, C^T is a 1×3 matrix (a row vector) and D^T is a 4×3 matrix.

For the product matrix M_1M_2 to be a row vector, i.e. a $1 \times p$ matrix, we need M_1 to have only 1 row, i.e. to also be a row vector, with dimensions $1 \times n$, and so M_2 must be an $n \times p$ matrix. The only row vector in the previous example is matrix C^T and so only the products of the form C^TM product a row vector. These are the products C^TC, C^TD and C^TA^T, which give a 1×1 matrix, a 1×4 matrix and a 1×2 matrix, respectively. But in our definition of row vector, we specified that a row vector must have more than one column, so C^TC isn't a row vector after all. Only C^TD and C^TA^T are row vectors.

Similarly, for the product matrix M_1M_2 to be a column vector, it must be an $m \times 1$ matrix, for some $m > 1$. That is, the second matrix in the product must be a column vector. Therefore we need M_2 to be an $n \times 1$ matrix, for any value n and so M_1 must be an $m \times n$ matrix, for some $m > 1$. (That is, M_1 cannot be a row vector.) Among the matrices in the example, C is the only column vector, and AC, C^TC and D^TC were the only products which could be formed with C as the second matrix in the product. But C^TC, the product of a 1×3 times a 3×1, produces a 1×1 matrix, and hence is not a column vector. That is, C^T, which is a 1×3 matrix, doesn't satisfy the requirement of not being a row vector, so the only products for which the product matrix is a column vector are AC (which gives a 2×1 column vector) and D^TC (which gives a 4×1 column vector).

At this point, we know which matrix products are defined, and what the dimensions of the product matrix are. Both things depend on the dimensions of the matrices in the product. We must have the number of columns in the first matrix of the product being the same as the number of rows of the second matrix in the product, and this number somehow disappears, so that the product matrix has the same number of rows as the first matrix in the product, and has the same number of columns as the second matrix in the product. Why? And how does that happen? Well, it's because we do what is effectively a dot product. And as you recall, a dot product is the product of 2 n-vectors, and its value is only a single number. We'll start by seeing how to multiply a matrix with only 1 row times a matrix with only one column, i.e. multiplying a row vector times a column vector. This involves only a single calculation. After that, we'll see that to multiply larger matrices, we just do a series of that same kind of calculation, one for each combination of a row of the first matrix in the product and a column of the second matrix.

Definition: The matrix product *row vector* times *column vector*
Let $A = [a_{1j}]$ be any $1 \times n$ row vector and $B = [b_{i1}]$ be any $n \times 1$ column vector. Then the matrix product, AB, is the 1×1 matrix whose one entry is the number $a_{11}b_{11} + a_{12}b_{21} + ...a_{1n}b_{n1}$. That is, if we let $\vec{a}$ be the n-vector whose components are, in order, the entries of the only row of A, and $\vec{b}$ be the n-vector whose components are, in order, the entries of the only column of B, then the one entry of AB is $\vec{a} \bullet \vec{b}$, so $AB = \left[\vec{a} \bullet \vec{b}\right]$.

Example 7.8. Find the following product matrices.

(a) AB, where $A = \begin{bmatrix} 1 & 2 \end{bmatrix}$ and $B^T = \begin{bmatrix} 3 & 4 \end{bmatrix}$.

(b) BA^T, where $A = \begin{bmatrix} 1 & 0 & -3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -5 & 2 & 5 \end{bmatrix}$.

(c) C^TC, where $C^T = \begin{bmatrix} 5 & 0 & -3 \end{bmatrix}$.

Solution:
(a) If we let $\vec{a} = (1, 2)$ and $\vec{b} = (3, 4)$, then we have

$$AB = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \left[\vec{a} \bullet \vec{b}\right] = [(1, 2) \bullet (3, 4)] = [(1)(3) + (2)(4)] = [3 + 8] = [11]$$

(b) Since A is a row vector, then A^T is a column vector, which is what we need here. Notice that we don't actually need to define vectors to calculate their dot product. We can just do the calculation, without the vectors. We need to calculate BA^T, where $B = \begin{bmatrix} 0 & -5 & 2 & 5 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 0 & -3 & 2 \end{bmatrix}$, which we can do by multiplying the k^{th} entry in the only row of B by the k^{th} entry in the only column of A^T and adding them up:

$$BA^T = \begin{bmatrix} 0 & -5 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix} = [0(1) + (-5)(0) + 2(-3) + 5(2)] = [0 + 0 - 6 + 10] = [4]$$

(c) For $C^T = \begin{bmatrix} 5 & 0 & -3 \end{bmatrix}$ we get

$$C^T C = \begin{bmatrix} 5 & 0 & -3 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ -3 \end{bmatrix} = [5(5) + 0(0) + (-3)(-3)] = [25 + 0 + 9] = [34]$$

Note: It's important to remember that, unlike a vector dot product in which the product of two vectors is a *scalar*, the product of two matrices is always a *matrix*. So when we find the product of a row vector and a column vector, we don't just get a number. We get a matrix, which contains only one number (i.e. entry).

Also Note: In order to calculate the product of a row vector times a column vector in the manner defined, the row of the first matrix must contain the same number of entries as the column of the second matrix does. The number of entries in a row of a matrix is the number of columns in the matrix, because a row has one entry for each column. Similarly, the number of entries in a column of a matrix is the number of rows in the matrix, because a column has one entry for each row. This is *why* the number of *columns* in the first matrix must be the same as the number of *rows* in the second matrix, in order for the product matrix to be defined.

Now we're ready to actually define the product of two matrices — any two matrices whose product is actually defined. That is, we're ready to learn how to calculate matrix products for larger matrices. All we do is "dot" each row with each column, to get a number which is then one of the entries in the product matrix.

Definition: Let A be any $m \times n$ matrix and B be any $n \times p$ matrix, for some values of m, n and p. Also, let $C = AB$. Then C is the $m \times p$ matrix whose (i, j)-entry is $a_{i1}b_{1j} + a_{i2}b_{2j} + ... + a_{in}b_{nj}$. That is, if we let $\vec{a}_i$ be the n-vector whose components are the entries from row i of matrix A, in order, and $\vec{b}_j$ be the n-vector whose components are the entries from column j of matrix B, in order, then we have $C = [c_{ij}]$, where $c_{ij} = \vec{a}_i \bullet \vec{b}_j$.

Note: In the matrix product AB, where A is an $m \times n$ matrix and B is an $n \times p$ matrix, we do one of these dot-product-like calculations for each combination of a row of A and a column of B. The calculation using row i and column j gives us the (i, j)-entry of the product matrix. This gives m different rows of entries (all the entries calculated using a particular row of A are in that row of the product matrix) in p different columns (all the entries calculated using a particular column of B are in that column of the product matrix). That's *why* the product matrix is an $m \times p$ matrix.

Example 7.9. Find the product matrix specified:

(a) AB, where $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ -3 & -1 \\ 2 & 4 \end{bmatrix}$.

(b) AB, where $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$.

(c) CC^T, where $C = \begin{bmatrix} 5 \\ 0 \\ -3 \end{bmatrix}$.

Solution:

(a) Let's do one with the vectors, calculating each entry separately. We define a vector for each row of the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}$ and a vector for each column of the matrix $B = \begin{bmatrix} 1 & 2 \\ -3 & -1 \\ 2 & 4 \end{bmatrix}$. We get $\vec{a}_1 = (1, 2, 1)$, $\vec{a}_2 = (1, 3, 2)$, $\vec{b}_1 = (1, -3, 2)$ and $\vec{b}_2 = (2, -1, 4)$. Now we dot each a vector with each b vector, with $\vec{a}_i \bullet \vec{b}_j$ giving us the (i, j)-entry of AB. Since we're multiplying a 2×3 matrix times a 3×2 matrix, the product matrix will be a 2×2 matrix. We get:

$$\begin{array}{lclclclclcl}
(1,1)\text{-entry} & = & \vec{a}_1 \bullet \vec{b}_1 & = & (1,2,1) \bullet (1,-3,2) & = & 1(1)+2(-3)+1(2) & = & 1-6+2 & = & -3 \\
(1,2)\text{-entry} & = & \vec{a}_1 \bullet \vec{b}_2 & = & (1,2,1) \bullet (2,-1,4) & = & 1(2)+2(-1)+1(4) & = & 2-2+4 & = & 4 \\
(2,1)\text{-entry} & = & \vec{a}_2 \bullet \vec{b}_1 & = & (1,3,2) \bullet (1,-3,2) & = & 1(1)+3(-3)+2(2) & = & 1-9+4 & = & -4 \\
(2,2)\text{-entry} & = & \vec{a}_2 \bullet \vec{b}_2 & = & (1,3,2) \bullet (2,-1,4) & = & 1(2)+3(-1)+2(4) & = & 2-3+8 & = & 7
\end{array}$$

Now, we put it all together in a matrix. We see that $AB = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -3 & -1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ -4 & 7 \end{bmatrix}$.

(b) We can do the calculations separately, as we did in the previous part, or we can do them right in a matrix. In this case, we're multiplying a 4×3 matrix times a 3×1 matrix, so the product will be a 4×1 matrix.

$$\begin{aligned}
AB &= \begin{bmatrix} 1 & 2 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} (1,2,3) \bullet (3,1,-2) \\ (1,-2,1) \bullet (3,1,-2) \\ (3,-1,2) \bullet (3,1,-2) \\ (-1,0,1) \bullet (3,1,-2) \end{bmatrix} \\
&= \begin{bmatrix} 1(3)+2(1)+3(-2) \\ 1(3)+(-2)(1)+1(-2) \\ 3(3)+(-1)(1)+2(-2) \\ (-1)(3)+0(1)+1(-2) \end{bmatrix} = \begin{bmatrix} 3+2-6 \\ 3-2-2 \\ 9-1-4 \\ -3+0-2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 4 \\ -5 \end{bmatrix}
\end{aligned}$$

(c) In part (b) we still used the vectors, even though we didn't name them. But we don't need to. We can just do the vector-like calculations. (Besides, for the product we're asked for this time, they would be 1-vectors, which we haven't actually defined.) We are asked to calculate CC^T, where $C^T = \begin{bmatrix} 5 & 0 & -3 \end{bmatrix}$. We are multiplying a 3×1 column vector times a 1×3 row vector, which will give us a 3×3 matrix (!). Because each row of the first matrix, C, contains only 1 entry, as does each column of the second matrix, C^T, each of the dot-product-type calculations is just a single product of two numbers. Remember, each of the row 1 entries of the product matrix is row 1 of C "times" a different column of C^T, and similarly for the other rows. We get:

$$CC^T = \begin{bmatrix} 5 \\ 0 \\ -3 \end{bmatrix} \begin{bmatrix} 5 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 5(5) & 5(0) & 5(-3) \\ 0(5) & 0(0) & 0(-3) \\ (-3)(5) & (-3)(0) & (-3)(-3) \end{bmatrix} = \begin{bmatrix} 25 & 0 & -15 \\ 0 & 0 & 0 \\ -15 & 0 & 9 \end{bmatrix}$$

Example 7.10. If $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$, find AA.

Solution:
We have a 2×2 matrix, so in the matrix product AA, the number of rows in the second matrix (2) is the same as the number of columns in the first matrix (also 2) and therefore the product is defined. Since we are multiplying a 2×2 matrix times a 2×2 matrix, the product matrix will also be a 2×2 matrix. We get:

$$AA = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1(1) + 2(-1) & 1(2) + 2(3) \\ (-1)(1) + 3(-1) & (-1)(2) + (3)(3) \end{bmatrix} = \begin{bmatrix} 1-2 & 2+6 \\ -1-3 & -2+9 \end{bmatrix} = \begin{bmatrix} -1 & 8 \\ -4 & 7 \end{bmatrix}$$

In this example, we multiplied a matrix by itself. We saw that the product was defined, and that the product matrix has the same dimensions as the original matrix. Whenever we have a square matrix, so that the number of columns is equal to the number of rows, the product of the matrix multiplied by itself is defined. And for a square matrix of order n, when we do this we're multiplying an $n \times n$ matrix times and $n \times n$, so the product matrix is also an $n \times n$ matrix, i.e. a square matrix of order n. And that means that we could also multiply this product matrix by the original matrix again. And the result would be another $n \times n$ matrix, so we could keep going ...

We use the same terminology and notation for this product of copies of the same matrix multiplied together as we use for numbers. If we have some number c and multiply it by itself, we get $cc = c^2$, which we express as "c squared". And then if we multiply by c again, we get $c^2c = ccc = c^3$, which we call "c cubed", and so forth. In general, if we multiply k copies of a number c together, we get c^k, which in general we refer to as "c to the power k". So when we multiply a matrix A by itself, we use A^2 to denote the product, and call it "A squared". And if we then multiply by this product matrix by A again, we get A^3, pronounced "A cubed". And in general, if we do this over and over, multiplying k copies of A together, we call the product matrix A^k, pronounced "A to the power k". Of course, this only works if A is a square matrix, so that these products are defined.

> **Definition:** Let A be any square matrix of order n. Multiplying several copies of A together is referred to as finding **powers** of the matrix A. We use A^2 to denote AA, and A^3 to denote AAA (i.e. A times A times A), and in general, we use A^k to denote k copies of matrix A multiplied together.

Example 7.11. For $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$, find A^4.

Solution:
In Example 7.10, we had this same matrix and we found AA, i.e. we calculated $A^2 = \begin{bmatrix} -1 & 8 \\ -4 & 7 \end{bmatrix}$.
We can use that in finding A^4.

Approach 1: We can find $A^3 = A^2A$ and then use that to find $A^4 = A^3A$. That is, we just keep multiplying by A until we have done it enough times. We get:

$$\begin{aligned} A^3 &= A^2A = \begin{bmatrix} -1 & 8 \\ -4 & 7 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} (-1)(1)+8(-1) & (-1)(2)+8(3) \\ (-4)(1)+7(-1) & (-4)(2)+7(3) \end{bmatrix} = \begin{bmatrix} -1-8 & -2+24 \\ -4-7 & -8+21 \end{bmatrix} = \begin{bmatrix} -9 & 22 \\ -11 & 13 \end{bmatrix} \end{aligned}$$

and then we get:

$$\begin{aligned} A^4 &= A^3A = \begin{bmatrix} -9 & 22 \\ -11 & 13 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} (-9)(1)+22(-1) & (-9)(2)+22(3) \\ (-11)(1)+13(-1) & (-11)(2)+13(3) \end{bmatrix} = \begin{bmatrix} -9-22 & -18+66 \\ -11-13 & -22+39 \end{bmatrix} = \begin{bmatrix} -31 & 48 \\ -24 & 17 \end{bmatrix} \end{aligned}$$

Approach 2: We could get to the answer more quickly by realizing that 4 copies of A multiplied together (i.e. A^4) could be considered as 2 copies of A multiplied by together, and then multiplied by another 2 copies of A multiplied together (i.e. A^2 times A^2). This way we get:

$$\begin{aligned} A^4 &= A^2A^2 = \begin{bmatrix} -1 & 8 \\ -4 & 7 \end{bmatrix}\begin{bmatrix} -1 & 8 \\ -4 & 7 \end{bmatrix} \\ &= \begin{bmatrix} (-1)(-1)+8(-4) & (-1)(8)+8(7) \\ (-4)(-1)+7(-4) & (-4)(8)+7(7) \end{bmatrix} = \begin{bmatrix} 1-32 & -8+56 \\ 4-28 & -32+49 \end{bmatrix} = \begin{bmatrix} -31 & 48 \\ -24 & 17 \end{bmatrix} \end{aligned}$$

Sometimes when a matrix is multiplied by itself repeatedly, a particular pattern can be seen in the various powers of the matrix, i.e. in the matrices that are the matrix powers. This allows us to express *all* powers of the matrix easily, or perhaps to find a particular power of the matrix without actually having to do all the matrix multiplication. Three different kinds of such patterns are observed in the following example.

Example 7.12. Determine what A^k looks like for all $k > 1$, and find A^{99}, for each of the following.
(a) $A = I_2$ (b) $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (c) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Solution:
(a) We start by finding A^2:

$$\begin{aligned} A^2 &= AA = I_2I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1(1)+0(0) & 1(0)+0(1) \\ 0(1)+1(0) & 0(0)+1(1) \end{bmatrix} = \begin{bmatrix} 1+0 & 0+0 \\ 0+0 & 0+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

We see that $A^2 = A$. But then if we multiply by A again, we're just doing the same calculation we already did. And so we'll get the same result. And then multiplying that matrix by A will give those same calculations again, with the same result. And so forth. So in this case, we see that

$$A^k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ for all } k \geq 1, \text{ so } A^{99} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) Again we start by finding A^2:

$$\begin{aligned} A^2 &= AA = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1(1)+0(0) & 1(0)+0(-1) \\ 0(1)+(-1)(0) & 0(0)+(-1)(-1) \end{bmatrix} = \begin{bmatrix} 1+0 & 0+0 \\ 0+0 & 0+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Once again we get $A^2 = I_2$. But since this time $A \neq I_2$, calculating A^3 won't involve repeating the same calculations. So we need to calculate A^3:

$$\begin{aligned} A^3 &= A^2A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1(1)+0(0) & 1(0)+0(-1) \\ 0(1)+(1)(0) & 0(0)+(1)(-1) \end{bmatrix} = \begin{bmatrix} 1+0 & 0+0 \\ 0+0 & 0-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = A \end{aligned}$$

But now, since $A^3 = A$, then if we multiply by A again we'll just be repeating the calculations we did in finding A^2. And those calculations will, of course, once again give A^2. So without doing any more calculations at all, we can see that $A^4 = A^2 = I_2$. And then when we multiply by A again, to get A^5, we'll be repeating the calculations that lead to A^3, so we'll get A again (because $A^3 = A$). And so forth. Everytime we multiply by A, we're going to get either A itself or the matrix that was the product of A multiplied by A. And so we see that:

For any $k > 1$ with k an even number, we get $A^k = A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

And for any $k > 1$ with k an odd number, we get $A^k = A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Therefore, since 99 is an odd number, we know that $A^{99} = A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

(c) As usual, we start by finding A^2:

$$\begin{aligned} A^2 &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1(1)+1(1) & 1(1)+1(1) \\ 1(1)+1(1) & 1(1)+1(1) \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \end{aligned}$$

Hmm. Well, let's calculate A^3 and see what that looks like:

$$\begin{aligned} A^3 = A^2A &= \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2(1)+2(1) & 2(1)+2(1) \\ 2(1)+2(1) & 2(1)+2(1) \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \end{aligned}$$

Let's see. We've got a matrix filled with 1's, then a matrix filled with 2's, and next a matrix filled with 4's, and ... Of course, $2 = 2^1$ and $4 = 2^2$, so ... Aha! Let's think for a minute. Suppose that we have a matrix filled with 2^m's, and we multiply that matrix by A. Then we have:

$$\begin{aligned} \begin{bmatrix} 2^m & 2^m \\ 2^m & 2^m \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} &= \begin{bmatrix} 2^m(1)+2^m(1) & 2^m(1)+2^m(1) \\ 2^m(1)+2^m(1) & 2^m(1)+2^m(1) \end{bmatrix} \\ &= \begin{bmatrix} 2^m(2) & 2^m(2) \\ 2^m(2) & 2^m(2) \end{bmatrix} \\ &= \begin{bmatrix} 2^{m+1} & 2^{m+1} \\ 2^{m+1} & 2^{m+1} \end{bmatrix} \end{aligned}$$

So each time we multiply by A, we multiply all the entries in the matrix by 2. And since we have $A^2 = \begin{bmatrix} 2^1 & 2^1 \\ 2^1 & 2^1 \end{bmatrix}$ and $A^3 = \begin{bmatrix} 2^2 & 2^2 \\ 2^2 & 2^2 \end{bmatrix}$, then we see that

$$A^k = \begin{bmatrix} 2^{k-1} & 2^{k-1} \\ 2^{k-1} & 2^{k-1} \end{bmatrix} \text{ and so } A^{99} = \begin{bmatrix} 2^{98} & 2^{98} \\ 2^{98} & 2^{98} \end{bmatrix}$$

That is, we had $A^2 = 2A$ and $A^3 = 4A = 2^2A$ and so in general $A^k = 2^{k-1}A$, so we get

$$A^{99} = 2^{98}A = 2^{98}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2^{98} & 2^{98} \\ 2^{98} & 2^{98} \end{bmatrix}$$

In our various examples and other calculations in this unit we have used some, and you may have observed others, of the following properties of the matrix operations we've been learning.

Theorem 7.1. Properties of Operations for Matrices

1. $A + B = B + A$ *for any matrices* A *and* B *of the same dimensions.*
   (i.e. matrix addition is commutative)
2. $A + (B + C) = (A + B) + C$ *for any matrices* A*,* B *and* C*, all with the same dimensions.*
   (i.e. matrix addition is also associative)
3. $A + 0 = A$ *for any matrix* A*, where* 0 *denotes the zero matrix whose dimensions is the same as the dimension of* A*.*
4. $A + (-A) = 0$ *for any matrix* A*, where again* 0 *is the zero matrix with the same dimensions as* A*.*
5. $A(BC) = (AB)C$ *for any matrices* A*,* B *and* C *of appropriate dimensions to perform this multiplication. That is if* A *is* $m \times n$ *and* C *is* $p \times q$*, then* B *must be* $n \times p$*.*
   (i.e. matrix multiplication is associative)
6. *For any* $m \times n$ *matrix* A*,* $AI_n = A$ *and* $I_m A = A$*.*
7. $A(B + C) = AB + AC$
   for any matrices A*,* B *and* C*, where* A *is* $m \times n$ *and* B *and* C *are both* $n \times p$*.*
   and also
   $(B + C)A = BA + CA$
   for any matrices A*,* B *and* C*, where* B *and* C *are both* $m \times n$ *and* A *is* $n \times p$*.*
   (i.e. matrix multiplication is distributive over matrix addition)
8. $a(B + C) = aB + aC$ *for any scalar* a *and any matrices* B *and* C *with the same dimensions.*
   (i.e. scalar multiplication of matrices is distributive over matrix addition)
9. $(a + b)C = aC + bC$ *for any scalars* a *and* b *and any matrix* C*.*
   (i.e. scalar multiplication of matrices is also distributive over scalar addition)
10. $(ab)C = a(bC)$ *for any scalars* a *and* b *and any matrix* C*.*
11. $1A = A$ *for any matrix* A*.* *Note: 1 is the scalar 1, of course.*
12. $A0 = 0$ *for any matrix* A*, where* 0 *denotes a zero matrix and the two zero matrices have appropriate dimensions.*
    and also
    $0A = 0$ *for any matrix* A*, where again the two zero matrices,* 0*, have appropriate dimensions.*
13. $a0 = 0$ *where* 0 *is any zero matrix and* a *is any scalar.*
14. $a(AB) = (aA)B = A(aB)$ *for any scalar* a *and any matrices* A *and* B *of appropriate dimensions to form the matrix product* AB*.*
    (i.e. scalar multiplication of a matrix product can either be performed after the matrices are multiplied or else be performed before the matrix multiplication, by multiplying **either one** *of the matrices in the product by the scalar.)*
15. $(A + B)^T = A^T + B^T$ *for any matrices* A *and* B *of the same dimensions.*
    (i.e. matrix transposition is distributive over matrix addition)
16. $(AB)^T = B^T A^T$ *for any matrices* A *and* B *of appropriate dimensions to be multiplied.*
    (i.e. matrix transposition is distributive over matrix multiplication, but the order of multiplication is reversed)

It is worth looking at an example, to demonstrate this last property as well as the reason for this reversal in the order of multiplication.

