

CONCORDIA UNIVERSITY
Department of Mathematics & Statistics

Course	Number	Section(s)
Mathematics	204	All (except EC)
Examination	Date	Pages
Alternate Midterm	October 2013	1

Special Instructions:

- ▷ Only approved calculators are allowed.
- ▷ Justify all your answers.
- ▷ All questions have equal value.

1. Solve by Gauss-Jordan elimination:

$$\begin{aligned}2x + y - 2z &= 8 \\3x + 2y - 4z &= 15 \\5x + 4y - z &= 1\end{aligned}$$

2. If $\begin{cases} x + y - z = 1 \\ 2x + 3y + kz = 3 \\ x + ky + 3z = 2 \end{cases}$,

find the values of k so that the system has

- (i) unique solution
- (ii) no solution
- (iii) an infinite number of solutions

3. If $(20I + 3A)^{-1} = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}$, find A

4. Find the inverse of A if $A = \begin{pmatrix} 1 & -2 & 2 \\ 2 & -3 & 6 \\ 1 & 1 & 7 \end{pmatrix}$,

5. Find the determinant of $A = \begin{pmatrix} 2 & 1 & 3 & 2 \\ 3 & 0 & 1 & -2 \\ 1 & -1 & 4 & 3 \\ 2 & 2 & -1 & 1 \end{pmatrix}$

6. Write the vector $v = (2, -5, 3)$ as a linear combination of $u_1 = (1, -3, 2)$, $u_2 = (2, -4, -1)$, $u_3 = (1, -5, 7)$.

Question 1

Augmented Coefficient Matrix (ACM):

$$\begin{array}{ccc|c} 2 & 1 & -2 & 8 \\ 3 & 2 & -4 & 15 \\ 5 & 4 & -1 & 1 \end{array} \quad R1 \leftrightarrow R2$$

$$\begin{array}{ccc|c} 3 & 2 & -4 & 15 \\ 2 & 1 & -2 & 8 \\ 5 & 4 & -1 & 1 \end{array} \quad R1 = R1 + (-1) \cdot R2$$

$$\begin{array}{ccc|c} 1 & 1 & -2 & 7 \\ 2 & 1 & -2 & 8 \\ 5 & 4 & -1 & 1 \end{array} \quad R2 = R2 + (-2) \cdot R1$$

$$\begin{array}{ccc|c} 1 & 1 & -2 & 7 \\ 0 & -1 & 2 & -6 \\ 5 & 4 & -1 & 1 \end{array} \quad R3 = R3 + (-5) \cdot R1$$

$$\begin{array}{ccc|c} 1 & 1 & -2 & 7 \\ 0 & -1 & 2 & -6 \\ 0 & -1 & 9 & -34 \end{array} \quad R2 = (-1) \cdot R2$$

$$\begin{array}{ccc|c} 1 & 1 & -2 & 7 \\ 0 & 1 & -2 & 6 \\ 0 & -1 & 9 & -34 \end{array} \quad R1 = R1 + (-1) \cdot R2$$

$$\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 6 \\ 0 & -1 & 9 & -34 \end{array} \quad R3 = R3 + (1) \cdot R2$$

$$\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 6 \\ 0 & 0 & 7 & -28 \end{array} \quad R3 = (1/7) \cdot R3$$

$$\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 6 \\ 0 & 0 & 1 & -4 \end{array} \quad R2 = R2 + (2) \cdot R3$$

$$\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -4 \end{array} \quad \begin{array}{l} \textcircled{1} \\ \textcircled{2} \\ \textcircled{3} \end{array}$$

$$\textcircled{3} \Rightarrow z = -4$$

$$\textcircled{2} \Rightarrow y = -2$$

$$\textcircled{1} \Rightarrow x = 1$$

Question 2

Form ACM and reduce it with ERO:

$$\begin{array}{cccc} 1 & 1 & -1 & 1 \\ 2 & 3 & k & 3 \\ 1 & k & 3 & 2 \end{array}$$

$$R_2 = R_2 + (-2)*R_1$$

$$\begin{array}{cccc} 1 & 1 & -1 & 1 \\ 0 & 1 & 2+k & 1 \\ 1 & k & 3 & 2 \end{array}$$

$$R_3 = R_3 + (-1)*R_1$$

$$\begin{array}{cccc} 1 & 1 & -1 & 1 \\ 0 & 1 & 2+k & 1 \\ 0 & -1+k & 4 & 1 \end{array}$$

$$R_1 = R_1 + (-1)*R_2$$

$$\begin{array}{cccc} 1 & 0 & -3-k & 0 \\ 0 & 1 & 2+k & 1 \\ 0 & -1+k & 4 & 1 \end{array}$$

$$R_3 = R_3 + (1-k)*R_2$$

$$\begin{array}{cccccc} 1 & 0 & -3-k & 0 & (1) \\ 0 & 1 & 2+k & 1 & (2) \\ 0 & 0 & 4+(1-k)(2+k) & 2-k & (3) \end{array}$$

