SAMPLE FINAL EXAMINATION FOR MATH 105

[42] **1.** **Short-Answer Questions.** Put your answer in the box provided but show your work also. Each question is worth 3 marks, but not all questions are of equal difficulty.

(a) Let $f(x, y) = 8x^{\frac{1}{5}}y^{\frac{4}{5}}$. Find $f(2x, 2y) - 2f(x, y)$.

Answer:

(b) Let $f(x, y) = xe^{2y} + y^2$. Evaluate $\dfrac{\partial^2 f}{\partial y^2}$.

Answer:

(c) If $F(x) = \displaystyle\int_0^{2x} \ln(3 + \sin t)dt$, find $F'(\frac{\pi}{2})$.

Answer:

(d) Find the function $f(x)$ for which $f'(x) = \dfrac{1}{\sqrt{x}} + x^2$ and $f(1) = 2$.

Answer:

(e) Identify and sketch the level curve corresponding to $z = e$ of the function

$$z = e^{x^2+4y^2}.$$

Label the axes of your graph and plot the coordinates of at least four points on the level curve.

(f) Use a Riemann sum with $n = 2$ and select the midpoints of subintervals to estimate the area under the graph of $f(x) = \dfrac{1}{1+\sqrt{x}}$ from 0 to 4.

Answer:

(g) Find a bound for the error in approximating

$$\int_0^1 \left[e^{-2x} + 3x^3\right] dx$$

using Simpson's rule with $n = 6$ subintervals. There is no need to simplify your answer. **Do not write down the Simpson's rule approximation S_n.**

Answer:

(h) Find $\displaystyle\int \frac{\sqrt{\ln x}}{x}\,dx$.

Answer:

(i) The Maclaurin series for $\arctan x$ is given by

$$\arctan x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1},$$

which has radius of convergence equal to 1. Use this fact to compute the exact value of the series below:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}.$$

Answer:

(j) Find the equation of the plane parallel to $3x - y + 4z = 13$ passing through the point $(2, 1, -1)$.

Answer:

(k) Solve the differential equation

$$\frac{dy}{dx} = xe^{x^2 - \ln(y^2)}.$$

Answer:

(l) Let k be a constant. Find the value of k such that $f(x) = kx^{\frac{3}{2}}$ is a probability density function on $1 \leq x \leq 4$.

Answer:

(m) Compute the cumulative distribution function corresponding to the density function $f(x) = 2(x-1)$, $1 \leq x \leq 2$.

Answer:

(n) Find the variance of the random variable X whose density function is $f(x) = \dfrac{1}{2\sqrt{x}}$, $1 \leq x \leq 4$. (You may need the fact that the expected value of the random variable X above is $\frac{7}{3}$.)

Answer:

Full-Solution Problems. In questions 2–6, justify your answers and show all your work.

[12] **2.** This problem contains three numerical series. For each of them, find out whether it converges or diverges. You should provide appropriate justification in order to receive credit.

(a)

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} e^{-\sqrt{k}}$$

(b)

$$\sum_{k=1}^{\infty} \frac{k^4 - 2k^3 + 2}{k^5 + k^2 + k}$$

(c)

$$\sum_{k=1}^{\infty} \frac{2^k (k!)^2}{(2k)!}$$

[12] 3. (a) Compute the following indefinite integral:

$$\int \sin(\ln x)\, dx.$$

(b) Evaluate the following definite integral:

$$\int_0^1 \frac{1}{x^2 - 5x + 6}\, dx.$$

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[12] 4. (a) Find all critical points of the function

$$f(x, y) = xye^y + \frac{1}{2}x^2 - 2.$$

(b) Classify each critical point you found as a local maximum, a local minimum, or a saddle point of $f(x, y)$.

[12] 5. The production function for a firm is $f(x, y) = 5x^{\frac{2}{3}}y^{\frac{1}{3}}$, where x and y are the number of units of labor and capital utilized respectively. Suppose that labor costs \$108 per unit and capital costs \$2 per unit and that the firm decides to produce 600 units of goods. Use Lagrange multipliers to determine the amounts of labor and capital that should be utilized in order to minimize the cost. You need not show that your solution minimizes the cost.