Example 7.13. For $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$, find $(AB)^T$.

Solution: According to property 16, we can either find the matrix product AB and then take its transpose, or else take the transposes of A and B and then find the product $(B^T)(A^T)$.

By the first approach, we have:

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1+2 & 0-1 \\ 1+4 & 0-2 \\ 1+6 & 0-3 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 5 & -2 \\ 7 & -3 \end{bmatrix}$$

$$\text{So } (AB)^T = \begin{bmatrix} 3 & -1 \\ 5 & -2 \\ 7 & -3 \end{bmatrix}^T = \begin{bmatrix} 3 & 5 & 7 \\ -1 & -2 & -3 \end{bmatrix}$$

For the second approach, we have $A^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ and $B^T = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$.
Notice that A^T is 2×3 and B^T is 2×2. The matrix product $(A^T)(B^T)$ is not defined, because the dimensions are not appropriate for multiplying these matrices. However, the matrix product $(B^T)(A^T)$ *is* defined (this is true whenever the matrix product AB is defined). And even if the dimensions were not wrong (without the reversal), because we have interchanged the roles of rows and columns in taking the transposes of the matrices we must reverse the order of multiplication, in order to be doing the same dot product calculations as we did using the first approach.

$$\text{We get } (B^T)(A^T) = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1+2 & 1+4 & 1+6 \\ 0-1 & 0-2 & 0-3 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 7 \\ -1 & -2 & -3 \end{bmatrix}$$

We see that this matrix is the same as the one found by the first approach. That is, we see that $(B^T)(A^T) = (AB)^T$.

It is equally important to recognize the properties that **do not** hold for matrix operations. Most notably, matrix multiplication is *not commutative*. That is, in general, $AB \neq BA$, even when both of these matrix products are defined and they have the same dimension.

First of all, notice that neither AB nor BA might be possible. (i.e., A and B might not have appropriate dimensions for either product to exist.) And even if one of these matrix products exists, the other may not. As well, if both AB and BA exist, they may have different dimensions, and therefore cannot be equal.

For instance, let A be a 4×2 matrix, B be a 3×2 matrix and C be a 2×3 matrix. Then neither AB nor BA is defined. The matrix product AC is defined, but the matrix product CA is not defined. And although both BC and CB are defined, BC is a 3×3 matrix, while CB is a 2×2 matrix.

However, even if A and B are square matrices of the same order, so that AB and BA are both defined and have the same dimension, it still will not usually be true that these are equal matrices. The next example demonstrates this.

Example 7.14. For $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}$, show that $AB \neq BA$.

Solution:

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1-1 & 2+2 \\ 2-1 & 4+2 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 1 & 6 \end{bmatrix} \\ BA &= \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+4 & 1+2 \\ -1+4 & -1+2 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 1 \end{bmatrix} \neq AB \end{aligned}$$

Let's look at one last example.

Example 7.15. If $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix}$, find ABC.

Solution:
Notice that we have a 2×3 mattrix times a 3×4 matrix times a 4×1 matrix, so the products are defined. Property 5 tells us that in that case, matrix multiplication is associative. That is, it doesn't matter where we put brackets — the answer is the same no matter what, which means that brackets aren't needed. And that also means it doesn't matter in what order we do the multiplications. We could calculate AB and then multiply AB times C, or we could find BC and then multiply A times BC.

We see that AB would be a 2×4 matrix, which can then be multiplied by C. Or BC would be a 3×1 matrix, which allows us to then multiply A by this. So which approach should we take? Well, it doesn't really matter. Let's calculate BC:

$$BC = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} (-1)(2) + 0(1) + 1(0) + 0(2) \\ 1(2) + 1(1) + 2(0) + 1(2) \\ 0(2) + 1(1) + 2(0) + 1(2) \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}$$

And now A times BC:

$$ABC = A(BC) = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 1(-2) + 0(5) + 1(3) \\ 2(-2) + 1(5) + 0(3) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Notice: This calculation involved 3 'dot products' to find BC (each involving 4 scalar products) and then 2 'dot products' to find $A(BC)$ (each involving 3 scalar products) for a total of 5 'dot products' (and 18 scalar multiplications). Finding AB, i.e. multiplying a 2×3 matrix times a 3×4 matrix, would require 8 'dot products' (each involving 3 scalar products) and then finding $(AB)C$, i.e. multiplying a 2×4 matrix times a 4×1 matrix, would require 2 more 'dot products' (each involving 4 scalar products) and so a total of 10 'dot products' (involving 32 scalar products) would be required. By eliminating the largest dimension first, we have significantly reduced the amount of arithmetic required (and thereby reduced the opportunities for making arithmetic mistakes).

Math 1229A/B

Unit 8:

The Inverse of a Matrix

(text reference: Section 3.2)

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8 The Inverse of a Matrix

Now that we know how matrix multiplication works, we can better understand where the matrices we used in solving systems of linear equations came from.

Suppose we have a system of m equations in the n unknowns $x_1, x_2, ..., x_n$. Let a_{ij} be the coefficient of the j^{th} variable in the i^{th} equation. Call the right hand side value of the i^{th} equation b_i. So we have the SLE:

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & \dots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & \dots & + & a_{mn}x_n & = & b_m \end{array}$$

Now, let $A = [a_{ij}]$ be the coefficient matrix for this SLE. Also, let X be a column vector whose $(j, 1)$-entry is simply the unknown x_j, i.e. X just looks like a vertical list of the unknowns. And let B be a column vector whose entries are the right hand side values, so that the $(i, 1)$-entry of B is b_i. Notice that X is an $n \times 1$ matrix and B is an $m \times 1$ matrix.

Consider the matrix product AX. We have an $m \times n$ matrix times an $n \times 1$ matrix, so this product is defined, and the product matrix will be an $m \times 1$ matrix. And because of the way matrix multiplication works, this product matrix is:

$$\begin{aligned} AX &= \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} \\ &= \begin{bmatrix} a_{11}x_1 & + & a_{12}x_2 & + & \dots & \dots & + & a_{1n}x_n \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & \dots & + & a_{2n}x_n \\ \vdots & & \vdots & & \vdots & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & \dots & + & a_{mn}x_n \end{bmatrix} \end{aligned}$$

Why look at that! AX is a column vector whose entries are the left hand sides of the equations. And so if we write a matrix equation $AX = B$, this simply says that the i^{th} entry of column vector AX must be equal to the i^{th} entry of column vector B, i.e. it says that for each equation in the system, the left hand side of the equation must equal the right hand side of the equation. That is, the matrix equation $AX = B$ exactly represents the SLE. We have:

$$\begin{aligned} AX = B \quad &\Rightarrow \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \\ &\Rightarrow \begin{bmatrix} a_{11}x_1 & + & a_{12}x_2 & + & \dots & \dots & + & a_{1n}x_n \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & \dots & + & a_{2n}x_n \\ \vdots & & \vdots & & \vdots & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & \dots & + & a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \end{aligned}$$

When we write the augmented matrix $[A|B]$, which is the coefficient matrix of the SLE with the column vector of right hand side values appended as an extra column, it is just a form of short-hand for this matrix equation $AX = B$. We assume that we know what the unknowns are, and so the interesting (i.e. important) parts of the equation are the coefficient entries in A and the right hand side values in B. It's easy to see how

$$\left[\begin{array}{cccccc|c} a_{11} & a_{12} & \cdots & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn} & b_m \end{array}\right] \text{ is shorthand for } \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Consider, for instance, the SLE from Example 6.9 back in Unit 6 (page 84). The system is:

$$\begin{array}{rcrcrcr} 3x & + & 3y & + & 12z & = & 6 \\ x & + & y & + & 4z & = & 2 \\ 2x & + & 5y & + & 20z & = & 10 \\ -x & + & 2y & + & 8z & = & 4 \end{array}$$

We can represent this SLE as a matrix equation:

$$\begin{bmatrix} 3 & 3 & 12 \\ 1 & 1 & 4 \\ 2 & 5 & 20 \\ -1 & 2 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 10 \\ 4 \end{bmatrix}$$

which says that:

$$\begin{bmatrix} 3x & + & 3y & + & 12z \\ x & + & y & + & 4z \\ 2x & + & 5y & + & 20z \\ -x & + & 2y & + & 8z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 10 \\ 4 \end{bmatrix}$$

and this matrix equation can be written in short-hand as the augmented matrix:

$$\left[\begin{array}{rrr|r} 3 & 3 & 12 & 6 \\ 1 & 1 & 4 & 2 \\ 2 & 5 & 20 & 10 \\ -1 & 2 & 8 & 4 \end{array}\right]$$

Notice that when we solve the SLE $AX = B$, the solution is a set of values for the unknowns in the column vector X. When we state a solution to a system of equations, it is usually more convenient to express it as a vector, rather than as a column vector. That is, if the solution to an SLE involving x, y and z is $x = 1$, $y = 2$ and $z = 3$, we generally write this as

$$(x, y, z) = (1, 2, 3) \quad \text{rather than} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Similarly, when we found the solution to the SLE shown above, in Example 6.9, we wrote it as $(x, y, z) = (0, 2 - 4t, t)$, and it makes sense that we would continue to do so, rather than having to write it as $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 2-4t \\ t \end{bmatrix}$. (See how messy that looks when we need to mention it in a paragraph! The vector form simply fits more easily.)

Because of this practice of writing an $n \times 1$ column vector in the form of an n-vector, we often give column vectors names using vector notation, rather than using matrix notation. That is, rather than talking about the column vectors X and B, we can refer to these as the vectors $\vec{x}$ and $\vec{b}$. So we usually write the matrix equation form of an SLE as $A\vec{x} = \vec{b}$, rather than as $AX = B$. This is the convention which we will use from now on. However, you should always keep in mind that when we write $A\vec{x} = \vec{b}$ for a system of m equations in n variables, since A is (as always) the $m \times n$ coefficient matrix, $\vec{x}$, the vector of variables, is really (i.e. represents) an $n \times 1$ column vector, and $\vec{b}$, the vector of right hand side values, is actually an $m \times 1$ column vector, so that this equation makes sense mathematically. (That is, "matrix times vector" is not defined. The vector is really a column vector, i.e. a matrix.)

> **Definition:** Any SLE involving m equations and n variables can be represented by the **matrix form** of the SLE $A\vec{x} = \vec{b}$, where A is the $m \times n$ coefficient matrix, $\vec{x}$ is the vector (technically, column vector) of the unknowns and $\vec{b}$ is the vector (technically, column vector) of right hand side values. Both $\vec{x}$ and $\vec{b}$ can be written out either as column vectors or as n- or m-vectors, whichever is most convenient in the present context.

This means that solving a system of linear equations is equivalent to solving the matrix equation $A\vec{x} = \vec{b}$ for the vector $\vec{x}$. When we have a single equation in one unknown, of the form $ax = b$, where x is a variable and a and b are scalars, we can solve this easily by multiplying both sides of the equation by the *inverse* of a, a^{-1}, to get $x = a^{-1}b$. That is, you may think of this as dividing both sides of the equation by a, but we can also express it as multiplying by $\frac{1}{a}$, which can also be written as a^{-1} and is called the *multiplicative inverse* of a. The equation $A\vec{x} = \vec{b}$ looks very much like $ax = b$, but it is much more complicated than it looks. We have more than one unknown, disguised as the vector $\vec{x}$, and there are several equations, hiding in the matrices A and $\vec{b}$. And yet ... it is *sometimes* possible to do something like $x = a^{-1}b$ to find the solution to the matrix equation $A\vec{x} = \vec{b}$. To do this, we need to define the *inverse* of a matrix.

> **Definition:** Let A be a square matrix. If there exists a matrix B with the same dimensions as A such that
> $$AB = BA = I$$
> then we say that A is **invertible** (or **nonsingular**) and that B is the **inverse** of A, written $B = A^{-1}$. If A has no inverse (i.e., if no such matrix B exists), then A is said to be **noninvertible** (or **singular**).

Notice: Only **square** matrices can have inverses.

Example 8.1. For $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$, show that B is the inverse of A.

Solution:
We must show that $AB = BA = I$. Notice that here, since AB and BA are both 2×2 matrices, I means I_2. We first calculate AB:

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} (1)(-2) + (2)\left(\frac{3}{2}\right) & (1)(1) + (2)\left(-\frac{1}{2}\right) \\ (3)(-2) + (4)\left(\frac{3}{2}\right) & (3)(1) + (4)\left(-\frac{1}{2}\right) \end{bmatrix} \\ &= \begin{bmatrix} -2+3 & 1-1 \\ -6+6 & 3-2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \end{aligned}$$

So we do indeed have $AB = I$, as required. Also, when we calculate BA we get:

$$\begin{aligned} BA &= \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} (-2)(1)+(1)(3) & (-2)(2)+(1)(4) \\ \left(\frac{3}{2}\right)(1)+\left(-\frac{1}{2}\right)(3) & \left(\frac{3}{2}\right)(2)+\left(-\frac{1}{2}\right)(4) \end{bmatrix} \\ &= \begin{bmatrix} -2+3 & -4+4 \\ \frac{3}{2}-\frac{3}{2} & 3-2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \end{aligned}$$

Therefore we do have $AB = BA = I$, and so $B = A^{-1}$. (*Notice:* This also means that A is the inverse of B, i.e. that $A = B^{-1}$.)

Do you suppose it's possible for a matrix to have more than one inverse? Could there be some other, different, matrix C such that if we find AC and CA for the matrix A in the example, we once again get I? Well, no. Just as $\frac{1}{a}$ is the only number for which $a\left(\frac{1}{a}\right) = 1$, so that the number a has only one inverse, it is also true that a matrix cannot have more than one inverse.

Theorem 8.1. *If A is invertible then its inverse is unique.*

Proof: Let A be any invertible square matrix. Suppose that matrix B is an inverse of A. And suppose that C is also an inverse of A. We will show that it must be true that $B = C$.

We know that A is a square matrix, as it must be to be invertible. Let n be the order of the square matrix A. Then B and C must also be square matrices of order n. Of course, the $n \times n$ matrix B can be multiplied by the identity matrix of order n, I_n. This matrix multiplication leaves the matrix unchanged (by property 6 of Theorem 7.1 in Unit 7, page 106). That is, we have:

$$B = BI_n$$

However, since C is an inverse of A, then by definition $AC = I_n$, that is $I_n = AC$, so we can write:

$$BI_n = B(AC)$$

Now, we know that matrix multiplication is associative (by property 5 of the same Theorem). This means that

$$B(AC) = (BA)C$$

But since B is also an inverse of A, then, again using the definition, $BA = I_n$ and so we have:

$$(BA)C = I_nC$$

Finally, we know that multiplying the square matrix C by the identity matrix of the same order leaves the matrix unchanged (as before), so we get:

$$I_nC = C$$

Putting this all together, we have:

$$B = BI_n = B(AC) = (BA)C = I_nC = C$$

and we see that, in fact, $B = C$. That is, we see that in order for B and C to both be inverses of A, they must be the *same* matrix, so any invertible matrix has a unique inverse.

There is another very useful theorem, which tells us that whenever $AB = I$ is true, it must also be true that $BA = I$, so we don't have to check. For instance, in Example 8.1 we only really needed to compute one of AB or BA to prove that B is the inverse of A.

Theorem 8.2. *Let A be a square matrix. If a square matrix B exists with $AB = I$, then it must always be the case that $BA = I$ as well, so that in fact $B = A^{-1}$.*

Procedure for Finding the Inverse of any non-singular square matrix A

At this point, we know what it means to say that a particular matrix is the inverse of some square matrix. And we know how to tell whether some particular (square) matrix is the inverse of some other particular (square) matrix. But suppose we have some matrix, and perhaps we even know that it is invertible (also known as nonsingular), that is we know that it *has* an inverse ... But how do we *find* the inverse matrix? Well, we can do it by setting up a particular augmented matrix (bigger than the ones we've used before) and row-reducing. If we want to find the inverse of some square matrix A of order n, then we set up an $n \times 2n$ augmented matrix which has all of the columns of A on the left side, and all of columns of the identity matrix I_n on the right side, with the line coming between the two sets of columns. (So we've just jammed A and I_n together, with the right bracket of A and the left bracket of I_n fused together to make the line.) And then we bring this matrix to RREF. When we're done, if the matrix A on the left has been transformed into the matrix I_n, which used to be on the right, then the matrix which is now on the right is A^{-1}. (Voila! It's magic!)

Note: The text contains an explanation of *why* this procedure finds the inverse, when it exists, in their Example 3 of this section. You should have a look at that (pages 103 - 104).

Procedure for finding the inverse of a matrix:
Let A be any square matrix and let I be the identity matrix of the same order as A. Form the augmented matrix $[A|I]$ and transform it to row-reduced echelon form. Let $[C|D]$ be the final augmented matrix in RREF.

Then

1. if $C = I$ then $D = A^{-1}$
2. if C is not the identity matrix, then A is not invertible, i.e., is singular.

That is, *if* A is invertible, we get:

$$[A \mid I] \xrightarrow{\text{ero's to get to RREF}} [I \mid A^{-1}]$$

But if A is singular then the left side of the RREF augmented matrix isn't I. That's how we can tell that A has no inverse.

Notice: This procedure does more than just *find* the inverse of the matrix, if it exists. It also gives us a way of knowing that no inverse exists, when we apply the procedure to a singular matrix.

Example 8.2. Use this procedure to find the inverse of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Solution:
We set up the augmented matrix, writing the columns of A, then a line, then the columns of I_2:

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]$$

And now we perform elementary row operations to bring this matrix to RREF. Row 1 already has a leading 1, so we clear out the rest of that column, and then get a leading one in row 2 and clear out that column, too.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - 3R_1} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right]$$
$$\xrightarrow{R_2 \leftarrow -\frac{1}{2}R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right]$$
$$\xrightarrow{R_1 \leftarrow R_1 - 2R_2} \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right]$$

The matrix is now in RREF. We see that the left side of the matrix does contain the columns of I_2, so matrix A is invertible and the columns in the right half of the matrix are the columns of A^{-1}. Therefore

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

Ordinarily we would check our arithmetic by finding the product AA^{-1} to confirm that we do get I_2. However the inverse we obtained in this case *is* the matrix that we already saw, in Example 8.1, is the inverse of this matrix A, so we don't need to check that again.

Example 8.3. Find A^{-1}, if it exists, for each matrix A.

$$\text{(a)} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 1 & 2 \end{bmatrix} \qquad\qquad (b) \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$$

Solution:
(a) We wish to find the inverse of a 3×3 matrix, so we start by forming the augmented matrix obtained by appending the 3×3 identity matrix. Then we row-reduce this augmented matrix to bring it to RREF.

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R3 \leftarrow R_3 - 2R_1]{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 1 \end{array}\right]$$
$$\xrightarrow[R3 \leftarrow R_3 + R_2]{R_1 \leftarrow R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 2 & -3 & 1 & 1 \end{array}\right]$$
$$\xrightarrow{R_3 \leftarrow \frac{1}{2}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$
$$\xrightarrow[R_2 \leftarrow R_2 - 2R_3]{R_1 \leftarrow R_1 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & -\frac{3}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

This matrix is now in RREF. We see that the matrix A (i.e. the left side of the augmented matrix) has been transformed into the 3×3 identity matrix. This tells us that A is invertible and that the columns on the right side of the RREF augmented matrix are the columns of A^{-1}. Thus we see that

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 2 & 0 & -1 \\ -\frac{3}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

We should check that we didn't make any arithmetic mistakes:

$$\begin{aligned}
AA^{-1} &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 2 & 0 & -1 \\ -\frac{3}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2}+2-\frac{3}{2} & -\frac{1}{2}+0+\frac{1}{2} & \frac{1}{2}-1+\frac{1}{2} \\ \frac{1}{2}+4-\frac{9}{2} & -\frac{1}{2}+0+\frac{3}{2} & \frac{1}{2}-2+\frac{3}{2} \\ 1+2-3 & -1+0+1 & 1-1+1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

Since the product of the given matrix A times the matrix we think is the inverse of A *does* give I_3, our inverse matrix is correct.

(b) Again, we form the augmented matrix $[A \mid I]$ and row-reduce it:

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - 2R_1]{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 1 & 2 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \leftarrow R_3 - R_2]{R_1 \leftarrow R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right]$$

This last matrix is not yet in RREF. However, we can see that the matrix on the left is not going to become I_3. The bottom row of the columns on the left side of the matrix has only 0's. If we continue performing ero's, we will next get a leading one in row 3, and it will be way over on the right, in the fourth column of the matrix. We'll get it by changing the signs in row 3, but that doesn't affect the 0's in the left half of row 3. When we clear out the rest of column 4, we'll be adding scalar multiples of row 3 to the other rows. So in column 3, we'll be adding 0's and nothing will change. And it certainly won't affect the fact that $(3,3)$-entry will be a 0, not a 1. So as previously stated, we're not going to end up with the columns on the left forming the 3×3 identity matrix. Even without finishing bringing this augmented matrix to RREF, we can stop, and conclude that A is singular, i.e. is not invertible. Therefore A^{-1} does not exist in this case, so A is singular. (Notice that when we stopped, the LHS of the augmented matrix was in RREF, and since that RREF was not I_3, we could see that A is singular.)

The Method of Inverses

Now, let's return to the problem of solving a system of linear equations. We have seen that we can represent any SLE by the matrix equation $A\vec{x} = \vec{b}$, where A is the $m \times n$ coefficient matrix of the SLE, $\vec{x}$ is actually an $n \times 1$ column vector, containing the unknowns from the SLE, and $\vec{b}$ is really another column vector, $m \times 1$ this time, which contains the right hand side values of the SLE. (Recall that we're now using b_i to denote the RHS value of equation i.)

In the particular situation where the coefficient matrix A happens to be an invertible square matrix, we can use the inverse of this matrix, A^{-1}, to easily solve the SLE. Consider multiplying each side of the matrix equation by this inverse matrix. In order to make the dimensions be right for this multiplication to be defined, we *pre-multiply* by A^{-1}, by which we mean that A^{-1} is the *first* matrix in the product. (As opposed to *post-multiplying* by a matrix, which means that the specified matrix is the second term in the matrix product.)