In (3) if $k=2$ we have $0 \ 0 \ 0 \ 0 \Rightarrow$ Infinite # sol.

if $k=-3$ " " " " $0 \ 0 \ 0 \ 5 \Rightarrow$ No solution

if $k \neq 2$ and $k \neq -3$ we can multiply Row 3 by $\frac{1}{(3+k)(2-k)}$ and then get 0 in $\begin{cases} \text{Row 2 col 3} \\ \text{Row 1 col 3} \end{cases}$ and we will then have unique solution

Question 3 let $\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} = B$

the solve $(20I + 3A)^{-1} = B$ for A

$$((20I + 3A)^{-1})^{-1} = B^{-1}$$

$$20I + 3A = B^{-1}$$

$$3A = B^{-1} - 20I$$

$$A = \frac{1}{3}(B^{-1} - 20I)$$

(Continued) ---

Question 3 continued

$$A = \frac{1}{3} \left(\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}^{-1} - 20 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

$$A = \frac{1}{3} \left(\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix} - \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix} \right)$$

$$A = \frac{1}{3} \begin{bmatrix} -15 & -2 \\ -7 & -17 \end{bmatrix}$$

$$A = \begin{bmatrix} -5 & -\frac{2}{3} \\ -\frac{7}{3} & -\frac{17}{3} \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}^{-1}$$

$$= \frac{1}{(3)(5) - (2)(7)} \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$$

Question 4 Set up Original matrix at LHS, Identity matrix at RHS and reduce the LHS to Identity. Then RHS is the Inverse

$$\begin{array}{cccccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 2 & -3 & 6 & 0 & 1 & 0 \\ 1 & 1 & 7 & 0 & 0 & 1 \end{array}$$

$$R2 = R2 + (-2)*R1$$

$$\begin{array}{cccccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 1 & 1 & 7 & 0 & 0 & 1 \end{array}$$

$$R3 = R3 + (-1)*R1$$

$$\begin{array}{cccccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 3 & 5 & -1 & 0 & 1 \end{array}$$

$$R1 = R1 + (2)*R2$$

$$\begin{array}{cccccc} 1 & 0 & 6 & -3 & 2 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 3 & 5 & -1 & 0 & 1 \end{array}$$

$$R3 = R3 + (-3)*R2$$

$$\begin{array}{cccccc} 1 & 0 & 6 & -3 & 2 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & -1 & 5 & -3 & 1 \end{array}$$

$$R3 = (-1)*R3$$

$$\begin{array}{cccccc} 1 & 0 & 6 & -3 & 2 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -5 & 3 & -1 \end{array}$$

$$R1 = R1 + (-6)*R3$$

$$\begin{array}{cccccc} 1 & 0 & 0 & 27 & -16 & 6 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -5 & 3 & -1 \end{array}$$

$$R2 = R2 + (-2)*R3$$

$$\begin{array}{cccccc} 1 & 0 & 0 & 27 & -16 & 6 \\ 0 & 1 & 0 & 8 & -5 & 2 \\ 0 & 0 & 1 & -5 & 3 & -1 \end{array}$$

$$\Rightarrow \begin{bmatrix} 1 & -2 & 2 \\ 2 & -3 & 6 \\ 1 & 1 & 7 \end{bmatrix}^{-1} = \begin{bmatrix} 27 & -16 & 6 \\ 8 & -5 & 2 \\ -5 & 3 & -1 \end{bmatrix}$$