[10] 6. Let

$$f(x) = \lim_{n\to\infty}\left[\left(\frac{1}{3} + \frac{1}{2+\left(1+\frac{x-1}{n}\right)^3} + \frac{1}{2+\left(1+\frac{2(x-1)}{n}\right)^3} + \ldots + \frac{1}{2+\left(1+\frac{(n-1)(x-1)}{n}\right)^3}\right)\frac{x-1}{n}\right]$$

where $x \geq 1$. Find the equation of the tangent line to the graph $y = f'(x)$ at $x = 2$.

Solutions to Sample Final Exam for M105

1. (a) $f(2x,2y)-2f(x,y)=8(2x)^{\frac{1}{5}}(2y)^{\frac{4}{5}}-2\cdot 8x^{\frac{1}{5}}y^{4/5}$

$=8\left(2^{\frac{1}{5}}x^{\frac{1}{5}}\right)\left(2^{\frac{4}{5}}y^{4/5}\right)-2\cdot 8x^{\frac{1}{5}}y^{\frac{4}{5}}=0.$

(b) $\frac{\partial f}{\partial y}=x\left(2e^{2y}\right)+2y$, $\frac{\partial^2 f}{\partial y^2}=2x\left(2e^{2y}\right)+2=4xe^{2y}+2$.

(c) $F'(x)=\ln(3+\sin(2x))\cdot(2x)'=2\ln(3+\sin(2x))$,

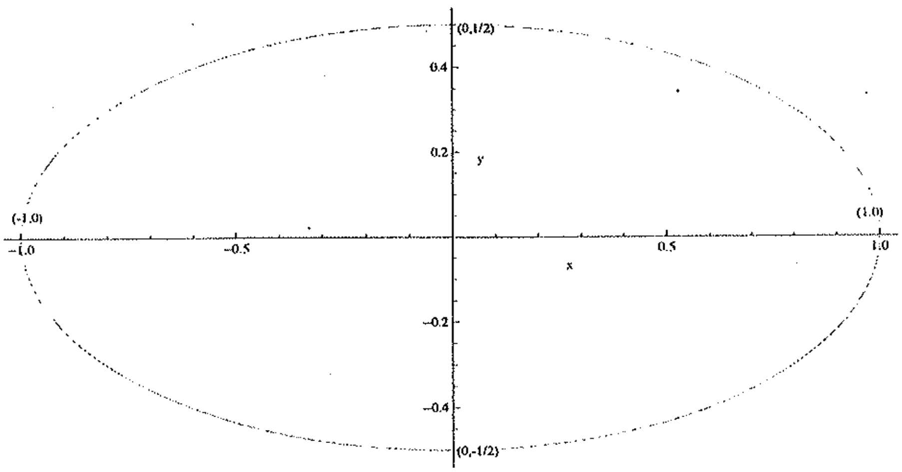
$F'\left(\frac{\pi}{2}\right)=2\ln\left(3+\sin\left(2\left(\frac{\pi}{2}\right)\right)\right)=2\ln 3$.

(d) $f(x)=\int\left(\frac{1}{\sqrt{x}}+x^2\right)dx=2x^{\frac{1}{2}}+\frac{1}{3}x^3+C$. $f(1)=2\Rightarrow C=-\frac{1}{3}$.

Hence, $f(x)=2x^{\frac{1}{2}}+\frac{1}{3}x^3-\frac{1}{3}$.

(e)

Substituting in $z=e$, we find that $e=e^{x^2+4y^2}$. Hence, $1=x^2+4y^2$. This is an *ellipse*. The sketch, with four labeled points, looks like:



(f) $\Delta X = \frac{4-0}{2} = 2$. The area $\approx \left(f(1)+f(3)\right)\Delta X = \left(\frac{1}{1+\sqrt{1}} + \frac{1}{1+\sqrt{3}}\right)2$

$= 1 + \frac{2}{1+\sqrt{3}}$.

(e) The fourth derivative of the integrand is $f^{(4)}(x) = 16e^{-2x}$. Since $|f^{(4)}(x)| \leq 16$ on $[0,1]$, the error is bounded by $E_S = K(b-a)(\Delta x)^4/180 = 16/(180 \cdot 6^4)$.

