THE UNIVERSITY OF OTTAWA

MAT1320 X - Calculus I
Midterm Exam 1 - July 21, 2009

INSTRUCTOR: Shonda Gosselin

STUDENT NAME: SOLUTIONS

STUDENT NUMBER:____________________

INSTRUCTIONS:

1. Print your name and student number in **ink**.
2. This examination is 80 minutes long.
3. NO books or notes are permitted. A basic scientific calculator (like the TI-30X) is permitted. Graphing and/or programmable calculators are NOT permitted.
4. Write clearly in the space provided. Show your work. Use the **backs of the pages** if more room is needed, or use the **extra page**.
5. This questionnaire has 5 questions (worth a total of 20 marks) and 7 pages.

Problem Number:	1	2	3	4	5	**TOTAL**
Value:	4	4	4	4	4	**20**
Your score:						

1. (a) **(2 marks)** Solve for x in the equation $\ln(x) - \ln\left(\frac{1}{x}\right) = 1$.

$$\ln(x) - \ln\left(\frac{1}{x}\right) = 1$$

$$\ln\left(x\left(\frac{1}{x}\right)^{-1}\right) = 1$$

$$\ln(x^2) = 1$$

$$x^2 = e$$

$$x = \pm\sqrt{e}$$

but $x > 0$ (for otherwise $\ln(x)$ is undefined), so $\boxed{x = \sqrt{e}}$.

(b) **(2 marks)** Find a formula for the inverse of the function $f(x) = \frac{1-e^x}{1+e^x}$. What is the domain of f^{-1}?

Set $y = \dfrac{1-e^x}{1+e^x}$ and solve for x:

$$y(1+e^x) = 1 - e^x \iff y + ye^x = 1 - e^x$$

$$\iff e^x(1+y) = 1 - y \iff e^x = \frac{1-y}{1+y}$$

$$\iff x = \ln\left(\frac{1-y}{1+y}\right).$$ Now interchange x and y:

$$y = \ln\left(\frac{1-x}{1+x}\right). \quad \therefore \quad \boxed{f^{-1}(x) = \ln\left(\frac{1-x}{1+x}\right)}.$$

Domain? Need $\dfrac{1-x}{1+x} > 0$

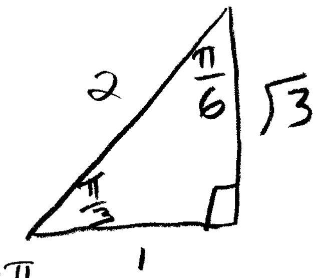
-1 1
— ? + 0 —

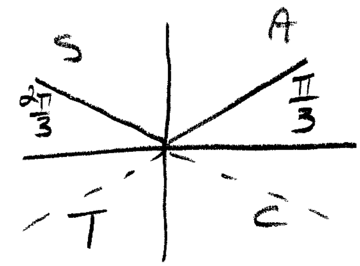
$$\boxed{\text{domain } f^{-1} = (-1, 1)}$$

2. (a) **(2 marks)** Find all values of x in the interval $[0, 2\pi]$ that satisfy the inequality $2\sin x \geq \sqrt{3}$.

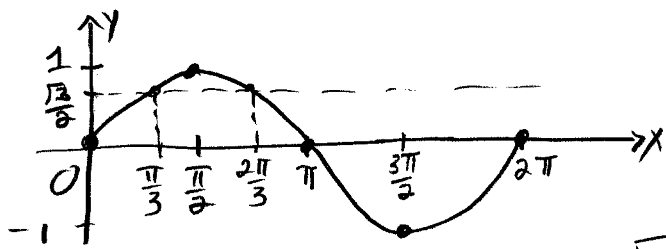
$$2\sin x \geqslant \sqrt{3}$$

$$\Longleftrightarrow \quad \sin x \geqslant \frac{\sqrt{3}}{2}.$$





$$\sin x = \frac{\sqrt{3}}{2} \quad \text{for} \quad x = \frac{\pi}{3}, \frac{2\pi}{3}.$$



Thus $2\sin x \geqslant \sqrt{3}$ for $\boxed{x \in \left[\frac{\pi}{3}, \frac{2\pi}{3}\right]}$

(b) **(2 marks)** Given $f(x) = \arcsin x$, $g(x) = \sin\sqrt{x}$, and $h(x) = x^2$, evaluate $(f \circ g \circ h)(1)$.