So we take our matrix equation that represents the SLE, and we write A^{-1} at the beginning of the LHS, and also at the beginning of the RHS. When we do this, we get:

$$\begin{aligned} A\vec{x} = \vec{b} \quad &\Rightarrow \quad A^{-1}(A\vec{x}) = A^{-1}(\vec{b}) \\ &\Rightarrow \quad (A^{-1}A)\vec{x} = A^{-1}\vec{b} \\ &\Rightarrow \quad I\vec{x} = A^{-1}\vec{b} \\ &\Rightarrow \quad \vec{x} = A^{-1}\vec{b} \end{aligned}$$

Notice: In an $m \times n$ SLE with a square coefficient matrix, we have A (which must have m rows) being an $m \times m$ matrix, and therefore so is A^{-1} (when it exists). And the vector of right hand side values, $\vec{b}$, is an $m \times 1$ column vector. So the matrix product $A^{-1}\vec{b}$ involves multiplying an $m \times m$ matrix times an $m \times 1$ matrix, and therefore (in this situation) this product is always defined.

This means that if we can find A^{-1}, we are able to solve the system simply by multiplying A^{-1} times $\vec{b}$. That is, the *method* we can use in this situation is:

The Method of Inverses
Given a SLE in which the coefficient matrix A is an invertible square matrix, if the inverse matrix A^{-1} is known, the solution to the SLE can be found as:

$$\vec{x} = A^{-1}\vec{b}$$

Remember: The method of inverses can **only** be used when A is a *square invertible* matrix. If the coefficient matrix A is not a square matrix, or is singular, then we *cannot* use this approach to find $\vec{x}$.

Example 8.4. Solve the following system using the method of inverses.

$$\begin{array}{ccccccc} x & + & y & + & z & = & 4 \\ x & + & 2y & + & 3z & = & 6 \\ 2x & + & y & + & 2z & = & 5 \end{array}$$

Solution:
The SLE can be represented as a matrix equation as:

$$A\vec{x} = \vec{b} \qquad \Rightarrow \qquad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 5 \end{bmatrix}$$

We have a coefficient matrix A which is a square matrix. Also, this is the same matrix whose inverse we found in Example 8.3(a). So we know that A is invertible, and that its inverse is

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 2 & 0 & -1 \\ -\frac{3}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Therefore the solution to this SLE is

$$\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 2 & 0 & -1 \\ -\frac{3}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 4 \\ 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 2-3+\frac{5}{2} \\ 8+0-5 \\ -6+3+\frac{5}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 3 \\ -\frac{1}{2} \end{bmatrix}$$

That is, the solution is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 3 \\ -\frac{1}{2} \end{bmatrix}$, which can be written as: $(x, y, z) = \left(\dfrac{3}{2}, 3, -\dfrac{1}{2}\right)$.

Notice: For any SLE for which the coefficient matrix A is square and invertible, A^{-1} is unique and therefore so is $A^{-1}\vec{b}$. This means that any system of linear equations for which the method of inverses can be used must have a unique solution. That is, if the coefficient matrix of an SLE is a non-singular square matrix, then the SLE has a unique solution.

Example 8.5. Show that the system

$$\begin{array}{rcrcrcl} x & + & y & + & z & = & b_1 \\ x & + & 2y & + & 3z & = & b_2 \\ x & + & 2y & + & 4z & = & b_3 \end{array}$$

has a unique solution for *any* values of b_1, b_2 and b_3.

Solution:
The coefficient matrix for this system is:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}$$

We try to find A^{-1}, to see if this matrix is invertible:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 1 & 3 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \leftarrow R_3 - R_2]{R_1 \leftarrow R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 \leftarrow R_2 - 2R_3]{R_1 \leftarrow R_1 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -2 & 1 \\ 0 & 1 & 0 & -1 & 3 & -2 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

We see that A *is* invertible, with $A^{-1} = \begin{bmatrix} 2 & -2 & 1 \\ -1 & 3 & -2 \\ 0 & -1 & 1 \end{bmatrix}$. But then, no matter what the vector $\vec{b} = (b_1, b_2, b_3)$ is, the SLE must have a unique solution, because we can find that unique solution using:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\vec{b} = \begin{bmatrix} 2 & -2 & 1 \\ -1 & 3 & -2 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 2b_1 - 2b_2 + b_3 \\ -b_1 + 3b_2 - 2b_3 \\ -b_2 + b_3 \end{bmatrix}$$

For instance, if we have $b_1 = b_2 = b_3 = 1$ (i.e. when the right hand side values are all 1), we find that the unique solution to the SLE is $(x, y, z) = (2(1) - 2(1) + 1, -(1) + 3(1) - 2(1), -(1) + 1) = (1, 0, 0)$. Similarly, we can find the unique solution for *any* values of b_1, b_2 and b_3.

Example 8.6. For what value(s) of c does the following SLE *not* have a unique solution?

$$\begin{array}{ccccccc} x & + & y & + & z & = & 1 \\ x & + & 2y & + & 3z & = & 1 \\ x & + & 2y & + & cz & = & 1 \end{array}$$

Solution:
We know that if the coefficient matrix, which is square, is invertible, then the system must have a unique solution. So the only time that this system will *not* have a unique solution is when the coefficient matrix has no inverse. We need to determine what value or values of c make the coefficient matrix singular. We do this by trying to find the inverse.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 2 & c & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 1 & c-1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \leftarrow R_3 - R_2]{R_1 \leftarrow R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & c-3 & 0 & -1 & 1 \end{array}\right]$$

As long as $c - 3 \neq 0$, i.e. as long as $c \neq 3$, we can multiply row 3 by $\dfrac{1}{c-3}$ to obtain a leading 1 in row 3, and then we will be able to clear out the rest of column 3 and get the columns of I_3 on the left side of the augmented matrix. Therefore for any $c \neq 3$, the coefficient matrix is invertible and therefore the SLE has a unique solution.

However, if $c = 3$, then the left side of the bottom row of the matrix is all 0's and therefore when we finish bringing the matrix to RREF, the left side of the matrix will not contain the columns of an identity matrix, meaning the the coefficient matrix has no inverse. So it is only in the case $c = 3$ that the SLE can fail to have a unique solution.

Notice that the ero's performed to bring the matrix $[A \mid I]$ to RREF are the same (up to the point at which the cooefficient matrix A is in RREF, at least) as those which would be performed to bring the augmented matrix of the SLE, $[A \mid \vec{b}]$ to RREF. And so when we solve the system, we will again have the bottom row on the left of the augmented matrix containing only 0's. Therefore column 3 of the RREF of the augmented matrix will not contain the leading one for any row. If the RHS value in the bottom row of the augmented matrix is not also 0, the system will be inconsistent and have no solution. But if the RHS value in the bottom row of the augmented matrix *is* also 0 (which will be the case for the given SLE), then the system will have a one-parameter family of solutions, i.e. will have infinitely many solutions.

Either way, when $c = 3$ the given SLE *does not* have a unique solution. But for any other value of c, it does. So $c = 3$ is the only such value.

Math 1229A/B

Unit 9:
Theory of SLE's

(text reference: Section 3.3)

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9 Theory of SLE's

We have seen that any system of linear equations can be written as a matrix equation, i.e. a statement about certain matrices. This means that characteristics of SLE's are very closely related to certain characteristics of matrices. Now that we know quite a bit about matrices, we can use what we know, and some other things that we'll learn in this unit, to recognize various properties of systems of linear equations. (In the units yet to come, we'll learn more about certain matrices, which we will relate to SLE's, too.)

We have seen examples of SLE's which have no solution, and others which have a unique solution, as well as some which have infinitely many solutions. We asserted before that these were the only possibilities. Now that we know how to express an SLE as a matrix equation, and how to perform matrix operations, and know the properties of those operations, we can actually **prove** that there are no other possibilities.

Reminder: In the following theorem, and throughout this unit, it is important to remember that when we express a system of linear equations as $A\vec{x} = \vec{b}$, the "vectors" $\vec{x}$ and $\vec{b}$ are actually *column vectors*. That is, they are really *matrices*, which happen to have only one column, so it's more convenient to express them as vectors. But they have the characteristics of matrices, *not* vectors, and the operations we perform on them are *matrix operations*. And of course it's also important to remember what the dimensions of the various matrices in the equation are. For a system of m equations in n unknowns, in the corresponding matrix equation $A\vec{x} = \vec{b}$, A is an $m \times n$ matrix, $\vec{x}$ is an $n \times 1$ matrix and $\vec{b}$ is an $m \times 1$ matrix. (In particular, keep in mind that $\vec{x}$ and $\vec{b}$ do not necessarily have the same dimensions, even though both are column vectors.)

Theorem 9.1. *Any system of linear equations which has more than one solution must have infinitely many solutions, i.e. must have a parametric family of solutions.*

Proof:
Let $A\vec{x} = \vec{b}$ be any SLE with m equations in n unknowns (so that A is $m \times n$, $\vec{x}$ is $n \times 1$ and $\vec{b}$ is $m \times 1$). Suppose that the $n \times 1$ column vectors $\vec{x}_1$ and $\vec{x}_2$, with $\vec{x}_1 \neq \vec{x}_2$, are both solutions to this system. That is, suppose that $\vec{x}_1$ and $\vec{x}_2$ are two *different* solutions to the SLE. Then $A\vec{x}_1 = \vec{b}$ and also $A\vec{x}_2 = \vec{b}$.

Consider some $\vec{w} = (1-t)\vec{x}_1 + t\vec{x}_2$ for any $t \in \Re$. Then $\vec{w}$ is an $n \times 1$ column vector, so the matrix product $A\vec{w}$ is defined, and we have:

$$
\begin{aligned}
A\vec{w} &= A\left[(1-t)\vec{x}_1 + t\vec{x}_2\right] && \text{(from our definition of } \vec{w}\text{)}\\
&= A(1-t)\vec{x}_1 + At\vec{x}_2 && \text{(by property 7 of Theorem 7.1)}\\
&= (1-t)A\vec{x}_1 + tA\vec{x}_2 && \text{(by property 14 of Theorem 7.1)}\\
&= (1-t)\vec{b} + t\vec{b} && \text{(because } A\vec{x}_1 = \vec{b} \text{ and } A\vec{x}_2 = \vec{b}\text{)}\\
&= \vec{b} - t\vec{b} + t\vec{b} && \text{(by property 9 of Theorem 7.1)}\\
&= \vec{b} && \text{(because we were adding and subtracting the same thing)}
\end{aligned}
$$

That is, we see that $A\vec{w} = \vec{b}$, so $\vec{w}$ is also a solution to $A\vec{x} = \vec{b}$.

We see that whenever the system $A\vec{x} = \vec{b}$ has two different solutions $\vec{x}_1$ and $\vec{x}_2$, it also has solutions of the form $\vec{w} = (1-t)\vec{x}_1 + t\vec{x}_2$ for any real value of t. And since $\vec{x}_1$ and $\vec{x}_2$ are different solutions, then these other solutions are all different from one another. And there are infinitely many of them, so $A\vec{x} = \vec{b}$ has infinitely many solutions.

Since the 3 possibilities mentioned above *are* possible, and this theorem proves that no other possibilities exist, then the following result follows directly from this theorem. (*Note: Corollary* means a result which follows directly from a result already proven.)

Corollary 9.2. *For any system of linear equations there are exactly 3 possibilities:*

- *the SLE may have no solution,*
- *the SLE may have a unique solution, or*
- *the SLE may have infinitely many solutions.*

There is a particular characteristic which a matrix has, and which we have not yet defined, which tells us a lot about how many solutions a system of linear equations can have when that matrix is the coefficient matrix of the system. This characteristic is determined by the RREF of the matrix.

> **Definition:** The **rank** of a matrix is the number of non-zero rows in the row-reduced echelon form of the matrix. We can use $r(A)$ to denote the rank of matrix A. Also, we say that the $m \times n$ matrix A **has full rank** if $r(A) = n$, i.e. if every column of the RREF of A contains the leading one for some row.

Of course, this means that if we want to find the rank of some matrix A, we need to find the RREF of A. And then we just count the number of non-zero rows. That number is the value of $r(A)$.

Example 9.1. Find the rank of each of the following matrices:

(a) $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -2 & -3 & 4 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$ (c) $\left[A \mid \vec{b}\right] = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 3 & 3 & 5 \end{array}\right]$

Solution:

We row-reduce each of the given matrices to get the RREF matrix. (You've had lots of practice with row-reducing by now, so the details of the reduction are not shown here. Of course, you should feel free to check that the RREF matrices shown here are correct.)

(a) $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -2 & -3 & 4 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

The RREF of A does not contain any zero rows (i.e. rows containing only zeroes), so there are 3 non-zero rows. Therefore $r(A) = 3$, and since A has 3 columns, A has full rank.

(b) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

This time, there *are* some zero rows in the RREF of A. In fact, the RREF of A has only one non-zero row, so $r(A) = 1$. (And since A has more than 1 column, A does not have full rank.)

(c) $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 3 & 3 & 5 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$

Notice that the given matrix, whose rank we are asked to find, is the augmented matrix, $\left[A \mid \vec{b}\right]$, *not* the matrix A which is embedded in this augmented matrix. The RREF of $\left[A \mid \vec{b}\right]$ has 3 non-zero rows, and so we have $r\left(\left[A \mid \vec{b}\right]\right) = 3$. And since the augmented matrix has 3 columns, this matrix has full rank.

Notice: From the RREF of the augmented matrix, we can also see that the matrix $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{bmatrix}$ has RREF $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, so $r(A) = 2$. And since A has only 2 columns, A also has full rank.

Since the RREF of a matrix has the same number of rows and columns as the original matrix, then clearly the rank of a matrix can never be larger than the number of rows in the matrix. But also, the RREF of a matrix cannot contain more leading ones than there are columns for them to occur in (since each row's leading one must be in a *different* column), and any row which does not have a leading one must contain only zeroes. And this means that the rank of the matrix also cannot be larger than the number of columns in the matrix. (It also means that we can think of the rank of a matrix as *the number of leading ones* in the RREF of the matrix.) So for any $m \times n$ matrix A, it must be true that $r(A) \leq m$ and also that $r(A) \leq n$ (and so $r(A)$ must be less than or equal to the *smaller* of m and n).

When we solve a system of linear equations, using elementary row operations to bring the augmented matrix to RREF (or at least to the point where the coefficient matrix is in RREF), we also look at the number and positioning of the leading ones to see what solutions the system has. And so the number of solutions that a system has is related to the rank – both the rank of the coefficient matrix and the rank of the augmented matrix. Let's think about the possibilities one at a time.

If the RREF of the coefficient matrix has at least one zero row, and at least one of those rows has a non-zero in the extra column, then we know that means that the SLE has no solution. Now think about: what does this mean about the rank of the coefficient matrix, and the rank of the augmented matrix? The final augmented matrix, in which the coefficient matrix is in RREF, has more non-zero rows than the RREF of the coefficient matrix has. (And if we bring the whole augmented matrix to RREF, it will have exactly one more non-zero row than the RREF of the coefficient matrix has.) So the rank of the augmented matrix is larger than the rank of the coefficient matrix.

If that's not the case, then clearly the final augmented matrix has the same number of non-zero rows as the RREF of the coefficient matrix (and the whole augmented matrix is in RREF) so this means that the rank of the augmented matrix is the same as the rank of the coefficient matrix. But if some column of the RREF of the coefficient matrix does not contain the leading one for any row, then we must introduce a parameter for the variable corresponding to that column and so the system has infinitely many solutions. (There may be more than one such column, but for now we'll just characterise the SLE as having infinitely many solutions, no matter how many parameters are needed to express those solutions.) In terms of rank, if there's a column of the RREF of the coefficient matrix which does not contain the leading one for any row, that means that the rank of the coefficient matrix is less than the number of columns of the coefficient matrix (i.e. the number of unknowns in the system).

On the other hand, if we don't have "no solution" and we don't have "infinitely many solutions", then the system must have a unique solution. That is, if we don't find that the RREF of the coefficient matrix has more zero rows than the final augmented matrix has (and therefore, as already observed, the rank of the coefficient matrix is the same as the rank of the augmented matrix), and every column of the RREF of the coefficient matrix contains the leading one for some row, then the rank of the coefficient matrix is equal to the number of columns it contains (i.e. the number of variables in the SLE). And the rank of the augmented matrix is the same.

All of this means that knowing *only* the rank of the coefficient matrix and the rank of the augmented matrix for some SLE, we can tell how many solutions the SLE has. We summarize our findings in the following theorem.

Theorem 9.3. *Consider any SLE with m equations in n unknowns. Let A be the coefficient matrix of the system, and $\left[A \mid \vec{b}\right]$ be the augmented matrix for the system. Then the SLE $A\vec{x} = \vec{b}$ has:*

- *no solution if $r(A) < r\left(\left[A \mid \vec{b}\right]\right)$,*
- *a unique solution if $r(A) = r\left(\left[A \mid \vec{b}\right]\right) = n$, or*
- *infinitely many solutions if $r(A) = r\left(\left[A \mid \vec{b}\right]\right)$ and $r(A) < n$.*

Example 9.2. Determine how many solutions $A\vec{x} = \vec{b}$ has in each of the following.
(a) A is a 3×4 matrix with $r(A) = 3$ and $r\left(\left[A \mid \vec{b}\right]\right) = 3$.

(b) A is a 4×3 matrix with $r(A) = 3$ and $r\left(\left[A \mid \vec{b}\right]\right) = 4$.

(c) A is a 4×3 matrix with $r(A) = 3$ and $r\left(\left[A \mid \vec{b}\right]\right) = 3$.

Solution:
(a) Since $r(A) = r\left(\left[A \mid \vec{b}\right]\right) = 3$, but A has 4 columns, so that the system has 4 unknowns, i.e. $n = 4$, then we have $r(A) = r\left(\left[A \mid \vec{b}\right]\right) < n$ and so the system has infinitely many solutions.

(Since $r(A) = r\left(\left[A \mid \vec{b}\right]\right)$, then the RREF of A has the same number of non-zero rows as the RREF of the augmented matrix, so there is no row in the final augmented matrix in which there are only 0's in the coefficient matrix part, and a non-zero in the extra column, so the system is consistent. And since $r(A) = 3$ is the number of leading ones in the RREF of A, but A has 4 columns, then clearly one column of the RREF of A does not contain a leading one for any row, so a parameter must be introduced for the variable corresponding to that column and so the system has infinitely many solutions.)

(b) Since $r(A) = 3 < r\left(\left[A \mid \vec{b}\right]\right) = 4$ then the system has no solution.

(Since $r(A) = 3$ but A has 4 rows, and the rank of A is the number of leading ones in the RREF of A, then clearly the RREF of A has one row which contains only zeroes, i.e. a zero row. But we have $r\left(\left[A \mid \vec{b}\right]\right) = 4$, and this gives the number of leading ones in the RREF of the augmented matrix. Since the coefficient matrix part of the RREF of the augmented matrix is just the RREF of A, and there are only 3 leading ones in the RREF of A, then the fourth leading one must be in the row in which the RREF of A has only zeroes. That is, in order for the rank of the augmented matrix to be 4 when the rank of A is only 3, the last row of the 4×4 augmented matrix $\left[A \mid \vec{b}\right]$ must have only zeroes in the coefficient matrix part, with its leading one in the extra column. And that tells us that the system has no solution.)

(c) Since A is a 4×3 matrix, then the system has only $n = 3$ unknowns. And since $r(A) = r\left(\left[A \mid \vec{b}\right]\right) = 3 = n$, then the system has a unique solution.

(Since $r(A) = 3$ and A has only 3 columns, then every column of the RREF of A contains the leading one for some row. However, A has 4 rows, so clearly the RREF of A has a row of only zeroes. But that row must also have a 0 in the extra column, because the rank of the augmented matrix is the same as the rank of A. Therefore there is not "no solution", and we don't need to introduce any parameters, so the system has a unique solution.)

Now let's think again about a system that has infinitely many solutions, and think about the *number* of columns of the RREF of A which don't contain the leading one for any row, i.e. the *number of parameters* needed to express those infinitely many solutions. We have already seen that for a system with infinitely many solutions, it must be true that the coefficient matrix and the augmented matrix have the same rank, and that this rank is less than the number of variables in the system.

Let n be the number of variables in the SLE (and hence also the number of columns of A) and let $p = r(A)$. Then we know that (since the system has infinitely many solutions) $p < n$. The rank of A is the number of leading ones in the RREF of A, so the RREF of A contains p leading ones, which appear in p different columns. But there are n columns in A, so there must also be $n - p$ columns which don't contain the leading one for any row. And to express the solution set for the system, we need to introduce a parameter for each of these columns. Therefore the system has an $(n-p)$-parameter family of solutions.

This means that if we know the rank of the coefficient matrix, the rank of the augmented matrix and the number of variables in the system, we can not only tell whether the system has infinitely many solutions, but also if it does, we can tell how many parameters are needed to express those solutions.

Theorem 9.4. *Let $A\vec{x} = \vec{b}$ be a system of m linear equations in n unknowns. Let $p = r(A)$. If $p < n$ and $r\left(\left[A \mid \vec{b}\right]\right) = p$ as well, then the system has an $(n-p)$-parameter family of solutions.*

For instance, if we know that $A\vec{x} = \vec{b}$ has 21 variables, where $r(A) = r\left(\left[A \mid \vec{b}\right]\right) = 17$, then the system must have a 4-parameter family of solutions. (That is, since the coefficient matrix and the augmented matrix have the same rank, the system is consistent. And we have $r(A) = 17$, with $n = 21$, so A does not have full rank and (since there are $r(A)$ leading ones in the RREF of A) there are $21 - 17 = 4$ columns of the RREF of A which do not contain the leading one for any row, and so 4 parameters are needed to express the infinitely many solutions to the system.)

So far, we have focussed only on the ranks of the coefficient matrix and the augmented matrix. That is, we have considered $\vec{b}$ only in conjunction with A, in comparing the rank of the augmented matrix to the rank of the coefficient matrix and the number of unknowns in the system. However, there is also one property of the right hand side values, i.e. the column vector $\vec{b}$, which some SLE's have, which can give us some information about the number of solutions a system can have, even without knowing anything at all about the coefficient matrix. We have a special name for a system in which $\vec{b}$ has this property.

Definition: The system of linear equations corresponding to the matrix equation $A\vec{x} = \vec{b}$ is called **homogeneous** whenever $\vec{b} = \vec{0}$, i.e. when the right hand side values are all 0. A system in which $\vec{b} \neq \vec{0}$ is said to be **nonhomogeneous**.

Example 9.3. Characterise each of the following SLE's as either homogeneous or nonhomogeneous.

(a) $\begin{array}{rcrcl} x & + & y & = & 0 \\ 5x & - & 4y & = & 0 \end{array}$ (b) $\begin{array}{rcrcr} x & = & y & - & z \\ z & = & 2y & - & 3x \end{array}$ (c) $\begin{array}{rcrcr} x & + & z & = & y \\ x & = & 2y & + & z \\ x & = & 4 & - & 3y \end{array}$

Solution:
(a) The SLE is in standard form, so we can clearly see that the right hand side values are all 0. Therefore this system is homogeneous.

