Solutions - Review Problems

1) We require continuous value, 1st derivative, and 2nd derivative at the knot at $x=1$.

$$\left.\begin{aligned} &S_1(1) = S_2(1) \\ &\frac{5}{3} + \frac{16}{3} + a + 1 = -\frac{7}{3} + b + \frac{22}{3} + \frac{2}{3} \end{aligned}\right\} \text{value}$$

$$\left.\begin{aligned} &S_1'(1) = S_2'(1) \\ &\frac{16}{3} + 2a(1) + 3(1)^2 = b + \frac{44}{3}(1) + 2(1)^2 \end{aligned}\right\} \text{1st deriv.}$$

$$\left.\begin{aligned} &S_1''(1) = S_2''(1) \\ &2a + 6(1) = \frac{44}{3} + 4(1) \end{aligned}\right\} \text{2nd deriv.}$$

Solve these equations for a and b, if possible. If no a and b satisfy all conditions, this cannot be a valid cubic spline.

2) Use our rule that $a \oplus b = (a+b)(1+\delta)$ for some $|\delta| \leq \varepsilon$ (machine eps.), for F.P. versions of arithmetic ops.

$$\frac{\left|(x^2(1+\delta_1)-1)(1+\delta_2) - (x^2-1)\right|}{|x^2-1|}$$

expand $\rightarrow$ $$= \frac{\left|\cancel{x^2} + \delta_1 x^2 \cancel{-1} + \delta_2 x^2 + \delta_1\delta_2 x^2 - \delta_2 \cancel{- x^2} \cancel{+1}\right|}{|x^2-1|}$$

cancel $\rightarrow$ $$= \frac{\left|\delta_1 x^2 + \delta_1\delta_2 x^2 + \delta_2 x^2 - \delta_2\right|}{|x^2-1|}$$

group $\rightarrow$ $$= \frac{\left|x^2(\delta_1 + \delta_1\delta_2) + (x^2-1)\delta_2\right|}{|x^2-1|}$$

triangle inequality $\rightarrow$ $$\leq \frac{|x^2|\,|\delta_1 + \delta_1\delta_2| + |x^2-1|\,|\delta_2|}{|x^2-1|}$$

$$\leq \frac{|x^2|(|\delta_1| + |\delta_1\delta_2|)}{|x^2-1|} + |\delta_2|$$

replace δ's w/ E $\longrightarrow$ $\leq \dfrac{|x^2|}{|x^2-1|}E + \dfrac{|x^2|}{|x^2-1|}E^2 + E$

Drop E^2 terms

$\longrightarrow \; = \left(\dfrac{|x^2|}{|x^2-1|} + 1\right)E$

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3) We know $p(x) = ax^3 + bx^2 + cx + d$

$p'(x) = 3ax^2 + 2bx + c$

$p''(x) = 6ax + 2b$

Plug in our data.

$p(0) = 2u \rightarrow \boxed{d = 2u}$

$p'(0) = u \rightarrow \boxed{c = u}$

$p(1) = 4v + 3u \rightarrow a + b + c + d = 4v + 3u$

$p''(1) = 20v \rightarrow 6a + 2b = 20v$

3rd line gives $a + b + \cancel{u} + \cancel{2u} = 4v + \cancel{3u} \rightarrow a + b = 4v$

or $a = 4v - b$

Then 4th line gives $6(4v - b) + 2b = 20v$

$24v - 6b + 2b = 20v$

$-4b = -4v$

$\boxed{b = v}$

$\therefore a = 4v - v$

$\boxed{a = 3v}$

$p(x) = 3vx^3 + vx^2 + ux + 2u$

4) Isolate derivative terms:

$$x'(t) = x(t)\,(w(t))^2\,t + 2$$
$$w'(t) = x(t) + y(t)\,t^2$$
$$y'(t) = x(t)$$

In equivalent vector form:

$$\begin{bmatrix} x' \\ w' \\ y' \end{bmatrix} = \begin{bmatrix} x w^2 t + 2 \\ x + y t^2 \\ x \end{bmatrix} \quad \text{with} \quad \begin{bmatrix} x_0 \\ w_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$\therefore$ Dynamics function $\vec{f}(t, \vec{y}(t)) = \begin{bmatrix} x w^2 t + 2 \\ x + y t^2 \\ x \end{bmatrix}$

Basic (scalar) improved Euler is

$$y^*_{n+1} = y_n + h f(t_n, y_n)$$
$$y_{n+1} = y_n + \frac{h}{2}\left(f(t_n, y_n) + f(t_{n+1}, y^*_{n+1})\right)$$

