

CONCORDIA UNIVERSITY

DEPARTMENT OF COMPUTER SCIENCE & SOFTWARE ENGINEERING

COMP 232/2

Mathematics for Computer Science

Fall 2014

Assignment 1 Solutions

1. For each of the following statements use a truth table to determine whether it is a tautology, a contradiction, or a contingency.

$$(a) \underbrace{(p \vee r)}_a \wedge \underbrace{(q \vee r)}_b \leftrightarrow \underbrace{((p \wedge q) \vee r)}_c$$

Solution: Tautology.

p	q	r	$\underbrace{p \vee r}_a$	$\underbrace{q \vee r}_b$	$a \wedge b$	$p \wedge q$	$\underbrace{(p \wedge q) \vee r}_c$	$(a \wedge b) \leftrightarrow c$
T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T	T
T	F	T	T	T	T	F	T	T
F	T	T	T	T	T	F	T	T
T	F	F	T	F	F	F	F	T
F	T	F	F	T	F	F	F	T
F	F	T	T	T	T	F	T	T
F	F	F	F	F	F	F	F	T

$$(b) \underbrace{(p \wedge (\neg(\neg p \vee q)))}_a \vee \underbrace{(p \wedge q)}_c$$

Solution: Contingency.

p	q	$\neg p$	$\neg p \vee q$	$\underbrace{\neg(\neg p \vee q)}_a$	$\underbrace{p \wedge a}_b$	$\underbrace{p \wedge q}_c$	$b \vee c$
T	T	F	T	F	F	T	T
T	F	F	F	T	T	F	T
F	T	T	T	F	F	F	F
F	F	T	T	F	F	F	F

$$(c) \underbrace{(p \wedge \underbrace{(\neg q \rightarrow \neg p)}_a)}_b \rightarrow q$$

Solution: Tautology.

p	q	$\neg p$	$\neg q$	$\underbrace{\neg q \rightarrow \neg p}_a$	$\underbrace{p \wedge a}_b$	$b \rightarrow q$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	F	T

$$(d) \underbrace{(p \rightarrow r)}_a \vee \underbrace{(q \rightarrow r)}_b \rightarrow \underbrace{((p \vee q) \rightarrow r)}_c$$

Solution: Contingency.

p	q	r	$\underbrace{p \rightarrow r}_a$	$\underbrace{q \rightarrow r}_b$	$a \vee b$	$p \vee q$	$\underbrace{(p \vee q) \rightarrow r}_c$	$(a \vee b) \rightarrow c$
T	T	T	T	T	T	T	T	T
T	T	F	F	F	F	T	F	T
T	F	T	T	T	T	T	T	T
F	T	T	T	T	T	T	T	T
T	F	F	F	T	T	T	F	F
F	T	F	T	F	T	T	F	F
F	F	T	T	T	T	F	T	T
F	F	F	T	T	T	F	T	T

2. For each of the following logical equivalences state whether it is valid or invalid. If invalid then give a counterexample (*e.g.*, based on a truth table). If valid then give an algebraic proof using logical equivalences from Tables 6, 7, and 8 from Section 1.3 of textbook.

(a) $p \rightarrow (q \rightarrow r) \equiv q \rightarrow (\neg p \vee r)$

Solution: Valid.

$$\begin{aligned}
 & p \rightarrow (q \rightarrow r) \\
 \equiv & \neg p \vee (q \rightarrow r) && \text{law for conditional} \\
 \equiv & \neg p \vee (\neg q \vee r) && \text{law for conditional} \\
 \equiv & (\neg p \vee \neg q) \vee r && \text{associativity} \\
 \equiv & (\neg q \vee \neg p) \vee r && \text{commutativity} \\
 \equiv & \neg q \vee (\neg p \vee r) && \text{associativity} \\
 \equiv & q \rightarrow (\neg p \vee r) && \text{law for conditional}
 \end{aligned}$$

(b) $(p \rightarrow r) \wedge (q \rightarrow r) \equiv ((p \wedge q) \rightarrow r)$

Solution: Invalid.

If $p = T$, $q = F$, and $r = F$ then the LHS is False, while the RHS is True.