(b) This time the SLE is not in standard form. If we put it into standard form, it is easier to see the RHS values:

$$\text{in standard form, } \begin{array}{rcrcr} x & = & y & - & z \\ z & = & 2y & - & 3x \end{array} \text{ becomes } \begin{array}{rcrcrcl} x & - & y & + & z & = & 0 \\ 3x & - & 2y & + & z & = & 0 \end{array}$$

Now we can easily see that this SLE is also homogeneous.

(c) Again, we start by putting the SLE into standard form (although you may already see what the answer will be):

$$\text{in standard form, } \begin{array}{rcrcr} x & + & z & = & y \\ x & = & 2y & + & z \\ x & = & 4 & - & 3y \end{array} \text{ becomes } \begin{array}{rcrcrcl} x & - & y & + & z & = & 0 \\ x & - & 2y & - & z & = & 0 \\ x & + & 3y & & & = & 4 \end{array}$$

This time, one of the RHS values is *not* 0, i.e. it is not the case that the RHS values are all 0, so this system is nonhomogeneous.

There is one property which every homogeneous SLE has. We know, from property 12 of Theorem 7.1, that for any matrix A, $A\vec{0} = \vec{0}$ (where, if A is $m \times n$, the first zero column vector mentioned is $n \times 1$ and the second is $m \times 1$, of course). But then that means that for any $m \times n$ matrix A, the $n \times 1$ column vector $\vec{0}$ is a solution to the homogeneous system with coefficient matrix A. So we already know one solution to any homogeneous SLE. That solution, which is so trivial that we already know what it is, without knowing anything about the system other than that it is homogeneous, is called the *trivial solution*.

Definition: In a homogeneous SLE, the solution $\vec{0}$ is called **the trivial solution**. Any other solutions which the system may have are referred to as **nontrivial** solutions.

So we know that any homogeneous SLE has at least one solution – the trivial solution. And that means that one of the 3 possibilities in Corollary 9.2 can't happen for a homogeneous system. That is, a homogeneous system *cannot "have no solution"*!

Theorem 9.5. *Any homogeneous system of linear equations either has exactly one solution, the trivial solution, or else has infinitely many solutions (including the trivial solution).*

It shouldn't be too hard for you to realize that in a homogeneous system, the rank of the augmented matrix is *always* the same as the rank of the coefficient matrix. That is, $r\left(\left[A \mid \vec{0}\right]\right) = r(A)$. The RREF of the augmented matrix cannot have more leading ones than the RREF of the coefficient matrix, because there cannot be a leading one in the extra column. No matter what elementary row operations we perform, the RHS values in the augmented matrix of a homogeneous system are always all zeroes. As we perform ero's we may move those 0's around, multiply them by non-zero constants, or add a multiple of one 0 to another, but none of that will make any of them be anything but 0.

This eliminates one of the situations in Theorem 9.3 when we're dealing with a homogeneous SLE. Of course it does, because we have eliminated one of the possibilities from Corollary 9.2, and the 3 situations in Theorem 9.3 exactly correspond to the 3 possibilities from Corollary 9.2. Thus we have a Corollary to Theorem 9.3 for the situation in which the SLE is homogeneous. And we might as well incorporate the number of parameters into this, as well, from Theorem 9.4.

Corollary 9.6. *For any $m \times n$ matrix A with rank $r(A) = p$, the homogeneous system $A\vec{x} = \vec{0}$ has:*

- *a unique solution if $p = n$, or*
- *an $(n - p)$-parameter family of solutions if $p < n$.*

You surely already realize, and we may have previously observed, that a system of linear equations in which there are fewer equations than unknowns cannot possibly have a unique solution. In terms of row-reducing, the RREF of a coefficient matrix in which there are more columns than rows cannot have a leading one in every column, because there are at most as many leading ones as there are rows. In terms of rank, we have already commented on the fact that an $m \times n$ matrix A must have $r(A) \leq m$ as well as $r(A) \leq n$, so if $m < n$ we certainly have $r(A) < n$.

In general, knowing that a SLE does not have a unique solution doesn't tell us very much about the number of solutions it does have. We cannot conclude that there are infinitely many solutions, because we cannot eliminate the possibility that the system might have no solution.

But for a homogeneous system, that other possibility is already eliminated. We know that every homogeneous system has at least one solution, and if there are more unknowns than there are equations then there cannot be "only the trivial solution". Therefore there *must* be infinitely many solutions. (Knowing only that the system is homogeneous and that there are more unknowns than equations doesn't tell us exactly how many parameters are needed to express the infinitely many solutions. We cannot find that without row-reducing, either to solve the system or to find the rank of the coefficient matrix, because the RREF of the coefficient matrix may have some zero rows.) This gives us another corollary, for the situation in which the coefficient matrix of a homogeneous system cannot have full rank.

Corollary 9.7. *Any homogeneous SLE in which the number of unknowns is larger than the number of equations has infinitely many solutions.*

This means that *sometimes* we can tell just at a glance how many solutions a system of linear equations has. We use this in the following example.

Example 9.4. If possible, determine how many solutions each of the following SLE's has just from looking at it.

(a) $$\begin{array}{rcrcrcl} 2x & + & 2y & - & 5z & = & 0 \\ 23x & + & 14y & - & z & = & 0 \end{array}$$

(b) $$\begin{array}{rcrcrcl} 2x & + & 2y & - & 5z & = & 0 \\ 23x & + & 14y & - & z & = & 1 \end{array}$$

(c) $$\begin{array}{rcrcrcl} 2x & + & 2y & - & 5z & = & 0 \\ 23x & + & 14y & - & z & = & 0 \\ 11x & - & 32y & + & 14z & = & 0 \end{array}$$

(d) $$\begin{array}{ccccccccccl} x_1 & & & & & + & 3x_4 & + & 5x_5 & - & 2x_6 & = & 1 \\ & & x_2 & & & + & 5x_4 & - & 3x_5 & + & 21x_6 & = & 3 \\ & & & & x_3 & - & 3x_4 & + & 7x_5 & & & = & 7 \end{array}$$

Solution:

(a) The system $\begin{array}{rcrcrcl} 2x & + & 2y & - & 5z & = & 0 \\ 23x & + & 14y & - & z & = & 0 \end{array}$ is homogeneous, with 3 variables and only 2 equations, so it must have infinitely many solutions.

(b) The system $\begin{array}{rcrcrcl} 2x & + & 2y & - & 5z & = & 0 \\ 23x & + & 14y & - & z & = & 1 \end{array}$ is not homogeneous. Since there are 3 variables and only 2 equations, we know that it does not have a unique solution, but we cannot tell how many solutions it does have. There may be no solution, or infinitely many solutions.

(c) The system $\begin{array}{rcrcrcl} 2x & + & 2y & - & 5z & = & 0 \\ 23x & + & 14y & - & z & = & 0 \\ 11x & - & 32y & + & 14z & = & 0 \end{array}$ is homogeneous. But since it does not have more variables than equations, we can't tell whether it has only the trivial solution, or infinitely many solutions.

(d) This one is different. The system

$$\begin{array}{ccccccccccl} x_1 & & & & & + & 3x_4 & + & 5x_5 & - & 2x_6 & = & 1 \\ & & x_2 & & & + & 5x_4 & - & 3x_5 & + & 21x_6 & = & 3 \\ & & & & x_3 & - & 3x_4 & + & 7x_5 & & & = & 7 \end{array}$$

is not homogeneous. However, because of the structure of the equations, we can see that this system has infinitely many solutions. If we were to form the augmented matrix for this system, there would be no work to do in bringing it to RREF, because it would already be there. We can tell just from looking at the system that both the coefficient matrix and the augmented matrix have rank 3, which is less than the number of unknowns (which is 6), and so the system must have a 3-parameter family of solutions. (We can even see that $(1, 3, 7, 0, 0, 0)$ is one solution, and that the parametric family of solutions is $(1 - 3r - 5s + 2t, 3 - 5r + 3s - 21t, 7 + 3r - 7s, r, s, t)$.)

In this unit, we have observed a number of theoretical results which tell us about the number of solutions a system of linear equations has. If you understand what the rank of a matrix is (just the number of non-zero rows in the RREF of the matrix, which can also be thought of as the number of leading ones in the RREF matrix), then the implications for the number of solutions aren't really very hard to keep track of. It's mostly the same as our rules for identifying how many solutions there are, and how many parameters we need to introduce, when we finish row-reducing the matrix.

But there have been a number of different pieces of information about a SLE or its coefficient matrix, both in this unit and earlier, that all lead to the same important conclusion – that the system has a unique solution. In the following theorem, we collect all of these pieces of information together. (And later, we'll add one more piece of information, having to do with something we haven't learnt about yet, that also leads to this same conclusion.)

Theorem 9.8. *If A is a square matrix of order n then the following statements are equivalent to one another.*

1. *A is invertible (i.e., nonsingular).*
2. *$r(A) = n$ (i.e., A has full rank).*
3. *The RREF of A is I (i.e., A is row-equivalent to the identity matrix).*
4. *The system $A\vec{x} = \vec{b}$ has an unique solution (for all $n \times 1$ column vectors $\vec{b}$).*
5. *The homogeneous system $A\vec{x} = \vec{0}$ has only the trivial solution.*

Notice: Saying that these statements are all equivalent tells us that if any one of them is true for a particular matrix A, then *all* of them are true for that matrix. So for instance knowing that A is invertible tells us that $r(A) = n$ and also that $A\vec{x} = \vec{b}$ has a unique solution no matter what $\vec{b}$ is, and so on. But it's important to remember that this only applies to square matrices.

Math 1229A/B

Unit 10:
Determinants

(text reference: Section 4.1)

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10 Determinants

Square matrices are a special class of matrices. We have already seen one instance of a concept which is defined only for square matrices — the *inverse matrix*. That is, only a square matrix may have an inverse. In this unit we will (begin to) learn about another concept which is defined only for square matrices — the *determinant* of a matrix.

Definition of Determinant: Part One
Every square matrix has associated with it a number, called the **determinant** of the matrix. The determinant of the $n \times n$ matrix A is denoted by **det** $\boldsymbol{A}$.

The number which is the determinant of a square matrix measures a certain characteristic of the matrix. In a more advanced study of matrix algebra, this characteristic is used for various purposes. In this course, the only way in which we will use this number is in its connection to the existence of the inverse of the matrix, and through that it's application to SLE's in which the coefficient matrix is a square matrix. For these purposes, what will matter to us is whether or not this number, the determinant of the matrix, is 0. But of course, in order to determine whether or not the determinant of a particular matrix is 0, we need to know how to calculate that number.

Calculating the determinant of a square matrix is somewhat complicated. The definition is *recursive*, meaning that the calculation is defined in a straightforward way for small matrices, and then for larger matrices, the determinant is defined as being a calculation involving the determinants of smaller matrices, which are certain *submatrices* of the matrix. We could express this recursive definition of the determinant of a square matrix of order n as applying for all $n \geq 2$, specifically defining only the determinant of a square matrix of order 1, i.e. a (square) matrix containing only a single number. However, the calculation for a 2×2 matrix is very straightforward — easier to think of as a special definition all on its own — so instead we use specific definitions for $n = 1$ and $n = 2$, and then define the determinant of a square matrix of order $n > 2$ in terms of determinants of submatrices of order $n - 1$, which are found by expressing them in terms of determinants of successively smaller submatrices until we get down to submatrices of order 2. The calculation of $\det A$ as defined in this way, when A is a square matrix of order $n > 2$, is not really as complicated as it will look. It's just a matter of applying a certain formula carefully, as many times as necessary until we have expressed $\det A$ in terms of the determinants of 2×2 matrices. Those determinants are easy to find.

So we start by defining $\det A$ for square matrices of order 1 and of order 2. When A is a 1×1 matrix, i.e. a matrix containing only one number, finding the particular number $\det A$ which is associated with that matrix is trivial. That number is the only number around — the single number that's in the matrix. For a square matrix of order 2, i.e. a matrix containing 4 numbers arranged in a square, we have to do a little more work. But it's a simple calculation. In fact, we can think of the calculation as "down products minus up products", which is something we have seen before. But this time there's only one down product, and only one up product, so it's actually just "down product minus up product".

Definition of Determinant: Part Two
If A is a square matrix of order 1, so that $A = [a]$ for some number a, then $\det A = a$. That is, if A is a 1×1 matrix, then $\det A = a_{11}$.

If A is a square matrix of order 2, so that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for some numbers a, b, c and

d, then $\det A = ad - cb$. That is, if A is a 2×2 matrix, then $\det A = a_{11}a_{22} - a_{21}a_{12}$.

Example 10.1. Find the determinants of the following matrices:

(a) $A = [5]$ (b) $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (c) $C = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$

Solution:

(a) Here, A is 1×1, so $\det A = a_{11} = 5$. That is, the determinant of this matrix is just the number that's in the matrix.

(b) For a 2×2 matrix, we use the formula $\det B = b_{11}b_{22} - b_{21}b_{12}$. That is, we take the product of the numbers going diagonally down to the right (i.e., on the main diagonal) and then subtract from that the product of the numbers going diagonally up to the right. So we take the down product minus the up product. Here, the down product is $b_{11}b_{22} = 1(4) = 4$ and the up product is $b_{21}b_{12} = 3(2) = 6$. Therefore $\det B = 1(4) - 3(2) = 4 - 6 = -2$.

(c) Again, we have a 2×2 matrix, so we do the same calculation as in (b). We get

$$\det C = c_{11}c_{22} - c_{21}c_{12} = (2)(3) - (1)(0) = 6 - 0 = 6$$

Before we can define how the determinant of a larger matrix is defined in terms of determinants of certain submatrices, we need to define what those submatrices are, and some notation for indicating what submatrix we're referring to. We will also define terminology which means the determinant of a particular submatrix, and for another number obtained from that determinant, which is that same number, but sometimes with the sign changed.

Notice that if we have a square matrix of order n, we can obtain various submatrices of order $n - 1$ by deleting both one row and one column of the larger matrix. In fact, the matrix doesn't have to be square. For any $m \times n$ matrix with $m > 1$ and $n > 1$ we can obtain an $(m - 1) \times (n - 1)$ submatrix by deleting one row and one column of the larger matrix. So we will define these submatrices for *any* matrix that has more than one row and more than one column, but we'll only be using them in the context of the original matrix being a square matrix. We simply need to indicate which row and which column are to be deleted.

Definition: For any $m \times n$ matrix A with $m > 1$ and $n > 1$, the **submatrix $\boldsymbol{A_{ij}}$** is the $(m - 1) \times (n - 1)$ submatrix of A obtained by deleting row i and column j.

Example 10.2. For $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ find A_{21} and B_{11}.

Solution:

We find A_{21} by deleting row 2 and column 1 of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$. Since A is a 3×2 matrix, A_{21} will be a 2×1 matrix, consisting of the parts of rows 1 and 3 of A which are not in column 1. That is, when we've deleted row 2 and column 1, all that's left is the column 2 entries for rows 1 and 3. We get

$$\begin{bmatrix} \cancel{1} & 2 \\ \cancel{3} & \cancel{4} \\ \cancel{5} & 6 \end{bmatrix} \text{ gives } A_{21} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

Similarly, we get B_{11} by deleting the first row and the first column from B. This will give the 1×1 submatrix which contains only the number that's in row 2, column 2. That is,

$$\text{For } B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \begin{bmatrix} \cancel{1} & \cancel{2} \\ \cancel{3} & 4 \end{bmatrix} \text{ gives } B_{11} = [4]$$

Example 10.3. For $A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 1 & 3 & 1 \\ 2 & 5 & 3 & 4 \\ 3 & 7 & 8 & 9 \end{bmatrix}$, find A_{11} and A_{23}.

Solution:
To find A_{11}, we delete both the first row and the first column. We get

$$\begin{bmatrix} \cancel{1} & \cancel{2} & \cancel{1} & \cancel{3} \\ \cancel{2} & 1 & 3 & 1 \\ \cancel{2} & 5 & 3 & 4 \\ \cancel{2} & 7 & 8 & 9 \end{bmatrix} \text{ gives } A_{11} = \begin{bmatrix} 1 & 3 & 1 \\ 5 & 3 & 4 \\ 7 & 8 & 9 \end{bmatrix}$$

For A_{23}, we delete row 2 and column 3 from the matrix A. We see that

$$\begin{bmatrix} 1 & 2 & \cancel{1} & 3 \\ \cancel{2} & \cancel{1} & \cancel{3} & \cancel{1} \\ 2 & 5 & \cancel{3} & 4 \\ 3 & 7 & \cancel{8} & 9 \end{bmatrix} \text{ gives } A_{23} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 7 & 9 \end{bmatrix}$$

Notice that A_{11} is a submatrix that we could see, intact, within matrix A, whereas the submatrix A_{23} contains non-contiguous parts of A, because things have been deleted from the midst of rows and columns. But the numbers from A that *are* in A_{23} still have the same relative positions to one another.

We have a special name for the determinant of submatrix A_{ij} of a matrix A. Also, in our calculation of the determinant of A we will use $\det A_{ij}$, but we will need the negative of this number whenever i and j are not both odd or both even. Notice that the sum of two odd numbers is even, as is the sum of two even numbers. But the sum of an odd number and an even number is odd. And of course, if we raise -1 to an odd power, the value is -1, whereas if we raise -1 to an even number, the value is 1. So we can accomplish "use $\det A$ if i and j are both odd or both even, but otherwise use the negative of $\det A_{ij}$" by multiplying $\det A_{ij}$ by $(-1)^{i+j}$. We have a special name for this product, too.

Definition: Let A be any square matrix of order $n > 1$ and let A_{ij} be the submatrix obtained by deleting row i and column j.

- The ***i*,*j*-minor** of A, denoted $\boldsymbol{M_{ij}}$, is given by $M_{ij} = \det A_{ij}$.
- The ***i*,*j*-cofactor** of A, denoted $\boldsymbol{C_{ij}}$, is given by $C_{ij} = (-1)^{i+j} \det A_{ij}$.

Example 10.4. For $A = \begin{bmatrix} -3 & 2 & 1 \\ 4 & 5 & 6 \\ 2 & -3 & 1 \end{bmatrix}$, find M_{11}, M_{13}, C_{23} and C_{31}.

Solution:
To find the specified minors we simply identify the corresponding submatrix and calculate its determinant. We have

$$\begin{bmatrix} \cancel{-3} & \cancel{2} & \cancel{1} \\ \cancel{4} & 5 & 6 \\ \cancel{2} & -3 & 1 \end{bmatrix} \Rightarrow A_{11} = \begin{bmatrix} 5 & 6 \\ -3 & 1 \end{bmatrix}$$

and so $M_{11} = \det A_{11} = 5(1) - (-3)(6) = 5 - (-18) = 23$. Similarly, deleting the first row and third column to find A_{13} we get

$$M_{13} = \det A_{13} = \det \begin{bmatrix} 4 & 5 \\ 2 & -3 \end{bmatrix} = 4(-3) - 2(5) = -12 - 10 = -22$$

To find a cofactor of A, we find the corresponding minor and multiply it by -1 raised to the power $i + j$ (so that we multiply by -1 only if one of the row number and column number is odd and the other is even). We get:

$$\begin{aligned} C_{23} &= (-1)^{2+3} M_{23} \\ &= (-1)^5 \det A_{23} \\ &= (-1) \det \begin{bmatrix} -3 & 2 \\ 2 & -3 \end{bmatrix} \\ &= -[(-3)(-3) - 2(2)] \\ &= -(9 - 4) \\ &= -5 \end{aligned}$$

$$\begin{aligned} C_{31} &= (-1)^{3+1} M_{31} \\ &= (-1)^4 \det A_{31} \\ &= (1) \det \begin{bmatrix} 2 & 1 \\ 5 & 6 \end{bmatrix} \\ &= (2)(6) - 5(1) \\ &= 12 - 5 \\ &= 7 \end{aligned}$$

Notice that we don't, yet, know how to find minors or cofactors of a square matrix of order 4, or of any order larger than 3, because we haven't yet defined how to find the determinant of a square matrix of order larger than 2. For instance, the 1,1-minor of a 4×4 matrix A is simply the determinant of A_{11}, but since A_{11} is a 3×3 matrix, we don't know how to calculate that. But we do, now, know everything we need to in order to define this. That is, we have assembled all the pieces to allow us to recursively define how to find the determinant of a square matrix of order bigger than 2. We define it in terms of the cofactors of certain submatrices. The definition we will now state will give a particular way of calculating $\det A$. But then afterwards, we'll have a theorem that shows other, similar, ways of calculating it, using a different series of cofactors. There are, in fact, $2n$ different ways that we could calculate the determinant of an $n \times n$ matrix, using cofactors determined by any one particular row or column. In our definition, we'll use the cofactors of row 1. We multiply each entry of this row by the cofactor with the same index. (That is, multiply a_{1j} by C_{1j}.) And then we add them all up.

Definition of Determinant: Part Three
For any square matrix A of order $n \geq 3$, the determinant of A is given by

$$\det A = a_{11}C_{11} + a_{12}C_{12} + ... + a_{1n}C_{1n}$$

That is, we multiply each entry of row 1 by the corresponding cofactor of A, which is to say that we multiply that entry by the determinant of the submatrix obtained by deleting the row and column in which that entry occurs, or the negative of that determinant if the row and column are not either both odd or both even. Using $\sum_{j=1}^{n}$ to denote "do this calculation for every value of j from 1 to n, and add them all up", we have

$$\det A = \sum_{j=1}^{n} a_{1j}(-1)^{1+j} \det A_{1j}$$

Example 10.5. Find $\det A$, where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$.

Solution:
We calculate $\det A$ as it says in the definition:

$$\begin{aligned}
\det A &= \sum_{j=1}^{3} a_{1j}C_{1j} = \sum_{j=1}^{3} a_{1j}(-1)^{1+j} \det A_{1j} \\
&= a_{11}(-1)^{1+1} \det A_{11} + a_{12}(-1)^{1+2} \det A_{12} + a_{13}(-1)^{1+3} \det A_{13} \\
&= (1)(-1)^2 \det \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} + (2)(-1)^3 \det \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} + (3)(-1)^4 \det \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \\
&= [(1)(1) - (2)(3)] - 2[(2)(1) - (3)(3)] + 3[(2)(2) - (3)(1)] \\
&= (1-6) - 2(2-9) + 3(4-3) \\
&= -5 - 2(-7) + 3(1) = -5 + 14 + 3 = 12
\end{aligned}$$

Notice that the $(-1)^{i+j}$ multipliers made the sign alternate across the row. That is, for the a_{11} term it is $+1$, then for the a_{12} term it is -1 and for the a_{13} term it is $+1$ again. So we could express $\det A$ as

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}$$

Example 10.6. If $A = \begin{bmatrix} 1 & -2 & -4 & 5 \\ 0 & 3 & 0 & 0 \\ 0 & -1 & 3 & 2 \\ 0 & 4 & -5 & 2 \end{bmatrix}$, find $\det A$.