(h) $\displaystyle\int \frac{\sqrt{\ln x}}{x}\,dx \underset{u=\ln x}{=\!=} \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(\ln x)^{3/2} + C.$

(i) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(\sqrt{3})^{2n}} = \sqrt{3}\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{\sqrt{3}}\right)^{2n+1}}{2n+1}$

$= \sqrt{3}\,\arctan\left(\frac{1}{\sqrt{3}}\right) = \sqrt{3}\left(\frac{\pi}{6}\right).$

(j) Since parallel planes share the same normal vector, the equation has to have the form $3x - y + 4z = C$ for some C. Substituting in $(2, 1, -1)$, we find that

$$C = 3 \cdot 2 - 1 + 4 \cdot -1 = 1.$$

So the equation is $3x - y + 4z = 1$.

(k) $\displaystyle\frac{dy}{dx} = xe^{x^2}\cdot\left(\frac{1}{y^2}\right) \Rightarrow \int y^2\,dy = \int xe^{x^2}\,dx$

$\displaystyle\frac{y^3}{3} = \frac{e^{x^2}}{2} + C \Rightarrow y = \sqrt[3]{\frac{3}{2}e^{x^2} + 3C}.$

(l) $1=\int_1^4 kx^{3/2}\,dx = k\frac{2}{5}x^{\frac{5}{2}}\Big|_1^4 = \frac{2k}{5}(2^5-1) = \frac{2k}{5}(31) \Rightarrow k=\frac{5}{62}$

(m) $F(x)=\int 2(x-1)\,dx = x^2-2x+c$. $\quad 0=F(1)=1-2+c \Rightarrow c=1$.

Hence, $F(x)=x^2-2x+1$.

(n) $Var(X)=\int_1^4 x^2 f(x)\,dx - E(X)^2 = \int_1^4 x^2\frac{1}{2\sqrt{x}}\,dx - \left(\frac{7}{3}\right)^2 = \int_1^4 \frac{1}{2}x^{\frac{3}{2}}\,dx - \frac{49}{9}$

$= \frac{1}{5}x^{5/2}\Big|_1^4 - \frac{49}{9} = \frac{1}{5}(32-1)-\frac{49}{9} = \frac{31}{5}-\frac{49}{9} \left(=\frac{34}{45}\right).$

2. (a) Since the summands are positive and decreasing with k, we may apply the integral test. Accordingly we compute

$$\int_1^\infty \frac{1}{\sqrt{x}}e^{-\sqrt{x}}\,dx = 2\int_1^\infty e^{-u}\,du = 2e^{-1}.$$

Since the integral above is convergent, the series is convergent as well.

(b) Set

$$a_k = \frac{k^4-2k^3+2}{k^5+k^2+k}, \qquad b_k=\frac{1}{k}.$$

Since

$$\lim_{k\to\infty}\frac{a_k}{b_k} = \lim_{k\to\infty}\frac{k^5-2k^4+2k}{k^5+k^2+k} = \lim_{k\to\infty}\frac{1-\frac{2}{k}+\frac{2}{k^4}}{1+\frac{1}{k^3}+\frac{1}{k^4}} = 1\neq 0,$$

by the limit comparison test, $\sum_k a_k$ and $\sum_k b_k$ either both converge or both diverge. But $\sum_k b_k$ is the harmonic series which diverges. So $\sum_k a_k$ diverges as well.

(c) By the ratio test,

$$\lim_{k\to\infty}\frac{a_{k+1}}{a_k} = \lim_{k\to\infty}\frac{\frac{2^{k+1}((k+1)!)^2}{(2k+2)!}}{\frac{2^k(k!)^2}{(2k)!}} = \lim_{k\to\infty}\frac{2(k+1)^2}{(2k+1)(2k+2)} = \frac{1}{2}<1,$$

which means that the series converges.

3. (a) For simplicity, we suppress constants of integration until the later steps.

Substitution of $\ln(x) = u$. Then $du = \frac{dx}{x}$ and $dx = x\,du = e^u\,du$.