$$\begin{aligned}
(f\circ g\circ h)(x) &= f(g(h(x))) \\
&= f(g(x^2)) \\
&= f(\sin\sqrt{x^2}) = f(\sin x) \\
&= \arcsin(\sin x) \\
&= x
\end{aligned}$$

$$\therefore (f\circ g\circ h)(1) = \boxed{1}$$

3. Consider the function $f(x) = e^{x-1}$.

(a) **(3 marks)** Find the average rate of change of f on the following intervals:

i. $[1, 1.1]$

$$\frac{e^{1.1-1} - e^{1-1}}{1.1-1} = \frac{e^{0.1} - 1}{0.1} \doteq \boxed{1.05171}$$

ii. $[1, 1.01]$

$$\frac{e^{1.01-1} - e^{1-1}}{1.01-1} = \frac{e^{0.01} - 1}{0.01} \doteq \boxed{1.00502}$$

iii. $[1, 1.001]$

$$\frac{e^{1.001-1} - e^{1-1}}{1.001-1} = \frac{e^{0.001} - 1}{0.001} \doteq \boxed{1.00050}$$

(b) **(1 mark)** Estimate the instantaneous rate of change of f at $x = 1$.

The instantaneous rate of change of f at $x=1$ is

$$f'(1) = \lim_{h \to 0} \frac{e^{(1+h)-1} - e^{1-1}}{h} = \boxed{1}.$$

(as $h \to 0$, the difference quotient $\to 1$)

4. **(4 marks)** Use the definition of the derivative to find $f'(x)$ if $f(x) = \frac{1}{x-1}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{(x+h)-1} - \frac{1}{x-1}}{h}$$

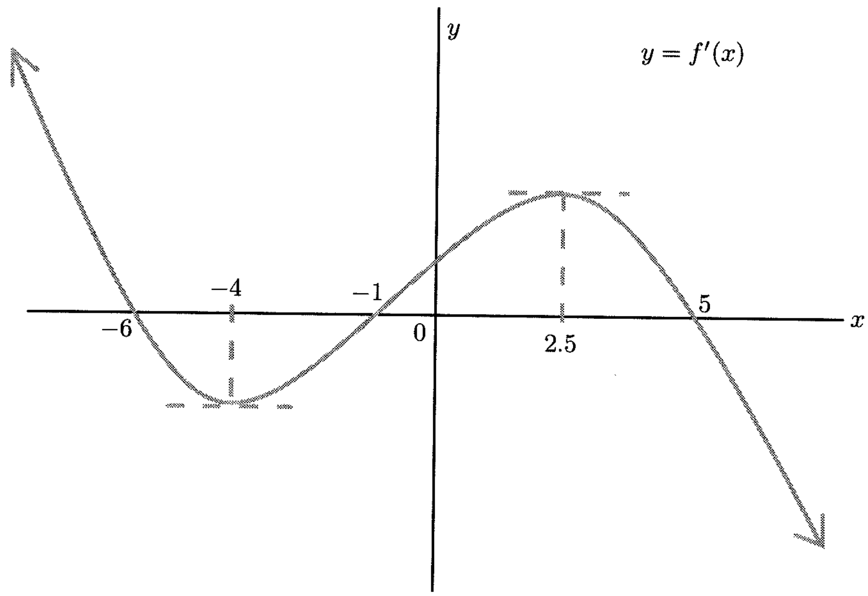
$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{(x-1) - (x+h-1)}{(x+h-1)(x-1)}\right)$$

$$= \lim_{h \to 0} \frac{1}{\cancel{h}}\left(\frac{-\cancel{h}}{(x+h-1)(x-1)}\right)$$

$$= \lim_{h \to 0} \frac{-1}{(x+h-1)(x-1)}$$

$$= \frac{-1}{(x+0-1)(x-1)} = \boxed{\frac{-1}{(x-1)^2}}.$$

5. **(4 marks)**
The graph of the derivative function $f'(x)$ of a function $f(x)$ is shown below.



(a) Determine the interval(s) on which the function $f(x)$ is

i. increasing. f is increasing where $f' > 0$: $\boxed{(-\infty, -6) \cup (-1, 5)}$

ii. decreasing. f is decreasing where $f' < 0$: $\boxed{(-6, -1) \cup (5, \infty)}$

iii. concave up. f is CCup where f' is increasing: $\boxed{(-4, 2.5)}$

iv. concave down. f is CC down where f' is decreasing: $\boxed{(-\infty, -4) \cup (2.5, \infty)}$

(b) Determine the value(s) of x at which the function $f(x)$ has

i. a local maximum point. Occurs where $f' = 0$ and f' changes from positive to negative: $\boxed{x = -6, x = 5}$

ii. a local minimum point. Occurs where $f' = 0$ and f' changes from negative to positive: $\boxed{x = -1}$

iii. a point of inflection.
Occurs at local maxima and local minima of f': $\boxed{x = -4, x = 2.5}$

EXTRA PAGE