Solution: Again, we use the definition to express $\det A$ in terms of the determinants of certain submatrices of A:

$$\begin{aligned}
\det A &= a_{11}(-1)^{1+1}\det A_{11} + a_{12}(-1)^{1+2}\det A_{12} + a_{13}(-1)^{1+3}\det A_{13} + a_{14}(-1)^{1+4}\det A_{14} \\
&= 1(-1)^2 \det \begin{bmatrix} 3 & 0 & 0 \\ -1 & 3 & 2 \\ 4 & -5 & 2 \end{bmatrix} + (-2)(-1)^3 \det \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & -5 & 2 \end{bmatrix} \\
&\quad + (-4)(-1)^4 \det \begin{bmatrix} 0 & 3 & 0 \\ 0 & -1 & 2 \\ 0 & 4 & 2 \end{bmatrix} + 5(-1)^5 \det \begin{bmatrix} 0 & 3 & 0 \\ 0 & -1 & 3 \\ 0 & 4 & -5 \end{bmatrix} \\
&= 1 \det \begin{bmatrix} 3 & 0 & 0 \\ -1 & 3 & 2 \\ 4 & -5 & 2 \end{bmatrix} - (-2) \det \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & -5 & 2 \end{bmatrix} \\
&\quad + (-4) \det \begin{bmatrix} 0 & 3 & 0 \\ 0 & -1 & 2 \\ 0 & 4 & 2 \end{bmatrix} - 5 \det \begin{bmatrix} 0 & 3 & 0 \\ 0 & -1 & 3 \\ 0 & 4 & -5 \end{bmatrix}
\end{aligned}$$

Notice that once again the $+$'s and $-$'s resulting from the $(-1)^{1+j}$'s are alternating. Now, to find the determinant of each of those 3×3 submatrices of A, we need to use the definition again, to express each in terms of determinants of 2×2 submatrices. Of course, this time the 1 and j indices in the $(-1)^{1+j}$ term (as well as the a_{1j} and A_{1j} terms) are the row and column numbers *in the submatrix whose determinant we are currently calculating*, not the row and column numbers from the original matrix.

$$\begin{aligned}
\det A &= 1 \det \begin{bmatrix} 3 & 0 & 0 \\ -1 & 3 & 2 \\ 4 & -5 & 2 \end{bmatrix} - (-2) \det \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & -5 & 2 \end{bmatrix} \\
&\quad + (-4) \det \begin{bmatrix} 0 & 3 & 0 \\ 0 & -1 & 2 \\ 0 & 4 & 2 \end{bmatrix} - 5 \det \begin{bmatrix} 0 & 3 & 0 \\ 0 & -1 & 3 \\ 0 & 4 & -5 \end{bmatrix} \\
&= 1 \left\{ 3(-1)^{1+1} \det \begin{bmatrix} 3 & 2 \\ -5 & 2 \end{bmatrix} + 0(-1)^{1+2} \det \begin{bmatrix} -1 & 2 \\ 4 & 2 \end{bmatrix} + 0(-1)^{1+3} \det \begin{bmatrix} -1 & 3 \\ 4 & -5 \end{bmatrix} \right\} \\
&\quad -(-2) \left\{ 0(-1)^{1+1} \det \begin{bmatrix} 3 & 2 \\ -5 & 2 \end{bmatrix} + 0(-1)^{1+2} \det \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} + 0(-1)^{1+3} \det \begin{bmatrix} 0 & 3 \\ 0 & -5 \end{bmatrix} \right\} \\
&\quad +(-4) \left\{ 0(-1)^{1+1} \det \begin{bmatrix} -1 & 2 \\ 4 & 2 \end{bmatrix} + 3(-1)^{1+2} \det \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} + 0(-1)^{1+3} \det \begin{bmatrix} 0 & -1 \\ 0 & 4 \end{bmatrix} \right\} \\
&\quad -5 \left\{ 0(-1)^{1+1} \det \begin{bmatrix} -1 & 3 \\ 4 & -5 \end{bmatrix} + 3(-1)^{1+2} \det \begin{bmatrix} 0 & 3 \\ 0 & -5 \end{bmatrix} + 0(-1)^{1+3} \det \begin{bmatrix} 0 & -1 \\ 0 & 4 \end{bmatrix} \right\}
\end{aligned}$$

Now, we have $\det A$ expressed in terms of determinants of 2×2 matrices, so for each of them we simply use the formula: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - cb$.

$$\begin{aligned}
\det A &= 1\{3(1)[3(2)-(-5)(2)] + 0(-1)[(-1)(2)-4(2)] + 0(1)[(-1)(-5)-4(3)]\} \\
&\quad +2\{0(1)[3(2)-(-5)(2)] + 0(-1)[0(2)-0(2)] + 0(1)[0(-5)-0(3)]\} \\
&\quad -4\{0(1)[(-1)(2)-4(2)] + 3(-1)[0(2)-0(2)] + 0(1)[0(4)-0(-1)]\} \\
&\quad -5\{0(1)[(-1)(-5)-4(3)] + 3(-1)[0(-5)-0(3)] + 0(1)[0(4)-0(-1)]\} \\
&= 1\{3[6-(-10)] - 0[-2-6] + 0[5-12]\} \\
&\quad +2\{0[6-(-10)] - 0[0-0] + 0[0-0]\} \\
&\quad -4\{0[-2-8] - 3[0-0] + 0[0-0]\} \\
&\quad -5\{0[5-12] - 3[0-0] + 0[0-0]\} \\
&= [3(16) - 0(-8) + 0(-7)] + 2[0(16) - 0(0) + 0(0)] \\
&\quad -4[0(-10) - 3(0) + 0(0)] - 5[0(-7) - 3(0) + 0(0)] \\
&= (48 - 0 + 0) + 2(0 - 0 + 0) - 4(0 - 0 + 0) - 5(0 - 0 + 0) \\
&= 48 - 0 - 0 - 0 \\
&= 48
\end{aligned}$$

Notice that once again, for each determinant, the effect of the $(-1)^{i+j}$ terms is to make the $+$'s and $-$'s alternate. And since we're always starting with $i = j = 1$, the pattern always starts with $+$.

Well, that was a lot of work! But as long as we take it slowly and carefully, paying attention to all the details of what we're doing, none of it is difficult. Notice, though, that the way the calculation was expressed above, many calculations were done unnecessarily, because we knew they were going to be multiplied by 0. Let's restate that calculation, exactly the same, but taking advantage of those zero multipliers to not bother expressing the calculations that don't matter because they won't be used.

Example 10.6. *Revisited:*
If $A = \begin{bmatrix} 1 & -2 & -4 & 5 \\ 0 & 3 & 0 & 0 \\ 0 & -1 & 3 & 2 \\ 0 & 4 & -5 & 2 \end{bmatrix}$, find $\det A$.

Solution:

$$\begin{aligned}
\det A &= a_{11}(-1)^{1+1}\det A_{11} + a_{12}(-1)^{1+2}\det A_{12} + a_{13}(-1)^{1+3}\det A_{13} + a_{14}(-1)^{1+4}\det A_{14} \\
&= 1(-1)^2 \det \begin{bmatrix} 3 & 0 & 0 \\ -1 & 3 & 2 \\ 4 & -5 & 2 \end{bmatrix} + (-2)(-1)^3 \det \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & -5 & 2 \end{bmatrix} \\
&\quad +(-4)(-1)^4 \det \begin{bmatrix} 0 & 3 & 0 \\ 0 & -1 & 2 \\ 0 & 4 & 2 \end{bmatrix} + 5(-1)^5 \det \begin{bmatrix} 0 & 3 & 0 \\ 0 & -1 & 3 \\ 0 & 4 & -5 \end{bmatrix} \\
&= 1(1)\left\{3(-1)^{1+1}\det \begin{bmatrix} 3 & 2 \\ -5 & 2 \end{bmatrix} - 0 + 0\right\} + (-2)(-1)\{0+0+0\} \\
&\quad +(-4)(1)\left\{0 + 3(-1)^{1+2}\det \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} + 0\right\} + 5(-1)\left\{0 + 3(-1)^{1+2}\det \begin{bmatrix} 0 & 3 \\ 0 & -5 \end{bmatrix} + 0\right\}
\end{aligned}$$

$$\begin{aligned}
\text{so } \det A &= 1\left\{3(1)[3(2)-(-5)(2)]\right\}-(-2)(0)+(-4)\left\{3(-1)[0(2)-0(2)]\right\}-5\left\{3(-1)[0(-5)-0(3)]\right\}\\
&= 1\left\{3[6-(-10)]\right\}+0-4\left\{(-3)[0-0]\right\}-5\left\{-3(0-0)\right\}\\
&= 3(16)-4[(-3)(0)]-5[(-3)(0)]\\
&= 48-0-0\\
&= 48
\end{aligned}$$

Well, that was somewhat better! But what if we could have taken advantage of more 0's earlier on? Consider, for instance, the determinant of A^T for the matrix A in that example. The transpose of the matrix has 3 zeroes in its first row. And that would mean that we would only need to calculate the determinant of one 3×3 matrix, because all of the others will simply be multiplied by 0 and therefore don't need to be calculated. That would certainly be convenient.

Actually, we could use that in the calculation of $\det A$ as given, too. Because the calculation of the determinant of A doesn't actually have to be done in the way stated in the definition. As previously stated, that's just one of the ways to calculate the determinant of a matrix. We call the method expressed in the definition *expansion along row 1* because we use the sum of the products of the row 1 entries times the row 1 cofactors. But in fact we can expand along any row of the matrix, instead of along row 1. (Just look at all those lovely 0's in row 2!) Or, we can expand along any column, instead of expanding along a row. (So we could expand along column 1, doing the same convenient calculation that we already observed would make calculating the determinant of the transpose of the matrix very easy.) We have a theorem which tells us that we can do any one of these expansions and we'll get the same answer, no matter which one we do.

Theorem 10.1. *Let A be any square matrix of order $n > 2$. Then the value of $\det A$ can be found by expanding along any row or column of A. That is:*

- *for any fixed value of i, with $1 \le i \le n$, $\det A$ can be calculated using* **expansion along row *i*** *as*

$$\begin{aligned}
\det A &= a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}\\
&= \sum_{j=1}^{n} a_{ij}(-1)^{i+j}\det A_{ij}
\end{aligned}$$

OR

- *for any fixed value of j, with $1 \le j \le n$, $\det A$ can be calculated using* **expansion along column *j*** *as*

$$\begin{aligned}
\det A &= a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}\\
&= \sum_{i=1}^{n} a_{ij}(-1)^{i+j}\det A_{ij}
\end{aligned}$$

and the value of $\det A$ obtained will always be the same.

(Notice that in the sum notation, i.e. the $\sum$'s, the difference shows up only in whether it is i or j at the bottom of the $\sum$. That is, for expansion along row i, i is a fixed value, so we sum over different values of j, as shown by the j at the bottom of the $\sum$. But for expansion along column j, the value of j is fixed and we're summing over different i values, as shown by the i at the bottom of the $\sum$.)

Theorem 10.1 means that we could have found $\det A$ in Example 10.6 much more easily, taking advantage of the zeroes in column 1, or in row 2. Let's see how that would work out.

Example 10.6. *One more time*

If $A = \begin{bmatrix} 1 & -2 & -4 & 5 \\ 0 & 3 & 0 & 0 \\ 0 & -1 & 3 & 2 \\ 0 & 4 & -5 & 2 \end{bmatrix}$, find $\det A$.

Solution:

Approach 1:

We can expand along row 2, which contains lots of 0's, instead of along row 1. Watch out for the effects of the $(-1)^{i+j}$ multipliers, though. We get:

$$\begin{aligned} \det A &= 0(-1)^{2+1}\det A_{21} + 3(-1)^{2+2}\det A_{22} + 0(-1)^{2+3}\det A_{23} + 0(-1)^{2+4}\det A_{24} \\ &= -0 + 3\det\begin{bmatrix} 1 & -4 & 5 \\ 0 & 3 & 2 \\ 0 & -5 & 2 \end{bmatrix} - 0 + 0 \end{aligned}$$

The effect of the $(-1)^{i+j}$ multipliers is shown, even for the 0's, to demonstrate that although the $+$'s and $-$'s still alternate, this time, since we had i being even, the pattern switched, and started with minus instead of with plus. Now, to calculate the determinant of A_{22}, we can expand along column 1, to once again take advantage of the 0's. So we have:

$$\begin{aligned} \det A &= 3\det\begin{bmatrix} 1 & -4 & 5 \\ 0 & 3 & 2 \\ 0 & -5 & 2 \end{bmatrix} \\ &= 3\left\{1(-1)^{1+1}\det\begin{bmatrix} 3 & 2 \\ -5 & 2 \end{bmatrix} - 0 + 0\right\} \\ &= 3(1)(-1)^2[3(2) - (-5)(2)] = 3(1)[6 - (-10)] = 3(16) = 48 \end{aligned}$$

That was much easier! We only ended up calculating the determinant of one 2×2 matrix!

Approach 2:

Instead, we can expand along column 1, and once again choose wisely for the next expansion:

$$\begin{aligned} \det A &= 1(-1)^{1+1}\det A_{11} - 0\det A_{21} + 0\det A_{31} - 0\det A_{41} \\ &= 1\det\begin{bmatrix} 3 & 0 & 0 \\ -1 & 3 & 2 \\ 4 & -5 & 2 \end{bmatrix} \\ &= 3(-1)^{1+1}\det\begin{bmatrix} 3 & 2 \\ -5 & 2 \end{bmatrix} - 0 + 0 \\ &= 3(-1)^2[3(2) - (-5)(2)] = 3[6 - (-10)] = 3(16) = 48 \end{aligned}$$

By choosing to expand along row 1 to find the determinant of the 3×3, we again only ended up calculating the determinant of one 2×2 matrix. And of course, both of these approaches give the same answer as the (much more painful) earlier calculation.

It won't always be possible to reduce the amount of work required in calculating the determinant of a large matrix by as much as we were able to here, and if the matrix doesn't contain any zeroes, it won't be possible to reduce the work at all. But it's always worthwhile to consider carefully in choosing which row or column to expand along, to take advantage of as many zeroes as possible.

If we can calculate the determinant of a matrix by expanding along a row, or by expanding along a column, and get the same value either way, then that means that the matrix and its transpose must have the same value for the determinant. That is, if we calculate the determinant of some matrix A by, for instance, expanding along row 1, and then we calculate the determinant of A^T by expanding along column 1, we're doing exactly the same calculations and must therefore end up with the same value. That's something worth remembering.

Corollary 10.2. *For any square matrix A,* $\det A = \det A^T$.

Let's recap what we've got so far. We have defined the definition of derivative in 3 separate pieces (referred to as Part One, Part Two and Part Three). The first piece just defined that there *is* a number called the determinant of A for any square matrix A. The second piece told us how to calculate the determinant of a 1×1 matrix and the determinant of a 2×2 matrix. Then we defined the minors and cofactors of a matrix (as well as the submatrix A_{ij}). And finally the third piece of the definition told us how to calculate the determinant of a square matrix of order larger than 2. But then we had Theorem 10.1 telling us other ways to calculate those determinants. Let's pull all of that (except for the other definitions, of the submatrix A_{ij}, the minors and the cofactors — we'll still need those definitions) together into a single *definition of determinant*, so that we've got it all in one place.

Definition of Determinant (Complete)
Every square matrix A has associated with it a number, called the **determinant** of the matrix, denoted by $\mathbf{det}\, \boldsymbol{A}$. Let A be a square matrix of order n.

1. If $n = 1$ then $\det A = a_{11}$.
2. If $n = 2$ then $\det A = a_{11}a_{22} - a_{21}a_{12}$.
3. If $n \geq 3$, the determinant of A may be obtained by expanding along any row i using

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \sum_{j=1}^{n} a_{ij}(-1)^{i+j} \det A_{ij}$$

or by expanding along any column j using

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} = \sum_{i=1}^{n} a_{ij}(-1)^{i+j} \det A_{ij}$$

Sometimes, it is possible to know what the value of the determinant of a matrix is without any calculations at all! Or to find the value with only very minimal calculation. That is, there are various properties that a matrix can have which make the value of the determinant obvious. We finish off this unit by observing these properties. For instance, consider the next example.

Example 10.7. Find the determinant of the matrix A below:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 3 & 2 & 1 & 0 & 5 & 2 \\ 2 & 1 & 3 & 6 & 2 & 4 & 5 & 2 & 0 & 9 \\ 1 & 2 & 4 & 6 & 0 & 7 & 6 & 3 & 9 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 10 & 32 & 45 & 2 & 9 & 8 & 5 & 21 & 112 \\ 7 & 3 & 93 & 38 & 0 & 239 & 34 & 9 & 35 & 13 \\ 1 & 4 & 6 & 3 & 90 & 47 & 2 & 9 & 3 & 53 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 342 \\ 4 & 3 & 4 & 5 & 0 & 2 & 8 & 2 & 0 & 2 \\ 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{bmatrix}$$

Solution: Oh my goodness! Let's count ... yes, that's a 10×10 matrix. Let's see. We'll have ten 9×9 submatrices, each of which will require nine 8×8 submatrices, for each of which we'll have eight 7×7 submatrices, for which we'll have to find the determinants of seven 6×6 submatrices, and for those ... it's exhausting just trying to say how much work this will be. And look at some of those numbers! Where did I put my calculator??? ... But wait! Look at row 4! Let's expand along row 4! Aha!

$$\det A = -0 + 0 - 0 + 0 - 0 + 0 - 0 + 0 - 0 + 0 = 0$$

Well that wasn't bad at all! We didn't even need to write down any of those submatrices. Or even observe the pattern of pluses and minuses, although it was good practice to do so.

Because A in this example had a row that contained nothing but 0's, we were able to choose to expand along this row to calculate the determinant without really doing any work at all. And similarly, if we had a matrix which contained a column that was all 0's, we could expand along that column to see that the value of the determinant was 0. This is an extremely useful thing to realize.

Theorem 10.3. *If square matrix A has any row which contains only 0's, or any column which contains only 0's, then* $\det A = 0$.

Example 10.8. Find $\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 4 & 5 & 6 \end{bmatrix}$ and $\det \begin{bmatrix} 5 & 5 \\ 11 & 11 \end{bmatrix}$.

Solution:
Well, no zeroes there. We'll have to crank through the work. For the first one, expanding along row 1 (since there's no other choice that looks easier) we have:

$$\begin{aligned} \det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 4 & 5 & 6 \end{bmatrix} &= 1 \det \begin{bmatrix} 5 & 6 \\ 5 & 6 \end{bmatrix} - 2 \det \begin{bmatrix} 4 & 6 \\ 4 & 6 \end{bmatrix} + 3 \det \begin{bmatrix} 4 & 5 \\ 4 & 5 \end{bmatrix} \\ &= 1[5(6) - 5(6)] - 2[4(6) - 4(6)] + 3[4(5) - 4(5)] \\ &= 1(30 - 30) - 2(24 - 24) + 3(20 - 20) \\ &= 1(0) - 2(0) + 3(0) \\ &= 0 \end{aligned}$$

Oh! Look at that! Every single one of those 2×2 matrices had determinant 0. Why was that? Well, look at the 2×2 matrices. In each case, the two rows are identical, which means that when we did the "down product minus up product" calculation, the down product and the up product were the same. That's where all those zeroes came from.

The second matrix whose determinant we're asked to find is a 2×2, but it doesn't have the two rows being identical. We better do the calculation:

$$\det \begin{bmatrix} 5 & 5 \\ 11 & 11 \end{bmatrix} = 5(11) - (11)(5) = 55 - 55 = 0$$

Again the determinant is 0! Although the rows were different, the columns were identical. And again that meant that the down product had the same value as the up product.

In this example, in the first matrix, the reason that all of the 2×2 submatrices had two identical rows is because the larger matrix they were submatrices of had two identical rows. And similarly, if we had a larger matrix in which there were two identical columns (for instance, the transpose of the matrix with two identical rows), then careful choosing of a column to expand along would result in all the 2×2 submatrices whose determinants we need to calculate having two identical columns. And as we have seen here, when we take the determinant of a 2×2 matrix in which *either* the rows are identical or the columns are identical, the determinant is just 0. So that means that if the larger matrix whose determinant we need to find has two identical rows, or has two identical columns, we can expand in such a way that we end up with 2×2 submatrices whose determinants are all 0, so that the larger matrix also has determinant 0. And that's another result which can save us a lot of work.

Theorem 10.4. *If a square matrix A has two rows which are identical, or has two columns which are identical, then* $\det A = 0$.

Example 10.9. Find $\det \begin{bmatrix} 2 & 4 & 2 \\ 1 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $\det \begin{bmatrix} 2 & 4 & 2 \\ 1 & 3 & 1 \\ -5 & 0 & -5 \end{bmatrix}$.

Solution:
For the first matrix, we see that there is a row containing only 0's, so the determinant is 0. And for the second one, there are two identical columns, so again the determinant is 0. That is, without doing any calculations at all we see that

$$\det \begin{bmatrix} 2 & 4 & 2 \\ 1 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix} = 0 \qquad \text{and} \qquad \det \begin{bmatrix} 2 & 4 & 2 \\ 1 & 3 & 1 \\ -5 & 0 & -5 \end{bmatrix} = 0$$

Notice: Look back at Example 10.5 from earlier. There, we found that $\det \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} = 12$. And in that matrix, there's a row that's identical to a column. So clearly having a row and a column identical *doesn't* have the same effect as having two rows which are identical, or having two columns which are identical. Make sure you remember that it's only *two identical rows* or *two identical columns* that cause the determinant to be 0, not just two identical "rows or columns".

Also Notice: **Warning!** In the last several determinant calculations we did in which we actually did expand along a row or a column, we didn't bother writing the $(-1)^{i+j}$ multipliers. We just used the alternating $+$ and $-$ pattern that we knew would hold. And that approach is fine *as long as* you're careful to figure out whether the pattern starts with a $+$ or a $-$. But if you don't think carefully about that, and get it wrong, then **all** of the signs will be wrong in your calculation, which will result in the sign in your final answer being wrong. (And in those multiple choice questions on quizzes and exams, you've gotta know that the right number with the wrong sign will *always* be one of the answer choices.) So be careful.

We've seen a couple of characteristics that a matrix can have that allow us to know, without doing any calculations at all, that the value of the determinant is 0. There are also some characteristics which allow us to find the value of the determinant, whatever it is, much more easily than by actually carrying out the row or column expansions. For instance, let's look at what happens in the calculations in the next example.

Example 10.10. Find $\det I_5$ and $\det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 10 \end{bmatrix}$.