We will need $\vec{f}$ evaluated at $(t_n, \vec{y}_n)$ and $(t_{n+1}, \vec{y}^*_{n+1})$

so these are:

$$\vec{f}(t_n, \vec{y}_n) = \begin{bmatrix} x_n w_n^2 t_n + 2 \\ x_n + y_n t_n^2 \\ x_n \end{bmatrix} \quad \text{and} \quad \vec{f}(t_{n+1}, \vec{y}^*_{n+1}) = \begin{bmatrix} x^*_{n+1} w^{*2}_{n+1} t_{n+1} + 2 \\ x^*_{n+1} + y^*_{n+1} t_{n+1}^2 \\ x^*_{n+1} \end{bmatrix}$$

Plugging in...

$$\begin{bmatrix} x^*_{n+1} \\ w^*_{n+1} \\ y^*_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ w_n \\ y_n \end{bmatrix} + h \begin{bmatrix} x_n w_n^2 t_n + 2 \\ x_n + y_n t_n^2 \\ x_n \end{bmatrix}$$

$$\begin{bmatrix} x_{n+1} \\ w_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ w_n \\ y_n \end{bmatrix} + \frac{h}{2} \begin{bmatrix} x_n w_n^2 t_n + 2 \\ x_n + y_n t_n^2 \\ x_n \end{bmatrix} + \frac{h}{2} \begin{bmatrix} x^*_{n+1} w^{*2}_{n+1} t_{n+1} + 2 \\ x^*_{n+1} + y^*_{n+1} t_{n+1}^2 \\ x^*_{n+1} \end{bmatrix}$$

(Note that $\vec{f}(t_n, \vec{y}_n)$ is needed twice, so we could (should) evaluate it once and reuse to save computation.)

5) Backwards Euler is $y_{n+1} = y_n + h f(t_{n+1}, y_{n+1})$

So for $y'(t) = 3y(t) \cdot t$ we get

$$y_{n+1} = y_n + h \cdot 3 (y_{n+1})(t_{n+1}) .$$

Since it is an implicit scheme we have y_{n+1} on both sides and have to solve...

$$y_{n+1} - 3h\, y_{n+1} t_{n+1} = y_n$$

$$\therefore \quad y_{n+1} = \frac{y_n}{1 - 3h\, t_{n+1}}$$

n	t	y
0	0	1
1	2	-1/11
2	4	1/253

$$y_1 = \frac{y_0}{1 - 3(h)(t_1)} = \frac{1}{1 - 3 \cdot 2 \cdot 2} = \frac{1}{1-12} = -\frac{1}{11}$$

$$y_2 = \frac{y_1}{1 - 3 \cdot h \cdot t_2} = \frac{-1/11}{1 - 3 \cdot 2 \cdot 4} = \frac{-1/11}{-23} = \frac{+1}{253}$$

6. a) Plug in $p_0 = 1$, $p_1 = -\frac{1}{2}$ to a an b.

$$a = \frac{5}{7}(1) - \frac{4}{7}\left(-\frac{1}{2}\right) = \frac{10}{14} + \frac{4}{14} = 1$$

$$b = \frac{2}{7}(1) + \frac{4}{7}\left(-\frac{1}{2}\right) = \frac{4}{14} - \frac{4}{14} = 0$$

$$\therefore \; p_n = 1 \cdot \left(-\frac{1}{2}\right)^n + 0 \cdot \left(\frac{5}{4}\right)^n = \left(-\frac{1}{2}\right)^n$$

b) Let $(p_n)_A$ be the approximate result (with effects of initial errors e_0, e_1) and $(p_n)_E$ be the exact value.

Then error at step n is

$$e_n = (p_n)_A - (p_n)_E.$$

Inserting the recursive definitions we get

$$e_n = \left[\frac{3}{4}(p_{n-1})_A + \frac{5}{8}(p_{n-2})_A\right] - \left[\frac{3}{4}(p_{n-1})_E + \frac{5}{8}(p_{n-2})_E\right]$$

$$= \frac{3}{4}\left((p_{n-1})_A - (p_{n-1})_E\right) + \frac{5}{8}\left((p_{n-2})_A - (p_{n-2})_E\right)$$

$$= \frac{3}{4} e_{n-1} + \frac{5}{8} e_{n-2}$$

So the error follows the same recurrence!

Applying the given exact solution, we have

$$e_n = a\left(-\frac{1}{2}\right)^n + b\left(\frac{5}{4}\right)^n$$

for $a = \frac{5}{7}e_0 - \frac{4}{7}e_1$ and $b = \frac{2}{7}e_0 + \frac{4}{7}e_1$

$\left|-\frac{1}{2}\right| < 1$, so this term does not "blow up."

$\left|\frac{5}{4}\right| > 1$, so this term causes instability if $b \neq 0$ ie. $\frac{2}{7}e_0 + \frac{4}{7}e_1 \neq 0$. In general this is the case, so the algorithm is unstable.