(c) $(p \rightarrow q) \wedge (p \rightarrow r) \equiv (p \rightarrow (q \wedge r))$

Solution: Valid.

$$\begin{aligned}
 & (p \rightarrow q) \wedge (p \rightarrow r) \\
 \equiv & (\neg p \vee q) \wedge (\neg p \vee r) && \text{law for conditional} \\
 \equiv & \neg p \vee (q \wedge r) && \text{distributivity} \\
 \equiv & (p \rightarrow (q \wedge r)) && \text{law for conditional}
 \end{aligned}$$

(d) $((p \vee q) \wedge (\neg p \vee r)) \equiv (q \vee r)$

Solution: Invalid.

If $p = T$, $q = T$, and $r = F$ then the LHS is False, while the RHS is True.

3. Five persons, anonymously known as P_1, P_2, P_3, P_4, P_5 , are suspected of being involved in a crime. Suppose we have the following information:
- (a) If P_5 is involved then so is P_3 .
 - (b) If P_2 is involved then so are P_5 and P_1 .
 - (c) Either P_1 or P_2 , or both, are involved.
 - (d) Either P_3 or P_4 , but not both, are involved.
 - (e) P_4 and P_1 are either both involved or neither is.

Determine which persons were involved in the crime. Explain your reasoning.

Solution:

We write the statements in propositional logic, using p_i to denote that P_i is involved, and $\neg p_i$ to denote that P_i is not involved,

- (a) $p_5 \rightarrow p_3$
- (b) $p_2 \rightarrow (p_5 \wedge p_1)$
- (c) $p_1 \vee p_2$
- (d) $p_3 \oplus p_4$
- (e) $p_4 \leftrightarrow p_1$

From (d) we have two possibilities, namely, $p_3 = T, p_4 = F$, or $p_3 = F, p_4 = T$. We consider each of these possibilities separately.

- i If $p_3 = T$ and $p_4 = F$ then it follows from (e) that $p_1 = F$. Since $p_1 = F$ it follows from (c) that $p_2 = T$. Since $p_2 = T$ it follows in particular from (b) that $p_1 = T$. So p_1 is both True and False, which is a contradiction. Thus the assumption that $p_3 = T, p_4 = F$ cannot be true, in other words, the assumption that $p_3 = T, p_4 = F$ does not lead to a solution of the crime problem.
- ii If $p_3 = F$ and $p_4 = T$ then it follows from (a) that $p_5 = F$. Since $p_5 = F$ it follows from (b) that $p_2 = F$. Since $p_2 = F$ it follows from (c) that $p_1 = T$. Finally, since both $p_4 = T$ and $p_1 = T$, we see that (e) is also satisfied.

Having considered the two cases above, we see that there is only one solution, namely p_1 and p_4 were involved, but p_2, p_3 , and p_5 are not.

4. Write the following statements in predicate form, using logical operators \wedge , \vee , \neg , and quantifiers \forall , \exists . Below \mathbb{Z}^+ denotes all positive integers $\{1, 2, 3, \dots\}$.

(a) For any $x, y \in \mathbb{Z}^+$ the equation $x^2 + y^2 - z = 0$ has a solution $z \in \mathbb{Z}^+$.

Solution:

$$\forall x, y \in \mathbb{Z}^+ \exists z \in \mathbb{Z}^+ (x^2 + y^2 - z = 0)$$

(b) The equation $x^3 + y^3 = z^3$ has no solutions $x, y, z \in \mathbb{Z}^+$.

Solution:

$$\neg \left(\exists x, y, z \in \mathbb{Z}^+ (x^3 + y^3 = z^3) \right)$$

or equivalently

$$\forall x, y, z \in \mathbb{Z}^+ (x^3 + y^3 \neq z^3)$$

(c) There is no greatest positive integer.

Solution:

$$\neg \left(\exists x \in \mathbb{Z}^+ \forall y \in \mathbb{Z}^+ (x \geq y) \right)$$

or equivalently

$$\forall x \in \mathbb{Z}^+ \exists y \in \mathbb{Z}^+ (x < y)$$

(d) The difference between two positive integers can be arbitrarily large.