Solution:
I_5 is the Identity Matrix of order 5. It's got lots of zeroes in it. Choosing to expand along row 1 each time (which has the $+/-$ pattern always starting with $+$), we take advantage of many of those zeroes. Likewise, the second matrix whose determinant we're asked to find also has lots of zeroes in it (although not as many as in an identity matrix). Because of where they're placed, when we calclate that determinant, we can take advantage of the zeroes by always expanding along ... well, row 1 would work again, but just to change things up a bit, let's expand along the last column each time, which will also use many of the 0's. (And will give us practice at being careful about the signs. For the 4×4, expanding along column 4 the first sign is given by $(-1)^{1+4}$, so the pattern starts with a minus. But then for the 3×3, expanding along column 3 the first sign is given by $(-1)^{1+3}$, so that pattern will start with a plus.)

For I_5 we get:

$$\begin{aligned} \det I_5 &= \det \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = 1 \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - 0 + 0 - 0 + 0 \\ &= 1 \left\{ 1 \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 0 + 0 - 0 \right\} + 0 = 1(1) \left\{ 1 \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 0 + 0 \right\} + 0 \\ &= 1(1)(1)[1(1) - 0(0)] + 0 = 1(1)(1)(1)(1) = 1 \end{aligned}$$

Hmm. We just ended up multiplying all the ones together. Those 5 ones that were along the main diagonal.

Let's see what we get for the other one. (Remember, we said we're gong to expand along the last column each time for this one.)

$$\begin{aligned} \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 10 \end{bmatrix} &= -0 + 0 - 0 + 10 \det \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix} \\ &= 0 + 10 \left\{ 0 - 0 + 6 \det \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \right\} \\ &= 10 \left\{ 0 + 6[1(3) - 2(0)] \right\} \\ &= 10(6)[1(3) - 0] \\ &= 10(6)(1)(3) \\ &= 1(3)(6)(10) = 180 \end{aligned}$$

Again, the value of the determinant ended up being just the product of the numbers along the main diagonal. Hmm.

In fact, any time we calculate the determinant of a diagonal matrix, such as an identity matrix, we will end up with (no matter how we do the expansion) the value of the determinant being just the product of the numbers along the main diagonal. And for a matrix that looks like the second one, or like the transpose of that matrix, the same thing will happen. We have special names to describe matrices that look like that one, or like its transpose.

Definition: A square matrix A is called **upper triangular** if all entries below the main diagonal are zero, and is called **lower triangular** if all entries above the main diagonal are zero.

(*Notice:* A diagonal matrix fits both definitions, i.e. could be said to be both upper triangular and lower triangular. But it's easier to just call it diagonal, as we have already been doing.)

The definition says concerning the matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 10 \end{bmatrix}$ and its transpose $\begin{bmatrix} 1 & 2 & 4 & 7 \\ 0 & 3 & 5 & 8 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix}$, that the first is *lower triangular* (because the non-zero entries form a triangle in the lower left part of the matrix), and the second is *upper triangular* (because the non-zero entries form a triangle in the upper right part of the matrix).

As we have already seen, calculating determinants of all of these kinds of matrices can be done very easily, without even thinking about how we do the expansion, as the following theorem tells us.

Theorem 10.5. *If a square matrix $A = [a_{ij}]$ is either upper or lower triangular, or is a diagonal matrix, then the determinant of A is the product of the elements lying on the main diagonal. That is,*

$$\det A = (a_{11})(a_{22})(a_{33})...(a_{nn})$$

An obvious consequence of this theorem is stated in the following corollary.

Corollary 10.6. *The determinant of any Identity Matrix is 1. That is, for any $n \geq 1$, $\det I_n = 1$.*

Example 10.11. Find the determinants of the following matrices.

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 0 & -3 & 1 \\ 0 & 0 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ -5 & 0 & -1 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -3 \end{bmatrix} \qquad D = -3\,(I_{24})$$

Solution:
A is an upper triangular matrix, so we simply find the product of the entries on the main diagonal. That is, we have

$$\det A = (a_{11})(a_{22})(a_{33}) = (1)(-3)(5) = -15$$

And we see that B is lower triangular, so again we just take the product of the entries on the main diagonal, to get:

$$\det B = (2)(2)(-1) = -4$$

Similarly, for diagonal matrix C we get

$$\det C = (2)(7)(-3) = -42$$

And finally we have $D = -3\,(I_{24})$. That is, matrix D is obtained by multiplying the Identity Matrix of order 24 by the scalar -3. Remember that when we multiply a matrix by a scalar, every entry in the matrix is multiplied by the scalar. Of course, the 0's will still just be 0's, so D is a diagonal matrix of order 24 in which each of the entries on the main diagonal is -3. Therefore the determinant is just the product of all those -3's. That is, we have:

$$\det D = (-3)(-3)...(-3) = (-3)^{24} = 3^{24}$$

(Note that because the exponent is even, the negatives all cancel out.)

Math 1229A/B

Unit 11:

Properties of Determinants

(text reference: Section 4.2)

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11 Properties of Determinants

In this section, we learn more about determinants. First, we observe some properties of determinants that allow us to calculate determinants more easily. We examine the effects on the determinant when the various kinds of elementary row operations are performed, so that we can easily see how the determinants of the various row-equivalent matrices are related to one another as we perform these operations. This allows us to calculate the determinant of a matrix by row-reducing the matrix (a procedure we already know well) to obtain a matrix whose determinant is easily calculated using facts we've already learnt in the previous section. We also learn some useful properties which allow us to calculate the determinant of a matrix from the determinants of one or more other matrices whose determinants we may already know. And finally we examine the relationship between determinants and inverses, which allows us to relate determinants to systems of linear equations, using what we already know about the implications of the existence of the inverse of a matrix for the number of solutions to the SLE which has that matrix as its coefficient matrix. Throughout all of this, of course, it is important to remember that we are only dealing with square matrices when we talk about determinants. That is, it is only for a square matrix that the characteristic "the determinant of the matrix" is defined.

First, let's think about what effect multiplying some row of a matrix by a non-zero scalar will have on the determinant. That is, let's think about the relationship between $\det A$ and $\det B$ if matrix B is identical to matrix A except that one of the rows in B is the corresponding row of A multiplied by some $c \neq 0$.

So suppose we have some $n \times n$ matrix $A = [a_{ij}]$. Let $B = [b_{ij}]$ be the matrix obtained by multiplying one row, row k, by some non-zero scalar c. Then we know that $b_{kj} = ca_{kj}$ and $b_{ij} = a_{ij}$ for all $i \neq k$. We can calculate $\det B$ by expanding along row k. Notice that when we form submatrices of B by deleting row k (and also some column of B), the one row that's different than in matrix A is deleted, so that in the submatrix of B obtained, each entry is just the corresponding entry from matrix A and therefore the entire submatrix of B is simply the corresponding submatrix of A. That is, we have $B_{kj} = A_{kj}$. So when we expand along row k we get:

$$\begin{aligned}
\det B &= \sum_{j=1}^{n} b_{kj}(-1)^{k+j}B_{kj} \\
&= \sum_{j=1}^{n} (ca_{kj})(-1)^{k+j}A_{kj} && \text{because } b_{kj} = ca_{kj} \\
& && \text{and } B_{kj} = A_{kj} \\
&= ca_{k1}(-1)^{k+1}A_{k1} + ca_{k2}(-1)^{k+2}A_{k2} + \cdots + ca_{kn}(-1)^{k+n}A_{kn} \\
&= c\left[a_{k1}(-1)^{k+1}A_{k1} + a_{k2}(-1)^{k+2}A_{k2} + \cdots + a_{kn}(-1)^{k+n}A_{kn}\right] && \text{i.e. factor } c \text{ out of} \\
& && \text{each term of the sum} \\
&= c\left[\textstyle\sum_{j=1}^{n} a_{kj}(-1)^{k+j}A_{kj}\right] \\
&= c\,[\det A]
\end{aligned}$$

Why look at that! When we multiply a row of matrix A by any non-zero scalar c, the effect is to multiply the value of the determinant by that same scalar. And notice that the same thing would happen if we were to multiply a column by c instead of a row, because we could calculate the determinant by expansion along that column. That is, we have already observed that $\det B = \det B^T$, so doing something to a column has the same effect on the determinant as doing something to a row. (We'll use that fact every time we look at the effect on the determinant of doing something to a row of a matrix.) So we have the following Theorem.

Theorem 11.1. *If matrix B is obtained from square matrix A by multiplying one row or column of A by some non-zero scalar c, then* $\det B = c(\det A)$.

Notice: We're specifying here that the scalar must be non-zero, but of course multiplying a row or column by 0 has the same effect – the value of the determinant is also multiplied by 0 – because we know that if a matrix has a row or column of only 0's (which is what multiplying a row or column by 0 would give) then the value of the determinant is 0.

We could also express the result from the theorem above as factoring a non-zero scalar multiplier out of a row or column. That is, we see that (in the case of row k being multiplied by c):

$$\det\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{(k-1)1} & a_{(k-1)2} & \cdots & a_{(k-1)n} \\ ca_{k1} & ca_{k2} & \cdots & ca_{kn} \\ a_{(k+1)1} & a_{(k+1)2} & \cdots & a_{(k+1)n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = c\left(\det\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{(k-1)1} & a_{(k-1)2} & \cdots & a_{(k-1)n} \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ a_{(k+1)1} & a_{(k+1)2} & \cdots & a_{(k+1)n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}\right)$$

And of course it's the same if some column k is multiplied by a non-zero scalar c. So the following result follows directly from Theorem 11.1.

Corollary 11.2. *If every entry of one row or column of a square matrix B has a common factor, c, and matrix A is obtained from matrix B by factoring that common factor out of that row or column, i.e. by multiplying the row or column by $\frac{1}{c}$, then* $\det B = c\det A$.

One of the ways in which this is useful is when the numbers in a matrix are obnoxious, but there are common factors in all the entries in some rows and/or columns which can be factored out to make the arithmetic easier.

Example 11.1. Find $\det A$ where $A = \begin{bmatrix} 42 & 5 & 1 \\ 84 & 0 & 0 \\ 63 & 5 & 2 \end{bmatrix}$

Solution:
Some of the numbers in this matrix look pretty obnoxious. But we see that the entries in the first column have a common factor of 21. The arithmetic will be easier if we factor it out. That is, we have the matrix

$$A = \begin{bmatrix} 42 & 5 & 1 \\ 84 & 0 & 0 \\ 63 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 21\times 2 & 5 & 1 \\ 21\times 4 & 0 & 0 \\ 21\times 3 & 5 & 2 \end{bmatrix}$$

so the matrix

$$B = \begin{bmatrix} 2 & 5 & 1 \\ 4 & 0 & 0 \\ 3 & 5 & 2 \end{bmatrix}$$

can be obtained from matrix A by factoring the 21 multiplier out of column 1. Therefore, according to our Corollary above, we have

$$\det A = \det \begin{bmatrix} 42 & 5 & 1 \\ 84 & 0 & 0 \\ 63 & 5 & 2 \end{bmatrix} = 21 \left(\det \begin{bmatrix} 2 & 5 & 1 \\ 4 & 0 & 0 \\ 3 & 5 & 2 \end{bmatrix} \right) = 21(\det B)$$

and the arithmetic for calculating $\det B$ is not difficult. Expanding along row 2, we have

$$\det B = 4(-1)^{2+1} \det \begin{bmatrix} 5 & 1 \\ 5 & 2 \end{bmatrix} - 0 + 0 = 4(-1)[5(2) - 5(1)] = -4(5) = -20$$

so we see that $\det A = 21 \det B = 21(-20) = -420$.

Of course, in that example, expanding along row 2 would have made the the arithmetic not all that complicated anyway, since only one of the column 1 entries would have been used. But we can have matrices in which we can factor out common factors from more than one row or column, having a much more profound effect in simplifying the arithmetic.

Example 11.2. Find $\det A$ where $A = \begin{bmatrix} 33 & 22 & 88 \\ 3 & 0 & 6 \\ 12 & 0 & 32 \end{bmatrix}$

Solution:
Clearly, we will want to expand along column 2 to calculate the determinant. But before we do so, we can make the arithmetic much easier by factoring out common factors in various rows and columns. We get:

$$\begin{aligned}
\det A &= \det \begin{bmatrix} 33 & 22 & 88 \\ 3 & 0 & 6 \\ 12 & 0 & 32 \end{bmatrix} && \text{(row 1 has common factor 11)} \\
&= 11 \left(\det \begin{bmatrix} 3 & 2 & 8 \\ 3 & 0 & 6 \\ 12 & 0 & 32 \end{bmatrix} \right) && \text{(column 1 has common factor 3)} \\
&= 3 \times 11 \left(\det \begin{bmatrix} 1 & 2 & 8 \\ 1 & 0 & 6 \\ 4 & 0 & 32 \end{bmatrix} \right) && \text{(row 3 has common factor 4)} \\
&= 4 \times 33 \left(\det \begin{bmatrix} 1 & 2 & 8 \\ 1 & 0 & 6 \\ 1 & 0 & 8 \end{bmatrix} \right) && \text{(column 3 has common factor 2)} \\
&= 2 \times 132 \left(\det \begin{bmatrix} 1 & 2 & 4 \\ 1 & 0 & 3 \\ 1 & 0 & 4 \end{bmatrix} \right) && \text{(now expand along column 2)} \\
&= 264 \left(2(-1)^{1+2} \det \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} - 0 + 0 \right) \\
&= 264[2(-1)(4-3)] \\
&= 264(-2)(1) \\
&= -528
\end{aligned}$$

Sometimes, the numbers are obnoxiously small, i.e. fractions or decimals, rather than obnoxiously big. Again, we can make the arithmetic easier by factoring out common factors from rows or columns, effectively multiplying by the common denominator of the fractions in a row or column, or by the power of ten that makes the decimal numbers into integers.

Example 11.3. Find $\det B$, where $B = \begin{bmatrix} \frac{1}{2} & -\frac{1}{6} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{4}{3} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \end{bmatrix}$

Solution:
Ugh! Fractions! (Not really, but ...) Not only are there no 0's, there aren't even any integers! But we can make the arithmetic easier by getting the fractions out in front. We can bring each row to a common denominator, and then factor out "1 over the common denominator" from each row. (Sometimes, we might want to factor from some columns as well as some rows, and/or factor out common factors of numerators as well, but not this time.) We get:

$$\det B = \det \begin{bmatrix} \frac{1}{2} & -\frac{1}{6} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{4}{3} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \end{bmatrix} = \det \begin{bmatrix} \frac{3}{6} & -\frac{1}{6} & \frac{2}{6} \\ -\frac{2}{3} & \frac{1}{3} & \frac{4}{3} \\ \frac{1}{4} & -\frac{1}{4} & \frac{2}{4} \end{bmatrix} = \left(\frac{1}{6}\right)\left(\frac{1}{3}\right)\left(\frac{1}{4}\right) \det \begin{bmatrix} 3 & -1 & 2 \\ -2 & 1 & 4 \\ 1 & -1 & 2 \end{bmatrix}$$

That is, at the last step we factored $\frac{1}{6}$ out of row 1, $\frac{1}{3}$ out of row 2 and $\frac{1}{4}$ out of row 3 (all at once). Now we can see that there's a 2 we can factor out of column 3, and column 2 is a reasonably good choice to expand along:

$$\begin{aligned} \det B &= \left(\frac{1}{6} \times \frac{1}{3} \times \frac{1}{4} \times 2\right) \det \begin{bmatrix} 3 & -1 & 1 \\ -2 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix} \\ &= \frac{2}{6 \times 3 \times 4} \times \left((-1)(-1)^{1+2} \det \begin{bmatrix} -2 & 2 \\ 1 & 1 \end{bmatrix} + (1)(-1)^{2+2} \det \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} + (-1)(-1)^{3+2} \det \begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix} \right) \\ &= \frac{1}{3 \times 3 \times 4} \{(-1)(-1)[-2-2] + (1)(1)[3-1] + (-1)(-1)[6-(-2)]\} \\ &= \frac{1}{36}(-4+2+8) \\ &= \frac{6}{36} \\ &= \frac{1}{6} \end{aligned}$$

When we have a matrix full of obnoxious decimals, it may be easier for you to think of simultaneously multiplying and dividing by a power of 10, rather than "factoring out" the decimal places. For instance, to eliminate a single decimal place in a row or column, we divide the determinant by 10, i.e. multiply by $\frac{1}{10} = 0.1$ while multiplying that row by 10. To eliminate 3 decimal places, we would divide the determinant by 1000, i.e. multiply by 0.001 while multiplying the row or column by 1000 to transform the numbers in the row or column to integers.

Example 11.4. Find the determinant of $\begin{bmatrix} 0.003 & 0.002 \\ 0.05 & 0.04 \end{bmatrix}$.

Solution:
Rather than fiddle around with multiplying all those decimals, we can factor 0.001 out of row 1 and 0.01 out of row 2. That is, we will multiply the determinant by 0.001 while multiplying row 1 by 1000 and then also multiply the determinant by 0.01 while multiplying row 2 by 100. We get:

$$\begin{aligned} \det \begin{bmatrix} 0.003 & 0.002 \\ 0.05 & 0.04 \end{bmatrix} &= (0.001) \det \begin{bmatrix} 3 & 2 \\ 0.05 & 0.04 \end{bmatrix} = (0.001)(0.01) \det \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix} \\ &= (0.00001)(3(4) - 5(2)) = (0.00001)(12 - 10) = (0.00001)(2) = 0.00002 \end{aligned}$$

Let's think again about what Theorem 11.1 told us. The effect of multiplying any row or column by a non-zero scalar is to multiply the determinant by that scalar. We know that if a matrix has two identical rows, or has two identical columns, then the determinant of the matrix is 0. And if we multiply one of those identical rows or columns by a scalar, the determinant of the resulting matrix will be 0 times that scalar, i.e. will also be 0. But then that means that we don't necessarily have to have two identical rows or columns in order to know that the determinant is 0. The determinant will be 0 whenever one row is a scalar multiple of another row, or one column is a scalar multiple of another column, because we can factor out the common scalar from that row or column to obtain a matrix with two identical rows, or two identical columns. That is, we get another Corollary from Theorem 11.1.

Corollary 11.3. *If a square matrix A has one row which is a scalar multiple of another, or one column which is a scalar multiple of another, then* $\det A = 0$.

Example 11.5. Find $\det A$ where $A = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 2 & 4 & 1 & 6 & 2 \\ 5 & 10 & 15 & 9 & 6 \\ -4 & -8 & 7 & 9 & -4 \\ -3 & -6 & 21 & 9 & 18 \end{bmatrix}$.

Solution:
Well, that matrix is both big and relatively ugly. There are no 0's, and so calculating the determinant of this 5×5 matrix will be a lot of work. But wait ... look at columns 1 and 2. We see that column 2 is 2 times column 1. And that means that we don't need to do any of those calculations. Since one column is a scalar multiple of another, we see that

$$\det A = 0$$

Next, let's think about what happens when we interchange 2 rows of a matrix. Let's start with the easiest case. Suppose we interchange the rows of a 2×2 matrix. Then we have

$$\det \begin{bmatrix} c & d \\ a & b \end{bmatrix} = cb - ad = -(ad - cb) = -\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Oh! Look at that! The sign of the determinant changes. In general, suppose we have an $n \times n$ matrix and we interchange rows 1 and 2. That is, consider some $n \times n$ matrix $A = [a_{ij}]$, and let $B = [b_{ij}]$ be the matrix obtained by interchanging rows 1 and 2 of matrix A, so that $b_{1j} = a_{2j}$ and

$b_{2j} = a_{1j}$, but for any other value i, $b_{ij} = a_{ij}$. Then deleting row 2 (and some column) of matrix B has the same result as deleting row 1 (and that same column) of matrix A, so we have $B_{2j} = A_{1j}$. Now, if we calculate $\det B$ by expanding along row 2, and use what we know about the way the $+$'s and $-$'s alternate when we do the expansion (instead of thinking about the powers of (-1)), recognizing that for expansion along row 2 the pattern starts with a negative, we have:

$$\begin{aligned}
\det B &= \textstyle\sum_{j=1}^{n} b_{2j}(-1)^{2+j}B_{2j} \\
&= -b_{21}B_{21} + b_{22}B_{22} + \cdots \pm b_{2n}B_{2n} \\
&= -a_{11}A_{11} + a_{12}A_{12} + \cdots \pm a_{1n}A_{1n} && \text{because } b_{2j} = a_{1j} \\
& && \text{and } B_{2j} = A_{1j} \\
&= -\left(a_{11}A_{11} - a_{12}A_{12} + \cdots \mp a_{1n}A_{1n}\right) && \text{factoring out a } - \\
& && \text{switches } \textit{all} \text{ the signs} \\
&= -\left[\det A\right]
\end{aligned}$$

That is, we were getting the terms of the row 1 expansion of $\det A$, but with the pattern of alternating pluses and minuses starting with a minus, whereas for expansion along row 1 the pattern should start with a plus. So we "factored out a minus", i.e. multiplied by -1 to switch all the signs. We see that once again, the effect of interchanging two rows of a matrix is that the sign of the determinant changes. The same thing would happen if we interchanged *any* two rows, although it's not quite as straightforward to see. And of course if we interchange two columns of a matrix, that's the same as interchanging two rows of the transpose of the matrix, so the effect on the determinant will be the same — the sign of the determinant will change.

Theorem 11.4. *Let A be any square matrix and let B be the matrix obtained either by interchanging two rows of A, or by interchanging two columns of A. Then* $\det B = -\det A$.

Example 11.6. Find $\det \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 0 & 0 & 5 \end{bmatrix}$.

Solution:
If we interchange columns 1 and 2, we'll get an upper triangular matrix:

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 0 & 0 & 5 \end{bmatrix} = -\det \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{bmatrix} = -(2)(1)(5) = -10$$

Example 11.7. Find $\det A$ where $A = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 3 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$.

Solution:
We can transform A to an upper triangular matrix by moving rows around. We have to be careful, though. We need to know how many interchanges of rows are done, so that we know whether or not to change the sign of the determinant. (That is, performing an even number of interchanges of rows switches the sign an even number of times, effectively not switching it at all, but performing an odd

number of interchanges *does* switch the sign.) It's important to remember that we can't just pick up a row and move it somewhere else. For instance, we can't just put the third row at the top of the matrix, leaving the relative positions of the other rows unchanged. We would have to interchange row 1 and row 3, changing the relative positions of what had been the first and second rows. Or move row 3 up gradually, first interchanging it with row 2 and then interchanging it with row 1, performing 2 interchanges to get row 1 to the top and leaving the relative positions of the original first and second rows unchanged. (That's what's done in the following.)

$$\begin{aligned}
\det A &= \det\begin{bmatrix} 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 3 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix} = -\det\begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 1 \end{bmatrix} && \text{(by interchanging rows 2 and 3)} \\
&= \det\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 1 \end{bmatrix} && \text{(by interchanging rows 1 and 2)} \\
&= -\det\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix} && \text{(by interchanging rows 3 and 4)} \\
&= -(1)(1)(2)(3) = -6
\end{aligned}$$

Notice: There's no one *right* pattern of row interchanges here. And some will be longer than others. But all the possible series of interchanges you can do to get from the matrix we started with to the matrix we finished with will involve an odd number of interchanges. And for any matrix, it will always be true that to get from the initial matrix to any particular rearranged matrix by a series of row or column interchanges, either all of the possible series of interchanges will require an even number of interchanges, or all of the possible series of interchanges will require an odd number of interchanges.

