MATH 1402A **Term test #2** **17 October 2013**

Maximum score: **35** marks

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Non-programmable calculators are permitted. This test is closed book.
<u>Supply your answers on this sheet</u>, but TA's have extra paper if you need it.

PLEASE PRINT ______________________ ______________________ ____________

First name Last name Student number

Please show your work where appropriate!

1. [5] You are asked to **<u>determine the (*K*, *L*) settings</u>** that will minimize the total cost *TC*, with the constraint that the production levels *Q* must remain at a constant value of $Q_0 = 32$. The model for total cost is $TC = 9L + 72K$ and the production function uses the standard Cobb-Douglas model, with $A = 1, \alpha = 1/4$, and $\beta = 1/2$. $\therefore Q = K^{1/4} L^{1/2}$ **USE ANY METHOD**

$Q_0 = 32 = K^{1/4} L^{1/2} \Rightarrow K = 32^4 / L^2$

$TC = 9L + 72\left(\frac{32^4}{L^2}\right) = 9L + 72 \cdot 32^4 L^{-2}$

$\frac{dTC}{dL} = 0 \Rightarrow 9 - 144 \cdot 32^4 L^{-3} = 0$

$\therefore L^3 = \frac{144 \cdot 32^4}{9}$

and: $L = 256$

$K = \frac{32^4}{(256)^2} = 16$

$\therefore (K, L) = (16, 256)$

2. [4] Let $y^3 + 3xy^2 + 3x^2y + x^3 - 8y + 4x = 0$. Determine *dy*/*dx* at the point (1, 1).

$f(x,y)$; so $f_x = 3y^2 + 6xy + 3x^2 + 4$; $f_y = 3y^2 + 6xy + 3x^2 - 8$

$\therefore \frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{3y^2 + 6xy + 3x^2 + 4}{3y^2 + 6xy + 3x^2 - 8}$

at $(1,1) \Rightarrow \frac{dy}{dx} = -\frac{3+6+3+4}{3+6+3-8} = \boxed{-4}$

3. [4] Let: $z = f(x, y) = x^3 - x^2y + xy^2 - y^3$ and $y = g(x) = 3x - 8$. Find *dz*/*dx* , the total derivative, **<u>using any method</u>**. <u>Express the answer as a function of *x* only</u>. Do not DEVELOP the expression.

$f_x = 3x^2 - 2xy + y^2$; $f_y = -x^2 + 2xy - 3y^2$; $dy/dx = 3$

$\frac{dz}{dx} = f_x + f_y\left(\frac{dy}{dx}\right) = 3x^2 - 2xy + y^2 + 3(-x^2 + 2xy - 3y^2)$

$\frac{dz}{dx} = 3x^2 - 2x(3x-8) + (3x-8)^2 - 3x^2 + 6x(3x-8) - 9(3x-8)^2$

leave as is ...

4. Let $f(w, x, y) = \sqrt{w^3 - x^2 - y}$ be a 3-variable function that needs to be optimized (i.e. you need to determine a max./min.). Suppose also that the independent variables are linked together by the equation $wxy^3 + w^3xy = wx^3y$. $\rightarrow wxy^3 + w^3xy - wx^3y = 0$

a. [1] WRITE the Langrangian Λ for the problem (without solving).

b. [3] Write down the equation $\Lambda_w = \frac{\partial \Lambda}{\partial w} = 0$

a. $\Lambda = \sqrt{w^3 - x^2 - y} - \lambda(wxy^3 + w^3xy - wx^3y)$

b. $\Lambda_w = 0 \Rightarrow \frac{3w^2}{2\sqrt{w^3 - x^2 - y}} - \lambda xy^3 - 3\lambda w^2xy + \lambda x^3y = 0$

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5. [6] Let $z = f(x, y) = \sqrt{2x^2 + xy + y^2}$. Use the differential dz to estimate the change in z (i.e. Δz) when moving from $(x, y) = (2, -2)$ to $(1.9, -1.9)$. Compare the result with the actual value of Δz.

$$f_x = \frac{4x+y}{2\sqrt{2x^2+xy+y^2}} \; ; \; f_y = \frac{x+2y}{2\sqrt{2x^2+xy+y^2}}$$

$$dx = -0.1 \qquad dy = 0.1$$

$$f_x\Big|_{(2,-2)} = \frac{4(2)-2}{2\sqrt{8-4+4}} = \frac{3}{\sqrt{8}}$$

$$f_y\Big|_{(2,-2)} = \frac{2-4}{2\sqrt{8}} = \frac{-1}{\sqrt{8}}$$

$$dz = f_x\,dx + f_y\,dy$$

$$dz = \frac{3}{\sqrt{8}}(-0.1) + \left(\frac{-1}{\sqrt{8}}\right)(0.1) \approx -0.141$$

and $\Delta z = f(1.9, -1.9) - f(2,2)$

$$= \sqrt{2(1.9)^2 + (1.9)(-1.9) + (-1.9)^2} - \sqrt{8} = -0.141$$

6. [5] Let $z = f(x, y) = 4x^3 + 3y^2 - 12xy + 144x - 120y + 16$. **Determine the** (x, y) **coordinates of the critical point(s)**. *Do NOT determine if these are a max, min, or saddle*.

$$f_x = 0 \Rightarrow 12x^2 - 12y + 144 = 0 \Rightarrow y = x^2 + 12$$

$$f_y = 0 \Rightarrow 6y - 12x - 120 = 0 \Rightarrow y = 2x + 20$$

$$x^2 + 12 = 2x + 20$$

$$x^2 - 2x - 8 = 0$$

$$(x-4)(x+2) = 0$$

$$x = 4 \Rightarrow y = 28$$

$$x = -2 \Rightarrow y = 16$$

$$\therefore (4, 28) \text{ and } (-2, 16)$$

7. [5] Let $z = f(x, y) = 2x^3 - y^3 - 6x + 12y + 18$. Determine the **nature** (i.e. min., max. or saddle) of the critical point (–1, 2). Make sure to do all the work necessary to justify your answer.

$$\frac{\partial z}{\partial x} = 6x^2 - 6 \; ; \; \frac{\partial^2 z}{\partial x^2} = 12x \Rightarrow \frac{\partial^2 z}{\partial x^2}\Big|_{x=-1} = -12 < 0$$

max. ?

$$\frac{\partial z}{\partial y} = -3y^2 + 12 \; ; \; \frac{\partial^2 z}{\partial x^2} = -6y \Rightarrow \frac{\partial^2 z}{\partial y^2}\Big|_{y=2} = -12 < 0$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(-3y^2 + 12\right) = 0$$

$$\therefore D = (-12)(-12) = 144 > 0$$

max. conf.

8. [2] Let $Q = f(K, L) = 4K^{1/4} L^{1/2}$ be the rule for a production function, where ***K*** corresponds to ***input capital***, ***L*** corresponds to ***input labour*** and ***Q*** corresponds to ***output production.*** Assuming $Q = Q_0 = 8$:

Determine *MRS* using ANY METHOD

$$Q_0 = 8 = 4K^{1/4}L^{1/2} \Rightarrow K^{1/4} = \frac{2}{L^{1/2}} \Rightarrow K = \frac{16}{L^2} = 16L^{-2}$$

$$\frac{dK}{dL} = -32L^{-3} = -\frac{32}{L^3}$$

$$\boxed{\therefore MRS = \frac{32}{L^3}}$$