Solution:

$$\forall z \in \mathbb{Z}^+ \exists x, y \in \mathbb{Z}^+ (x - y > z)$$

5. Let the Universe of Discourse for x be the set of all students in this class and for y be the set of all countries in the world. Let $P(x, y)$ denote student x has visited country y and $Q(x, y)$ denote student x has a friend in country y . Express each of the following using logical operations and quantifiers, and the propositional functions $P(x, y)$ and $Q(x, y)$.

- (a) Carlos has visited Bulgaria.

Solution:

$$P(\text{Carlos}, \text{Bulgaria})$$

- (b) Every student in this class has visited the United States.

Solution:

$$\forall x P(x, \text{US})$$

- (c) Every student in this class has visited some country in the world.

Solution:

$$\forall x \exists y P(x, y)$$

- (d) There is no country that every student in this class has visited.

Solution:

$$\forall y \exists x (\neg P(x, y))$$

- (e) There are two students in this class, who between them, have a friend in every country in the world.

Solution:

$$\exists x_1 \exists x_2 (x_1 \neq x_2 \wedge \forall y (Q(x_1, y) \vee Q(x_2, y)))$$

- (f) Nobody in this class has visited a country in which they did not have a friend.

Solution:

$$\neg (\exists x \exists y (P(x, y) \wedge \neg Q(x, y)))$$

or equivalently

$$\forall x \forall y (P(x, y) \rightarrow Q(x, y))$$

- (g) There is a country in which nobody in this class has a friend.

Solution:

$$\exists y \forall x (\neg Q(x, y))$$

- (h) Everyone in this class has visited every country in which they have a friend.

Solution:

$$\forall x \forall y (Q(x, y) \rightarrow P(x, y))$$

6. Negate the following statements and transform the negation so that negation symbols immediately precede predicates.

(a) $\exists x \exists y P(x, y) \wedge \forall x \forall y Q(x, y)$

Solution:

$$\begin{aligned} & \neg \left(\exists x \exists y P(x, y) \wedge \forall x \forall y Q(x, y) \right) \\ \equiv & \neg \left(\exists x \exists y P(x, y) \right) \vee \neg \left(\forall x \forall y Q(x, y) \right) \\ \equiv & \left(\forall x \forall y \neg P(x, y) \right) \vee \left(\exists x \exists y \neg Q(x, y) \right) \end{aligned}$$

(b) $\exists x \exists y (Q(x, y) \leftrightarrow Q(y, x))$

Solution:

$$\begin{aligned} & \neg \left(\exists x \exists y (Q(x, y) \leftrightarrow Q(y, x)) \right) \\ \equiv & \forall x \forall y \left(\neg (Q(x, y) \leftrightarrow Q(y, x)) \right) \\ \equiv & \forall x \forall y \neg \left((Q(x, y) \rightarrow Q(y, x)) \wedge (Q(y, x) \rightarrow Q(x, y)) \right) \\ \equiv & \forall x \forall y \left(\neg (Q(x, y) \rightarrow Q(y, x)) \vee \neg (Q(y, x) \rightarrow Q(x, y)) \right) \\ \equiv & \forall x \forall y \left(\neg (\neg Q(x, y) \vee Q(y, x)) \vee \neg (\neg Q(y, x) \vee Q(x, y)) \right) \\ \equiv & \forall x \forall y \left((Q(x, y) \wedge \neg Q(y, x)) \vee (Q(y, x) \wedge \neg Q(x, y)) \right) \end{aligned}$$

(c) $\forall y \exists x \exists z (T(x, y, z) \vee Q(x, y))$

Solution:

$$\begin{aligned} & \neg \left(\forall y \exists x \exists z (T(x, y, z) \vee Q(x, y)) \right) \\ \equiv & \exists y \forall x \forall z \neg (T(x, y, z) \vee Q(x, y)) \\ \equiv & \exists y \forall x \forall z (\neg T(x, y, z) \wedge \neg Q(x, y)) \end{aligned}$$

7. Let P and Q be predicates on the set S , where S has two elements, say, $S = \{a, b\}$. Then the statement $\forall xP(x)$ can also be written in full detail as $P(a) \wedge P(b)$. Rewrite each of the statements below in a similar fashion, using P , Q , and logical operators, but without using quantifiers.