In Theorems 11.1 and 11.4, we've seen the effects of 2 of our 3 types of elementary row operations on the determinant of a matrix. What is the effect of the one remaining ero on the determinant? Actually, nothing at all. It's harder to demonstrate this in the general case, i.e. for an $n \times n$ matrix, but easy enough to see for a 2×2 matrix, and since evaluating a determinant can always be expressed in terms of evaluating determinants of 2×2 submatrices, you should then be willing to accept that the same result is true for a larger matrix.

Of course, the ero we're talking about here is "replace a row by that row plus a scalar multiple of another row". Let's see what happens to the determinant of a non-specific 2×2 matrix when we replace row 2 by itself plus some non-specific scalar multiple of row 1, i.e. by row 2 plus k times row 1 for any $k \neq 0$. For instance, if row 1 contains a and b, while row 2 contains c and d, then the new row 2 will contain $c + ka$ and $d + kb$, and we have:

$$\det\begin{bmatrix} a & b \\ c + ka & d + kb \end{bmatrix} = a(d + kb) - (c + ka)b = ad + kab - cb - kab = ad - cb = \det\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

We see that the value of the determinant is not affected at all by this ero. The determinant has the same value as the determinant of the matrix before the ero is performed. And of course, if we did a similar transformation using columns instead of rows, it would be like performing the ero on the transpose of the matrix and would again make no difference to the value of the determinant.

Theorem 11.5. *Let $A = [a_{ij}]$ be any square matrix. Consider the matrix $B = [b_{ij}]$ obtained by adding any scalar multiple of one row (or column) to another row (or column) in A and leaving the other rows (or columns) unchanged. Then* $\det B = \det A$.

Example 11.8. Find $\det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 7 \\ -2 & -4 & 1 \end{bmatrix}$.

Solution:
In this case, row 3 is almost, but not quite, a scalar multiple of row 1. That is, the first two entries of row 3 are -2 times the first two entries of row 1, but the third entry is different. This means that if we add 2 times row 1 to row 2, those first two entries will become zeroes and we'll have an upper triangular matrix. And doing this will not change the value of the determinant. So we see that:

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 7 \\ -2 & -4 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 7 \\ -2+2(1) & -4+2(2) & 1+2(3) \end{bmatrix} = \det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & 7 \end{bmatrix} = (1)(5)(7) = 35$$

Example 11.9. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 2 & 5 & 3 \end{bmatrix}$, find $\det A$.

Solution:
Well, this one is less obvious. But maybe we can transform the matrix to an upper triangular matrix. We'll start by getting a zero in the $(3,1)$-entry, by subtracting 2 times row 1 from row 3. According to Theorem 11.5, the value of the determinant of this new matrix will be the same as the value of the determinant of the original matrix. So we have:

$$\det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 2 & 5 & 3 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 2-2(1) & 5-2(1) & 3-2(1) \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix}$$

And now, we can get a zero in the $(3,2)$-entry by subtracting 3 times row 2 from row 3, again leaving the value of the determinant unchanged:

$$\det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0-3(0) & 3-3(1) & 1-3(1) \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix} = (1)(1)(-2) = -2$$

Using the results from Theorems 11.1, 11.4 and 11.5 together, we can find the determinant of *any* matrix by row-reducing. We don't need to row-reduce all the way to RREF. We simply need to get to an upper (or lower) diagonal matrix. We track the effect of the ero's on the determinant as we perform them.

Example 11.10. Find the determinant of $A = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 3 & 1 & 4 & 1 \\ 2 & 1 & -1 & 2 \\ 1 & 1 & -2 & 1 \end{bmatrix}$.

Solution:
A does not have any row that's helpfully filled with, or nearly filled with, zeroes, so there's no row or column we can expand along that won't require a lot of work. And no row or column is duplicated,

or is a scalar multiple of any other, so we can't easily conclude that the determinant is 0. But rather than crunching through the work of expansion, we can perform the relatively friendlier task of row-reducing the matrix to get to a triangular matrix. We don't necessarily need to worry about getting leading ones (although we may want to, to make the arithmetic easier), and we don't need to entirely clear out columns, we simply need to clear out the entries below (or alternatively above) the main diagonal. Every time we perform an ero, we indicate what the effect on the determinant is. We get:

$$\det A = \det\begin{bmatrix} 1 & -1 & 1 & 2 \\ 3 & 1 & 4 & 1 \\ 2 & 1 & -1 & 2 \\ 1 & 1 & -2 & 1 \end{bmatrix}$$

Start by clearing out column 1 below row 1, by subtracting multiples of row 1, which leaves the determinant unchanged.

$$= \det\begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 4 & 1 & -5 \\ 0 & 3 & -3 & -2 \\ 0 & 2 & -3 & -1 \end{bmatrix}$$

Clearing out column 2 below the main diagonal will be easier with a leading 2 in row 2, so we interchange rows 2 and 4

$$= -\det\begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & -3 & -1 \\ 0 & 3 & -3 & -2 \\ 0 & 4 & 1 & -5 \end{bmatrix}$$

The sign of the determinant changed. Now we subtract 2 times row 2 from row 4 and subtract $\frac{3}{2}$ row 2 from row 3. (No effect on the determinant.)

$$= -\det\begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & -3 & -1 \\ 0 & 0 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 7 & -3 \end{bmatrix}$$

The last calculation will be easier if we get rid of the leading fraction in row 3. Factor $\frac{1}{2}$ out of row 3.

$$= -\tfrac{1}{2}\det\begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & -3 & -1 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 7 & -3 \end{bmatrix}$$

Now subtract $\frac{7}{3}$ row 3 from row 4, which doesn't change the value of the determinant.

$$= -\tfrac{1}{2}\det\begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & -3 & -1 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & -\frac{2}{3} \end{bmatrix}$$

$$= -\left(\tfrac{1}{2}\right)(1)(2)(3)\left(-\tfrac{2}{3}\right) = 2$$

Notice: We could avoid that last, awkward, fraction calculation (i.e. row 4 - $\frac{7}{3}$ row 3) by factoring a 3 out of row 3 (to get a leading one, but introducing a fraction in the (3, 4)-entry), or by calculating the determinant at that point (i.e. almost upper triangular except for a 2×2 in the bottom right corner) by repeatedly expanding on column 1 as follows:

$$\begin{aligned} -\frac{1}{2}\det\begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & -3 & -1 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 7 & -3 \end{bmatrix} &= -\frac{1}{2}\left(1\det\begin{bmatrix} 2 & -3 & -1 \\ 0 & 3 & -1 \\ 0 & 7 & -3 \end{bmatrix} - 0 + 0 - 0\right) \\ &= -\frac{1}{2}\left(1(2)\det\begin{bmatrix} 3 & -1 \\ 7 & -3 \end{bmatrix} - 0 + 0\right) \\ &= -\frac{1}{2}(2)[(3(-3) - 7(-1)] \\ &= -(-9+7) = -(-2) = 2 \end{aligned}$$

Notice that in that example, we had some fractional multipliers, but they miraculously disappeared. That's because they were caused by the leading non-ones appearing on the main diagonal. Of course, the determinant of a matrix which contains only integers must always be integer-valued. So if fractions are introduced as you row-reduce, those fractions *must* cancel out at the end. If they don't, there's something wrong in your arithmetic.

There are also some theoretical results relating the effect on the determinant of certain operations of matrix arithmetic. First of all, suppose a matrix is multiplied by a scalar. What effect does that have on it's determinant? Look out ... I can hear you thinking "Well, that's easy. Obviously, the determinant is multiplied by ..." and up to that point you're right. But if you were going to finish the sentence with "that same scalar." then you're wrong. Think about it for a moment. We've already seen (in Theorem 11.1) that multiplying *one row* of a matrix by a scalar has the effect of multiplying the determinant by that scalar. And for any matrix A, we obtain the matrix cA by multiplying *every element* of the matrix by the scalar c. So for an $n \times n$ matrix, we're multiplying *each of the n rows by c*. When we multiply row 1 by c, that multiplies the determinant by c. And then when we multiply row 2 by c, that multiplies the determinant by c **again!** So now the determinant has been multiplied by $c \times c = c^2$. And by the time we've multiplied each of the n rows by c, we've multiplied the determinant by c n times over, i.e. we've multiplied the determinant by c^n. So the sentence actually needs to be finished with "by that same scalar, raised to the power n, where n is the order of the square matrix.". So we have another Corollary from Theorem 11.1.

Corollary 11.6. *Let A be any square matrix of order n and c be any scalar. Then* $\det cA = c^n \det A$.

Example 11.11. If A is a 4×4 matrix with $\det A = 2$, and B is a 3×3 matrix with $\det B = -1$, find $\det(-A)$ and $\det(5B)$.

Solution:
Since A is a square matrix of order 4, then for any scalar c we have $\det(cA) = (c^4)\det A$. So we get:

$$\det(-A) = \det(-1)A = (-1)^4 \det A = 1(2) = 2$$

Similarly, since matrix B has 3 rows, then $\det(cB) = (c^3)\det B$, so

$$\det(5B) = 5^3 \det B = 125(\det B) = 125(-1) = -125$$

Example 11.12. Find $\det \begin{bmatrix} 24 & 12 \\ 18 & -6 \end{bmatrix}$.

Solution:
Well, it's only a 2×2 matrix, so we know how to "easily" calculate the determinant, but ... the arithmetic looks a bit non-trivial. However, every element in the matrix is a multiple of 6. So we can factor a 6 out of each row, to make the arithmetic friendlier. That is, letting A be the matrix which when multiplied by 6 gives this matrix, the determinant of this matrix is $6^2 \det A$. And we get A by dividing each element by 6. So we see that

$$\det \begin{bmatrix} 24 & 12 \\ 18 & -6 \end{bmatrix} = 6^2 \det \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} = 36[4(-1) - 3(2)] = 36(-4-6) = 36(-10) = -360$$

There's another result, that's harder to explain, that tells us about the determinant of the product of two matrices. We won't worry about *why* it's true. We'll just accept that it *is* true.

Theorem 11.7. *Let A and B be two square matrices of the same order. That is, both are $n \times n$ matrices for some value n, so that the product matrix AB is defined. Then the determinant of the product matrix is the product of the determinants of the 2 matrices. That is,*

$$\det(AB) = (\det A)(\det B)$$

Example 11.13. Let $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 6 \\ 3 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 6 & -3 \\ -4 & 2 \end{bmatrix}$. Find $\det(ABC)$.

Solution:
Since each of the matrices is a 2×2 matrix, then the matrix ABC is defined and is also a 2×2 matrix. That means that once we find what this matrix is, it will be easy to calculate its determinant. But actually calculating ABC will be a lot of work. Instead, we can just use Theorem 11.7 to save us all that work. We simply need to find the determinants of the 3 matrices, and multiply the determinants together, instead of multiplying the matrices. That is, applying the theorem twice, we get:

$$\det(ABC) = \det[(AB)C] = (\det AB)(\det C) = [(\det A)(\det B)](\det C) = (\det A)(\det B)(\det C)$$

Therefore we have:

$$\begin{aligned} \det(ABC) &= (\det A)(\det B)(\det C) = \left(\det \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}\right)\left(\det \begin{bmatrix} -1 & 6 \\ 3 & 4 \end{bmatrix}\right)\left(\det \begin{bmatrix} 6 & -3 \\ -4 & 2 \end{bmatrix}\right) \\ &= [3(5) - 7(2)] \times [(-1)(4) - 3(6)] \times [6(2) - (-4)(-3)] \\ &= (15 - 14)(-4 - 18)(12 - 12) = (1)(-22)(0) = 0 \end{aligned}$$

Example 11.14. If $A = \begin{bmatrix} 1 & 5 & 34 \\ 0 & -2 & 47 \\ 0 & 0 & 10 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 0 \\ 9 & -1 & 0 \\ 15 & -7 & 2 \end{bmatrix}$, find $\det AB$.

Solution:
Notice that although A is an upper triangular matrix, and B is a lower triangular matrix, the product matrix, AB will not be triangular. (If they were both upper triangular, or both lower triangular, then the product would be, too. But not when we have one of each type. Go ahead and experiment. You don't have to experiment with these. Try it with matrices whose non-zeroes are all 1's. You'll see.) So not only will finding AB be a lot of work, but then we'll have to do a lot of work to find it's determinant, too. However finding $\det A$ and $\det B$ is easy. Fortunately we can use Theorem 11.7 and not do much more work than that.

$$\det AB = (\det A)(\det B) = [1(-2)(10)] \times [2(-1)(2)] = -20 \times -4 = 80$$

Example 11.15. If $A = \begin{bmatrix} 5 & 1 \\ 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 2 \\ 7 & 1 \end{bmatrix}$, find $\det(A + B)$.

Solution:
This time it's not a product, it's a sum. We don't have a theorem telling us about the determinant of the sum of two matrices. (Maybe it's coming next? Better not count on it, or make any foolish

assumptions. In the absence of a theorem stating otherwise, we need to do the work.) So we need to find the sum matrix and then calculate its determinant.

$$A+B=\begin{bmatrix} 5 & 1 \\ 4 & 3 \end{bmatrix}+\begin{bmatrix} -3 & 2 \\ 7 & 1 \end{bmatrix}=\begin{bmatrix} 2 & 3 \\ 11 & 4 \end{bmatrix}$$

so we get

$$\det(A+B)=\det\begin{bmatrix} 2 & 3 \\ 11 & 4 \end{bmatrix}=2(4)-11(3)=8-33=-25$$

Notice that $\det A = 11$ and $\det B = -17$ and so $\det A + \det B = -6 \neq \det(A+B)$. (So in fact we aren't going to have a theorem saying anything nice about the determinant of the sum of two matrices. It is always necessary to calculate the sum matrix and find its determinant.)

Theorem 11.7 also has a nice corollary about the determinant of the inverse of a matrix. We know that for any nonsingular matrix A, $AA^{-1} = I$. And we also know that $\det I = 1$. So that means that $\det(AA^{-1}) = 1$. And now from the theorem we also know that $\det(AA^{-1}) = (\det A)(\det A^{-1})$. So we see that:

$$AA^{-1}=I \Rightarrow \det AA^{-1}=1 \Rightarrow (\det A)(\det A^{-1})=1 \Rightarrow \det A^{-1}=\frac{1}{\det A}$$

(Notice that since $(\det A)(\det A^{-1}) = 1$ then it cannot be true that $\det A = 0$, because that would give $(\det A)(\det A^{-1}) = 0 \neq 1$, and so we can divide through by $\det A$.)

Corollary 11.8. *For any nonsingular square matrix A,* $\det A^{-1} = \dfrac{1}{\det A}$.

Of course, if we have to *find* A^{-1} in order to know whether or not A has an inverse, so that we know whether or not we can apply the corollary, then maybe the corollary is of limited usefulness. If only there was an easier way to know whether or not a matrix is invertible.

Let's think again about what we said above. (The parenthetical remark just before the statement of the corollary.) We saw that if A is nonsingular, i.e. if A^{-1} exists, then $\det A$ cannot be 0. Of course, we can also take that the other way. If $\det A = 0$, then it cannot be true that A is invertible, i.e. is nonsingular. (Using the same reasoning as above.) And in fact, whenever $\det A \neq 0$, it turns out that A is nonsingular. That doesn't follow from the reasoning above, or a variation of it, but it's not too hard to see.

Consider any square matrix A. Let B be the RREF of A. Then we know that B can be obtained from A by transforming A using elementary row operations. And we know what the effect on the determinant of those operations is. Some operations don't change the determinant at all. Others change the sign of the determinant. And the only other possibility for ero's is to multiply the determinant by a non-zero scalar. None of those effects will cause the determinant to be 0 if it wasn't already. That is, the net effect on the determinant of all the ero's performed in transforming A to the RREF matrix B can be expressed as $\det A = c \det B$ for some non-zero scalar c. (c is the product of all the non-zero scalars which rows were multiplied by, times the product of all the -1 multipliers resulting from interchanging rows.)

And what do we know about the RREF of a square matrix A? From our Procedure for Finding the Inverse of a Matrix, we know that if the RREF of A is an identity matrix, then A is nonsingular, and otherwise, it's not. That is, if A is nonsingular, then $B = I$, so $\det B = \det I = 1$ and

$\det A = c \det B = c$ for some non-zero value c. So we see that if A is nonsingular, i.e. if A^{-1} exists, then $\det A \neq 0$. (We had already realized that, above.)

Now suppose that $\det A \neq 0$. Then since $\det A = c \det B$ for some non-zero value c, it must also be true that $\det B \neq 0$. Therefore B, which is the RREF of A, doesn't contain a row of only 0's. But for a square matrix, if the RREF is not an identity matrix then it must contain a row of zeroes. So knowing that B does not contains a row of only zeroes tells us that it must be true that $B = I$. And therefore (according to the Procedure for Finding the Inverse of a Matrix) A is invertible, i.e. A is nonsingular.

And so we see that if A is nonsingular then $\det A \neq 0$, and if $\det A \neq 0$ then A is nonsingular. And so "$\det A \neq 0$" and "A is nonsingular" can only occur together. Knowing that one is true also tells us that the other is true.

Theorem 11.9. *Let A be any square matrix. Then A is invertible if and only if* $\det A \neq 0$.

Example 11.16. Prove that the matrix $A = \begin{bmatrix} 4 & 5 & 9 & 12 & 15 \\ 0 & 9 & 6 & 10 & 14 \\ 0 & 0 & -13 & 8 & 11 \\ 0 & 0 & 0 & -21 & 26 \\ 0 & 0 & 0 & 0 & -5 \end{bmatrix}$ has an inverse.

Solution:
Until now, the only way we knew to prove that this square matrix is invertible was to actually find the inverse. (That's not quite true. We have Theorem 9.8 which told us about various other things which are equivalent to knowing that A is nonsingular, so we could prove any of those. For instance, we could prove that $A\vec{x} = \vec{0}$ has only the trivial solution. But we would have to do just as much work to prove that any of those results are true.) And since we weren't asked to actually find A^{-1}, but just to prove that it exists, and finding the inverse would be a lot of work, we'd really rather not do it. Now, because of Theorem 11.9, we don't have to. We only have to calculate $\det A$ and show that it's not 0. And since A is triangular, that's easy. Or would be if those were nicer numbers along the main diagonal of A. But in fact, we don't even have to find the value of the determinant. We just have to show that it's not 0. And we know that the product of a bunch of non-zero numbers, no matter how many, and no matter how ugly, cannot be 0. So here we have

$$\det A = 4(9)(-13)(-21)(-5) \neq 0 \Rightarrow A \text{ has an inverse.}$$

Example 11.17. For $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 7 \\ 0 & 0 & -1 \end{bmatrix}$, find $\det A^{-1}$.

Solution:
We see that $\det A = 1(-2)(-1) = 2 \neq 0$, so A^{-1} exists and using Corollary 11.8 we get

$$\det A^{-1} = \frac{1}{\det A} = \frac{1}{2}$$

Example 11.18. Prove that $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ are both invertible, but the matrix $A - B$ is not.

Solution:
We see that $\det A = 1(2)(5) = 10 \neq 0$, so A is invertible, and that $\det B = (2)(2)(3) = 12 \neq 0$, so B is also invertible. And we have

$$A - B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Since $A - B$ is upper triangular, $\det(A - B) = (-1)(0)(2) = 0$ and therefore $A - B$ is not invertible.

In Example 11.16 above, we mentioned Theorem 9.8, which states several equivalent statements, including "square matrix A is invertible". We now know that "$\det A \neq 0$" is equivalent to "A is invertible", which means that it's also equivalent to all those other statements. So we can add another piece to that theorem. That is, combining Theorem 9.8 and Theorem 11.9 we get the following Corollary.

Corollary 11.10. *If A is a square matrix of order n then the following statements are equivalent to one another.*

1. *A is invertible (i.e., nonsingular).*
2. *$r(A) = n$ (i.e., A has full rank).*
3. *The RREF of A is I (i.e., A is row-equivalent to the identity matrix).*
4. *The system $A\vec{x} = \vec{b}$ has an unique solution (for all $n \times 1$ column vectors $\vec{b}$).*
5. *The homogeneous system $A\vec{x} = \vec{0}$ has only the trivial solution.*
6. $\det A \neq 0$.

Example 11.19. If $A = \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}$, find all solutions to the homogeneous system $A^3\vec{x} = \vec{0}$.

Solution:
We see that $\det A = 3(4) - 5(2) = 12 - 10 = 2$, so $\det A^3 = (\det A)(\det A)(\det A) = 2^3 = 8 \neq 0$. Therefore the homogeneous system with coefficient matrix A^3 has only the trivial solution. That is, the only solution to the given SLE is $\vec{x} = \vec{0}$.

Example 11.20. How many solutions does the SLE $A\vec{x} = \vec{0}$ have, if $A = \begin{bmatrix} 1 & 2 & 5 & 7 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 5 \end{bmatrix}$?

Solution:
We see that $\det A = 1(3)(0)(5) = 0$ so this homogeneous SLE does *not* have only the trivial solution, it has infinitely many solutions. Furthermore, we can see that when we row-reduce A, only column 3 will not contain the leading one for any row, so the system has a one-parameter family of solutions.

Math 1229A/B

Unit 12:
Applications of the Determinant

(text reference: Section 4.3)

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12 Applications of the Determinant

We shall finish up the course by looking at a couple of other ways that the determinant of a square matrix can be used, that is, a couple of applications of the determinant of a square matrix. The first is a method of finding the solution to the SLE $A\vec{x} = \vec{b}$ when A is a nonsingular (i.e. invertible) square matrix, without row reducing or finding the inverse matrix. Instead, we calculate $\det A$ as well as the determinant of certain other matrices obtained from A and $\vec{b}$. After that, we will learn how to find the inverse of square matrix A, when it exists, using $\det A$ and another matrix which is obtained using the cofactors of A. Again, this gives us another method for doing something which previously we could only do by row reducing.

Cramer's Rule

Consider a SLE $A\vec{x} = \vec{b}$ in which A is a square matrix of order n with $\det A \neq 0$. We can form n new $n \times n$ matrices by replacing different columns of A by the column vector $\vec{b}$. And if we do so, then we can directly find the value of x_j in the unique solution to $A\vec{x} = \vec{b}$ using the determinant of one of these new matrices and the determinant of A. A fellow named Cramer developed a rule for doing this. Before we get to the rule, though, we need to define these new matrices and the notation we use to refer to them.