Question 5

Step 1 Get more zeros in C2. Note the ERO below do not alter the determinant value of the original matrix

$$\begin{array}{cccc} 2 & 1 & 3 & 2 \\ 3 & 0 & 1 & -2 \\ 1 & -1 & 4 & 3 \\ 2 & 2 & -1 & 1 \end{array}$$

$$R3 = R3 + (1)*R1$$

$$\begin{array}{cccc} 2 & 1 & 3 & 2 \\ 3 & 0 & 1 & -2 \\ 3 & 0 & 7 & 5 \\ 2 & 2 & -1 & 1 \end{array}$$

$$R4 = R4 + (-2)*R1$$

$$\begin{array}{cccc} 2 & 1 & 3 & 2 \\ 3 & 0 & 1 & -2 \\ 3 & 0 & 7 & 5 \\ -2 & 0 & -7 & -3 \end{array}$$

Step 2 Use Cofactor expansion about C2 then 3x3 shortcut

$$\begin{aligned} \det &= 1(-1)^{1+2} \begin{vmatrix} 3 & 1 & -2 \\ 3 & 7 & 5 \\ -2 & -7 & -3 \end{vmatrix} \\ &= - \begin{pmatrix} \cancel{3} & \cancel{1} & \cancel{-2} & \cancel{3} & \cancel{1} \\ \cancel{3} & \cancel{7} & \cancel{5} & \cancel{3} & \cancel{7} \\ \cancel{-2} & \cancel{-7} & \cancel{-3} & \cancel{-2} & \cancel{-7} \end{pmatrix} \\ &= - \left[(3)(7)(-3) + (1)(5)(-2) + (-2)(3)(-7) \right. \\ &\quad \left. - ((-2)(7)(-2) + (-7)(5)(3) + (-3)(3)(1)) \right] \\ &= -55 \end{aligned}$$

Question 6

Step 1 Form the linear combination

$$k_1 \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + k_2 \begin{bmatrix} 2 \\ -4 \\ -1 \end{bmatrix} + k_3 \begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}$$

Step 2 Simplify Step 1 and make a system of three equations

$$\begin{bmatrix} k_1 \\ -3k_1 \\ 2k_1 \end{bmatrix} + \begin{bmatrix} 2k_2 \\ -4k_2 \\ -k_2 \end{bmatrix} + \begin{bmatrix} k_3 \\ -5k_3 \\ 7k_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} k_1 + 2k_2 + k_3 \\ -3k_1 - 4k_2 - 5k_3 \\ 2k_1 - k_2 + 7k_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}$$

$$\begin{aligned} k_1 + 2k_2 + k_3 &= 2 \\ -3k_1 - 4k_2 + 5k_3 &= -5 \\ 2k_1 - k_2 + 7k_3 &= 3 \end{aligned}$$

Scalar mult

Add. of matrices

DEF. of Equal Matrices

Step 3 Solve the system in Step 2 with Gauss-Jordan

$$\begin{array}{ccc} 1 & 2 & 1 & 2 \\ -3 & -4 & -5 & -5 \\ 2 & -1 & 7 & 3 \end{array}$$

$$R_2 = R_2 + (3) \cdot R_1$$

$$\begin{array}{ccc} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 1 \\ 2 & -1 & 7 & 3 \end{array}$$

$$R_3 = R_3 + (-2) \cdot R_1$$

$$\begin{array}{ccc} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 1 \\ 0 & -5 & 5 & -1 \end{array}$$

$$R_1 = R_1 + (-1) \cdot R_2$$

$$\begin{array}{ccc} 1 & 0 & 3 & 1 \\ 0 & 2 & -2 & 1 \\ 0 & -5 & 5 & -1 \end{array}$$

$$R_3 = (2) \cdot R_3$$

$$\begin{array}{ccc} 1 & 0 & 3 & 1 \\ 0 & 2 & -2 & 1 \\ 0 & -10 & 10 & -2 \end{array}$$

$$R_2 = (5) \cdot R_2$$

$$\begin{array}{ccc} 1 & 0 & 3 & 1 \\ 0 & 10 & -10 & 5 \\ 0 & -10 & 10 & -2 \end{array}$$

$$R_3 = R_3 + (1) \cdot R_2$$

$$\begin{array}{ccc} 1 & 0 & 3 & 1 \\ 0 & 10 & -10 & 5 \\ 0 & 0 & 0 & 3 \end{array}$$

(1)

(2)

(3)

③ \Rightarrow there is no solution to system
 \Rightarrow It is not possible to write
 \vec{v} as a linear Comb. of $\vec{u}_1, \vec{u}_2, \vec{u}_3$.