(a) $\forall x\forall y(P(x) \vee Q(y))$

Solution:

$$\begin{aligned} & \forall x \left(P(x) \vee (Q(a) \wedge Q(b)) \right) \\ \equiv & (P(a) \wedge P(b)) \vee (Q(a) \wedge Q(b)) \\ \equiv & (P(a) \vee Q(a)) \wedge (P(a) \vee Q(b)) \wedge (P(b) \vee Q(a)) \wedge (P(b) \vee Q(b)) \end{aligned}$$

(b) $\exists xP(x) \wedge \exists xQ(x)$

Solution:

$$(P(a) \vee P(b)) \wedge (Q(a) \vee Q(b))$$

(c) $\exists x\exists y(P(x) \wedge Q(y))$

Solution:

$$\begin{aligned} & \exists y \left((P(a) \vee P(b)) \wedge Q(y) \right) \\ \equiv & (P(a) \vee P(b)) \wedge (Q(a) \vee Q(b)) \\ \equiv & (P(a) \wedge Q(a)) \vee (P(a) \wedge Q(b)) \vee (P(b) \wedge Q(a)) \vee (P(b) \wedge Q(b)) \end{aligned}$$

(d) $\forall x\exists y(P(x) \wedge Q(y))$

Solution:

$$\begin{aligned} & \forall x \left(P(x) \wedge (Q(a) \vee Q(b)) \right) \\ \equiv & \left(P(a) \wedge (Q(a) \vee Q(b)) \right) \wedge \left(P(b) \wedge (Q(a) \vee Q(b)) \right) \\ \equiv & \left((P(a) \wedge Q(a)) \vee (P(a) \wedge Q(b)) \right) \wedge \left((P(b) \wedge Q(a)) \vee (P(b) \wedge Q(b)) \right) \end{aligned}$$

8. For each of the following equivalences, determine if it is valid for all predicates P and Q . If yes then give a full explanation. If not then provide a counterexample.

(a) $\exists x(P(x) \vee Q(x)) \equiv \exists xP(x) \vee \exists xQ(x)$

Solution: *True.*

If the LHS is *True* then there is a constant, say, a such that (i) $P(a) = T$, or (ii) $Q(a) = T$. In case (i) $\exists xP(x)$ is *True*, while in case (ii) $\exists xQ(x)$ is *True*, so that in both cases the RHS is *True*,

If the RHS is *True*, then (i) there is a constant, say, a such that $P(a) = T$, so that the LHS is also *True* with $x = a$, or (ii) there is a constant, say, b such that $Q(b) = T$, so that the LHS is also *True* with $x = b$.

(b) $\exists x(P(x) \wedge Q(x)) \equiv \exists xP(x) \wedge \exists xQ(x)$

Solution: *False.*

Suppose $P(a)$ is *True* for some constant a , and *False* for all constants $b \neq a$. Suppose also $Q(b)$ is *True* for some constant b , and *False* for all constants $a \neq b$. Then RHS is *True* and LHS is *False*.

(c) $\forall x\forall y(P(x) \wedge Q(y)) \equiv \forall xP(x) \wedge \forall yQ(y)$

Solution: *True.*

If the RHS is *True* then P is always *True* and Q is always *True*. Then the LHS is also always *True*.

If the RHS is *False* then there is a constant, say a , such that $P(a) = F$, or there is a constant, say b , such that $Q(b) = F$. In the first case $P(a) \wedge Q(a)$ is *False*, and in the second case $P(b) \wedge Q(b)$ is *False*, so in any case the LHS is also *False*.

(d) $\forall x\forall y(P(x) \vee Q(y)) \equiv \forall xP(x) \vee \forall yQ(y)$

Solution: *True.*

If the RHS is *True* then we must consider two cases: (i): P is always *True* or (ii) Q is always *True*. In either case the LHS is *True* also.

If on the other hand the RHS is *False* then there is a constant, say a , such that $P(a) = F$, and there is a constant, say b , such that $Q(b) = F$. Then the LHS is seen to be *False* also, namely by picking $x = a$ and $y = b$.