Definition: Let $A\vec{x} = \vec{b}$ be any SLE in which A is a square matrix. We define the matrix $\boldsymbol{A(j)}$ to be the matrix obtained by replacing column j of A with the column vector $\vec{b}$.

First, let's look at an example of forming these matrices, to make sure you understand what the definition is saying.

Example 12.1. Find $A(1)$, $A(2)$ and $A(3)$, where A is the coefficient matrix of the linear system:

$$\begin{array}{ccccccc} x_1 & + & x_2 & - & x_3 & = & 6 \\ x_1 & - & x_2 & + & x_3 & = & 2 \\ x_1 & & & - & 2x_3 & = & 0 \end{array}$$

Solution:
We have the SLE $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix} \qquad \text{and} \qquad \vec{b} = \begin{bmatrix} 6 \\ 2 \\ 0 \end{bmatrix}$$

We form the matrix $A(j)$ by replacing the j^{th} column of A by the column vector $\vec{b}$. So for instance to form $A(1)$ we write the numbers from $\vec{b}$ instead of the first column of A, and then write columns 2 and 3 of A as usual. And so forth. We get:

$$A(1) = \begin{bmatrix} 6 & 1 & -1 \\ 2 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \qquad A(2) = \begin{bmatrix} 1 & 6 & -1 \\ 1 & 2 & 1 \\ 1 & 0 & -2 \end{bmatrix} \qquad A(3) = \begin{bmatrix} 1 & 1 & 6 \\ 1 & -1 & 2 \\ 1 & 0 & 0 \end{bmatrix}$$

Now, how do we use these matrices? Recall that as long as $\det A \neq 0$, the SLE $A\vec{x} = \vec{b}$ has a unique solution. According to Mr. Cramer, the values of the x_j's in the unique solution to such a system are the quotients of the determinants of these new matrices over the determinant of the coefficient matrix. Neat trick, eh? We won't attempt to prove or even explain *why* this works. We'll just take Mr. Cramer's word for it. (And that means that you can just ignore pages 170 and 171 in the text.)

Theorem 12.1. Cramer's Rule
Let A be any square matrix of order n with $\det A \neq 0$ and let $\vec{b}$ be any $n \times 1$ column vector. Then in the unique solution to the system $A\vec{x} = \vec{b}$, the value of the j^{th} unknown, x_j, is given by:

$$x_j = \frac{\det A(j)}{\det A}$$

This means that if these determinants are reasonably easy to find, using Cramer's Rule can be an easier way to find the solution to a SLE than row reducing. (However if the determinants require a lot of work to calculate, then using Cramer's Rule involves more work than row reducing. So that's just obnoxious.) For instance determinants of 2×2 matrices are always easy to find, so Cramer's Rule is a reasonably good way to solve a system of 2 equations in 2 unknowns, as long as the coefficient matrix is nonsingular. And if there are 0's around then sometimes the determinants of larger square matrices are reasonably easy to find. The following examples show how Cramer's Rule can be used to find the unique solution to a SLE with a nonsingular square coefficient matrix. It's important to remember, though, that Cramer's Rule simply doesn't apply to a system whose coefficient matrix isn't square, or has determinant 0.

Example 12.2. Use Cramer's Rule to find the unique solution to the SLE in Example 12.1.

Solution:
We have the SLE:

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & - & x_3 & = & 6 \\ x_1 & - & x_2 & + & x_3 & = & 2 \\ x_1 & & & - & 2x_3 & = & 0 \end{array}$$

with coefficient matrix $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix}$ and we found the matrices

$$A(1) = \begin{bmatrix} 6 & 1 & -1 \\ 2 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \qquad A(2) = \begin{bmatrix} 1 & 6 & -1 \\ 1 & 2 & 1 \\ 1 & 0 & -2 \end{bmatrix} \qquad A(3) = \begin{bmatrix} 1 & 1 & 6 \\ 1 & -1 & 2 \\ 1 & 0 & 0 \end{bmatrix}$$

First, we find $\det A$. We can zero out column 1 and then expand along that column:

$$\begin{aligned} \det A &= \det \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 2 \\ 0 & -1 & -1 \end{bmatrix} = 1 \det \begin{bmatrix} -2 & 2 \\ -1 & -1 \end{bmatrix} - 0 + 0 \\ &= (-2)(-1) - (-1)(2) = 2 - (-2) = 4 \end{aligned}$$

Notice that $\det A \neq 0$, so the SLE does indeed have a unique solution. Now we find the determinants of the $A(j)$ matrices. For $A(1)$ we can expand along row 3:

$$\begin{aligned} \det A(1) &= \det \begin{bmatrix} 6 & 1 & -1 \\ 2 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix} = 0 - 0 + (-2) \det \begin{bmatrix} 6 & 1 \\ 2 & -1 \end{bmatrix} \\ &= (-2)[(6)(-1) - 2(1)] = (-2)[-6 - 2] = (-2)(-8) = 16 \end{aligned}$$

For $A(2)$, it will be easiest to zero out column 1 again:

$$\begin{aligned}\det A(2) &= \det\begin{bmatrix} 1 & 6 & -1 \\ 1 & 2 & 1 \\ 1 & 0 & -2 \end{bmatrix} = \det\begin{bmatrix} 1 & 6 & -1 \\ 0 & -4 & 2 \\ 0 & -6 & -1 \end{bmatrix} = 1\det\begin{bmatrix} -4 & 2 \\ -6 & -1 \end{bmatrix} - 0 + 0 \\ &= (-4)(-1) - (-6)(2) = 4 - (-12) = 16\end{aligned}$$

Finally, for $A(3)$ we can expand along row 3 again:

$$\det A(3) = \det\begin{bmatrix} 1 & 1 & 6 \\ 1 & -1 & 2 \\ 1 & 0 & 0 \end{bmatrix} = 1\det\begin{bmatrix} 1 & 6 \\ -1 & 2 \end{bmatrix} - 0 + 0 = (1)(2) - (-1)(6) = 2 - (-6) = 8$$

So using Cramer's Rule, we see that the values of x_1, x_2 and x_3 in the unique solution to the system are:

$$x_1 = \frac{\det A(1)}{\det A} = \frac{16}{4} = 4, \qquad x_2 = \frac{\det A(2)}{\det A} = \frac{16}{4} = 4, \qquad x_3 = \frac{\det A(3)}{\det A} = \frac{8}{4} = 2$$

That is, the unique system to this SLE is $(x_1, x_2, x_3) = (4, 4, 2)$.

As always, it's a good idea to check. For $\vec{x} = (4, 4, 2)$ we have:

$$A\vec{x} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1(4) + 1(4) + (-1)(2) \\ 1(4) + (-1)(4) + 1(2) \\ 1(4) + 0(4) + (-2)(2) \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 0 \end{bmatrix} = \vec{b}$$

Example 12.3. Use Cramer's Rule, if possible, to solve the system $\begin{array}{rcrcr} 3x_1 & - & x_2 & = & 5 \\ 2x_1 & + & x_2 & = & -2 \end{array}$.

Solution:
We have the coefficient matrix $A = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$ and the RHS column vector $\vec{b} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$. We see that $\det A = (3)(1) - (2)(-1) = 3 - (-2) = 5 \neq 0$, so Cramer's Rule can be used. We get:

$$\begin{aligned} x_1 &= \frac{\det A(1)}{\det A} = \frac{\det\begin{bmatrix} 5 & -1 \\ -2 & 1 \end{bmatrix}}{\det A} = \frac{(5)(1) - (-2)(-1)}{5} = \frac{5-2}{5} = \frac{3}{5} \\ \text{and } x_2 &= \frac{\det A(2)}{\det A} = \frac{\det\begin{bmatrix} 3 & 5 \\ 2 & -2 \end{bmatrix}}{\det A} = \frac{(3)(-2) - (2)(5)}{5} = \frac{-6-10}{5} = -\frac{16}{5}\end{aligned}$$

Therefore the unique solution to the SLE is $(x_1, x_2) = \left(\frac{3}{5}, -\frac{16}{5}\right)$.

Check:

$$\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} \frac{3}{5} \\ -\frac{16}{5} \end{bmatrix} = \begin{bmatrix} 3\left(\frac{3}{5}\right) - 1\left(-\frac{16}{5}\right) \\ 2\left(\frac{3}{5}\right) + 1\left(-\frac{16}{5}\right) \end{bmatrix} = \begin{bmatrix} \frac{9}{5} + \frac{16}{5} \\ \frac{6}{5} - \frac{16}{5} \end{bmatrix} = \begin{bmatrix} \frac{25}{5} \\ -\frac{10}{5} \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix} = \vec{b}$$

Example 12.4. Use Cramer's Rule, if possible, to solve the following SLE:

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & x_3 & = & 0 \\ & & 3x_2 & + & 2x_3 & = & 2 \\ & & x_2 & + & 3x_3 & = & 0 \end{array}$$

Solution:

We have $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ 0 & 1 & 3 \end{bmatrix}$, with $\vec{b} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$. We get

$$\det A = \det \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ 0 & 1 & 3 \end{bmatrix} = 1 \det \begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix} - 0 + 0 = 3(3) - 1(2) = 7$$

so $\det A \neq 0$ and the system has a unique solution which we may find using Cramer's Rule. We get:

$$x_1 = \frac{\det A(1)}{\det A} = \frac{\det \begin{bmatrix} 0 & 2 & 1 \\ 2 & 3 & 2 \\ 0 & 1 & 3 \end{bmatrix}}{7} = \frac{0 - 2\det \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} + 0}{7} = \frac{-2(6-1)}{7} = -\frac{10}{7}$$

$$x_2 = \frac{\det A(2)}{\det A} = \frac{\det \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}}{7} = \frac{(1)(2)(3)}{7} = \frac{6}{7}$$

$$x_3 = \frac{\det A(3)}{\det A} = \frac{\det \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 2 \\ 0 & 1 & 0 \end{bmatrix}}{7} = \frac{1\det \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} - 0 + 0}{7} = \frac{0-2}{7} = -\frac{2}{7}$$

We get the unique solution $(x_1, x_2, x_3) = \left(-\frac{10}{7}, \frac{6}{7}, -\frac{2}{7}\right)$.

Check:

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} -\frac{10}{7} \\ \frac{6}{7} \\ -\frac{2}{7} \end{bmatrix} = \begin{bmatrix} -\frac{10}{7} + \frac{12}{7} - \frac{2}{7} \\ 0 + \frac{18}{7} - \frac{4}{7} \\ 0 + \frac{6}{7} - \frac{6}{7} \end{bmatrix} = \begin{bmatrix} \frac{0}{7} \\ \frac{14}{7} \\ \frac{0}{7} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \vec{b}$$

The Adjoint form of the Inverse

As we know, if square matrix A has $\det A \neq 0$, then A is invertible. We can use something called the *adjoint* of A, also called the *classical adjoint* of A to express a formula for A^{-1}. The adjoint of A is just a matrix whose elements are the cofactors of A^T. That is, the (i,j)-entry of the matrix called the adjoint of A is the (i,j)-cofactor of A^T, which can also expressed as the (j,i)-cofactor of A. Recall from Unit 10 that the (i,j)-cofactor of a matrix B is defined to be $(-1)^{i+j} \det B_{ij}$ (see Unit 10 page 130), where B_{ij} is the matrix obtained by deleting row i and column j from matrix B (see Unit 10 page 129). Of course, if $B = A^T$, then $B_{ij} = A_{ji}$. So the easiest way to express the (i,j)-cofactor of A^T is as $(-1)^{i+j} \det A_{ji}$. (Otherwise we would have to write $(A^T)_{ij}$, which is awkward.)

Definition: The **Adjoint**, or **Classical Adjoint**, of a square matrix A is denoted ***Adj A*** and is the matrix whose (i,j)-entry is the (i,j)-cofactor of A^T, so that

$$(i,j)\text{-entry of } Adj A = (-1)^{i+j} \det A_{ji}$$

Therefore, to find $Adj A$, we simply make a matrix in which the (i,j)-entry is the (i,j)-cofactor of A^T. To do this, it's usually easiest to start by finding A^T. (Alternatively, we could find the matrix of cofactors of A, and then take the transpose of this matrix, but if we do the transposing at the beginning, it's done with and we don't have to remember to do it at the end.) Of course, since the cofactors are what we use to calculate the determinant, the effect of the (-1) multipliers is to make the signs alternate, just the way they do for determinant calculations.

Example 12.5. Find $Adj A$ where $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Solution:
We have $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$, so we get:

$$Adj A = \begin{bmatrix} +\det A_{11} & -\det A_{21} \\ -\det A_{12} & +\det A_{22} \end{bmatrix} = \begin{bmatrix} \det[4] & -\det[2] \\ -\det[3] & \det[1] \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

Example 12.6. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$, find $Adj A$.

Solution:
We have $A^T = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{bmatrix}$, so we get:

$$\begin{aligned} Adj A &= \begin{bmatrix} \det\begin{bmatrix} 4 & 0 \\ 5 & 6 \end{bmatrix} & -\det\begin{bmatrix} 2 & 0 \\ 3 & 6 \end{bmatrix} & \det\begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix} \\ -\det\begin{bmatrix} 0 & 0 \\ 5 & 6 \end{bmatrix} & \det\begin{bmatrix} 1 & 0 \\ 3 & 6 \end{bmatrix} & -\det\begin{bmatrix} 1 & 0 \\ 3 & 5 \end{bmatrix} \\ \det\begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix} & -\det\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} & \det\begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 24 & -12 & -2 \\ 0 & 6 & -5 \\ 0 & 0 & 4 \end{bmatrix} \end{aligned}$$

Example 12.7. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

(a) Find $A(Adj A)$. (b) Find $(Adj A)A$. (c) Find $\det A$. (d) Show that $\left(\frac{1}{\det A}\right) Adj A = A^{-1}$.

Solution:
In Example 12.5 we found $Adj A = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$. We use that to calculate the matrix products in parts (a) and (b).

(a) We get:

$$A(Adj A) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 4-6 & -2+2 \\ 12-12 & -6+4 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

Hmm. A diagonal matrix. With the same entry repeated along the diagonal. (Must be a fluke.)

(b) If we reverse the order of the matrices, we get:

$$(Adj\,A)A = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4-6 & 8-8 \\ -3+3 & -6+4 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

Why look! It's the same matrix! We know that generally $AB \neq BA$, so that's surprising.

(c) We have $\det A = \det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 1(4) - 3(2) = 4 - 6 = -2$. Goodness! That's the number that was all along the diagonal of the 2 (same) product matrices.

(d) We already found, in part (b), that $(Adj\,A)A = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$, and from part (c) we have $\det A = -2$, so we see that

$$\left[\left(\frac{1}{\det A}\right) Adj\,A\right] A = \left(\frac{1}{-2}\right)[(Adj\,A)A] = \left(-\frac{1}{2}\right)\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Since $\left[\left(\frac{1}{\det A}\right) Adj\,A\right] A = I_2$, then by Theorem 8.2 $\left(\frac{1}{\det A}\right) Adj\,A$ is the inverse of A.

What we found in that example was not, of course, just a fluke. (If it was just a fluke, we either wouldn't have done that example, or would immediately follow it with another example that didn't turn out that way, to show it wasn't a general rule.) It can be proved (although we're not going to do it) that for *any* square matrix A, it is *always* true that

$$A(Adj\,A) = (Adj\,A)A = (\det A)I$$

And that means that if $\det A \neq 0$, then A is invertible and we can post-multiply through by A^{-1} in the equation $(Adj\,A)A = (det A)I$ to get

$$(Adj\,A)A(A^{-1}) = (\det A)I(A^{-1}) \quad \Rightarrow \quad Adj\,A = (\det A)A^{-1} \quad \Rightarrow \quad A^{-1} = \left(\frac{1}{\det A}\right) Adj\,A$$

So this gives us a very different way to find A^{-1}, when it exists, than the way we were doing it before.

Theorem 12.2. The Adjoint form of the Inverse
If A is any nonsingular square matrix, then $A^{-1} = \left(\frac{1}{\det A}\right) Adj\,A$.

Example 12.8. Find the inverse of the matrix A in Example 12.6.

Solution:
We have $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$, and in Example 12.6 we found that $Adj\,A = \begin{bmatrix} 24 & -12 & -2 \\ 0 & 6 & -5 \\ 0 & 0 & 4 \end{bmatrix}$. We see that $\det A = (1)(4)(6) = 24$ and so using the adjoint to find the inverse matrix we get:

$$A^{-1} = \left(\frac{1}{\det A}\right) Adj\,A = \left(\frac{1}{24}\right)\begin{bmatrix} 24 & -12 & -2 \\ 0 & 6 & -5 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{12} \\ 0 & \frac{1}{4} & -\frac{5}{24} \\ 0 & 0 & \frac{1}{6} \end{bmatrix}$$

Notice that besides giving us a new way to find the inverse matrix, it also gives us a way to check our calculation of $Adj A$. We check that this really is the inverse of A:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{12} \\ 0 & \frac{1}{4} & -\frac{5}{24} \\ 0 & 0 & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} 1+0+0 & -\frac{1}{2}+\frac{2}{4}+0 & -\frac{1}{12}-\frac{10}{24}+\frac{3}{6} \\ 0+0+0 & 0+\frac{4}{4}+0 & 0-\frac{20}{24}+\frac{5}{6} \\ 0+0+0 & 0+0+0 & 0+0+\frac{6}{6} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 12.9. Use the adjoint to find the inverse of $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix}$.

Solution:
Expanding along row 2 we get $\det A = -0 + 3 \det \begin{bmatrix} 1 & 2 \\ 1 & 5 \end{bmatrix} - 0 = 3(5-2) = 9$, so A^{-1} exists. We have $A^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 2 & 0 & 5 \end{bmatrix}$ so we get

$$\begin{aligned} Adj A &= \begin{bmatrix} \det\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} & -\det\begin{bmatrix} 0 & 0 \\ 2 & 5 \end{bmatrix} & \det\begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix} \\ -\det\begin{bmatrix} 0 & 1 \\ 0 & 5 \end{bmatrix} & \det\begin{bmatrix} 1 & 1 \\ 2 & 5 \end{bmatrix} & -\det\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \\ \det\begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix} & -\det\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} & \det\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 15 & 0 & -6 \\ 0 & 3 & 0 \\ -3 & 0 & 3 \end{bmatrix} \\ \text{and } A^{-1} &= \left(\frac{1}{\det A}\right) Adj A = \frac{1}{9} \begin{bmatrix} 15 & 0 & -6 \\ 0 & 3 & 0 \\ -3 & 0 & 3 \end{bmatrix} = \begin{bmatrix} \frac{15}{9} & 0 & -\frac{6}{9} \\ 0 & \frac{3}{9} & 0 \\ -\frac{3}{9} & 0 & \frac{3}{9} \end{bmatrix} = \begin{bmatrix} \frac{5}{3} & 0 & -\frac{2}{3} \\ 0 & \frac{1}{3} & 0 \\ -\frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} \end{aligned}$$

Check:

$$AA^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} \frac{5}{3} & 0 & -\frac{2}{3} \\ 0 & \frac{1}{3} & 0 \\ -\frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{5}{3}+0-\frac{2}{3} & 0+0+0 & -\frac{2}{3}+0+\frac{2}{3} \\ 0+0+0 & 0+1+0 & 0+0+0 \\ \frac{5}{3}+0-\frac{5}{3} & 0+0+0 & -\frac{2}{3}+0+\frac{5}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 12.10. If $\det A = 6$ and $Adj A = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, find A.

Solution:
We find A^{-1} and then find its inverse, which is A.

$$A^{-1} = \left(\frac{1}{\det A}\right) Adj A = \left(\frac{1}{6}\right) \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

Notice that in this case the easiest way to find $(A^{-1})^{-1} = A$ is to row reduce. We just multiply row 2 by 2 and row 3 by 3:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 \end{array}\right]$$

We see that $A = (A^{-1})^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

Example 12.11. If $\det A = 2$ and $A^{-1} = \begin{bmatrix} 3 & 2 & 1 \\ 0 & \frac{1}{2} & 2 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$, find $Adj A$.

Solution:
Since $A^{-1} = \left(\frac{1}{\det A}\right) Adj A$, then $Adj A = (\det A)A^{-1}$. (That is, just multiply through the equation by $\det A$.) So we see that:

$$Adj A = (\det A)A^{-1} = 2 \begin{bmatrix} 3 & 2 & 1 \\ 0 & \frac{1}{2} & 2 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 6 & 4 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & \frac{2}{3} \end{bmatrix}$$

Example 12.12. If A is a 5×5 matrix with $\det A = -2$, find $\det(Adj A)$.

Solution:
As we have already seen, the formula $A^{-1} = \left(\frac{1}{\det A}\right) Adj A$ can be rearranged to $Adj A = (\det A)A^{-1}$, so we just need to take the determinant of both sides. That is, since $Adj A = (\det A)A^{-1}$ then $\det(Adj A) = \det[(\det A)A^{-1}]$. We use the facts that (1) when we factor a scalar multiplier out of a determinant calculation, the scalar gets raised to the power n, where n is the order of the square matrix whose determinant we're calculating (i.e. $\det(cA) = c^n \det A$, from Corollary 11.6 on page 152), and of course the $\det A$ multiplier is a scalar, and (2) the determinant of the inverse is the inverse (i.e. reciprocal) of the determinant (that is, $\det\left(A^{-1}\right) = \frac{1}{\det A}$, from Corollary 11.8 on page 154). Here, since A is a square matrix of order 5, then so is A^{-1} and so we get:

$$\det(Adj A) = \det[(\det A)A^{-1}] = (\det A)^5 \det(A^{-1}) = (-2)^5 \left(\frac{1}{\det A}\right) = \frac{(-2)^5}{-2} = (-2)^4 = 16$$

We'll finish up the course by looking at one last example, to tie what we've just been learning into solving systems of equations.

Example 12.13. If $\det A = 4$ and $Adj A = \begin{bmatrix} 4 & -2 & -2 \\ -8 & 6 & 2 \\ 2 & -2 & 2 \end{bmatrix}$, find the unique solution to the SLE $A\vec{x} = \vec{b}$ where $\vec{b} = (2, -1, 1)$.

Solution:
Since we know the adjoint matrix and the value of $\det A$, then we actually know A^{-1} and so we can use the method of inverses. We have $A^{-1} = \left(\frac{1}{4}\right) Adj A$ and so we get:

$$\vec{x} = A^{-1}\vec{b} = \left(\frac{1}{4}\right)(Adj A)\vec{b} = \frac{1}{4}\begin{bmatrix} 4 & -2 & -2 \\ -8 & 6 & 2 \\ 2 & -2 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 8+2-2 \\ -16-6+2 \\ 4+2+2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 8 \\ -20 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 2 \end{bmatrix}$$

Notice that as usual we can save ourselves some hassle by waiting until the end to apply the fractional scalar multiplier.