Therefore
$$\int \sin(\ln(x))\ dx = \int \sin(u)e^u\ du$$

Integration by parts:
$$\int \sin(u)e^u\ du = \sin(u)e^u - \int \cos(u)e^u\ du$$

Integration by parts (again):
$$\int \cos(u)e^u\ du = \cos(u)e^u + \int \sin(u)e^u\ du$$

Solving for the integral $\int \sin(u)e^u\ du$.
$$\begin{aligned}\int \sin(u)e^u\ du &= \sin(u)e^u - \int \cos(u)e^u\ du\\ &= \sin(u)e^u - \cos(u)e^u - \int \sin(u)e^u\ du\end{aligned}$$

Therefore,
$$\int \sin(u)e^u\ du = \frac{e^u}{2}(\sin(u) - \cos(u)) + C$$

Final step.
$$\begin{aligned}\int \sin(\ln(x))\ dx &= \int \sin(u)e^u\ du\\ &= \frac{e^u}{2}(\sin(u) - \cos(u)) + C\\ &= \frac{x}{2}(\sin(\ln(x)) - \cos(\ln(x)) + C\end{aligned}$$

(b) Partial fraction of the integrand:
$$\frac{1}{x^2-5x+6} = \frac{1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$$

Solving for A and B: $A + B = 0$ and $-3A - 2B = 1$. Hence, $A = -1$ and $B = 1$.

Solving the integral
$$\begin{aligned}\int_0^1 \frac{1}{x^2-5x+6}\ dx &= \int_0^1 \frac{1}{x-3} - \frac{1}{x-2}\ dx\\ &= \int_0^1 \frac{1}{x-3}\ dx - \int_0^1 \frac{1}{x-2}\ dx\\ &= (\ln|x-3|)_0^1 - (\ln|x-2|)_0^1\\ &= \ln(2) - \ln(3) - \ln(1) + \ln(2)\\ &= 2\ln(2) - \ln(3)\end{aligned}$$

Solution P4

4. (a) The partials of f are

$$f_x = ye^y + x, \quad f_y = xye^y + xe^y = xe^y(y+1).$$

For (x, y) to be a critical point we need $f_y = xe^y(y+1) = 0$, which requires that either $y = -1$ or $x = 0$ (since e^y is never 0).
In the case $y = -1$ we have $f_x = ye^y + x = -1/e + x = 0$, so that $x = 1/e$. This yields the critical point $(1/e, -1)$.
In the case $x = 0$ we have $f_x = ye^y + x = ye^y = 0$, thus $y = 0$. This yields the critical point $(0, 0)$.
There are exactly two critical points: $(0, 0)$ and $(1/e, -1)$.

(b) The second partials of f are

$$f_{xx} = 1, \quad f_{xy} = e^y(y+1), \quad f_{yy} = xe^y(y+2).$$

We now test each point with the Second Derivative Test, taking $D = f_{xx}f_{yy} - f_{xy}^2$.

- Point $(0, 0)$: $f_{xx} = 1, f_{xy} = 1, f_{yy} = 0$ so $D = -1 < 0$. This is a saddle point.
- Point $(1/e, -1)$: $f_{xx} = 1, f_{xy} = 0, f_{yy} = 1/e^2$ so $D = 1/e^2 > 0$. This is a local extremum, and since $f_{xx} > 0$ it is a local minimum.

solution – Ps

5. We need to minimize $C(x,y) = 108x + 2y$

subject to the constraint $g(x,y) = 5x^{\frac{2}{3}}y^{\frac{1}{3}} - 600 = 0$

Since $5x^{\frac{2}{3}}y^{\frac{1}{3}} = 600$, $xy \neq 0$. Hence, $x > 0$ and $y > 0$.

The system we have to solve is

$$\begin{cases} \dfrac{\partial C}{\partial x} = \lambda \dfrac{\partial g}{\partial x} \\ \dfrac{\partial C}{\partial y} = \lambda \dfrac{\partial g}{\partial y} \\ g = 0 \end{cases} \quad \text{or} \quad \begin{cases} 108 = \lambda\left(5 \cdot \frac{2}{3} x^{-\frac{1}{3}} y^{\frac{1}{3}}\right) & (1) \\ 2 = \lambda\left(5 \cdot \frac{1}{3} x^{\frac{2}{3}} y^{-\frac{2}{3}}\right) & (2) \\ 5x^{\frac{2}{3}} y^{\frac{1}{3}} = 600 & (3) \end{cases}$$

By (1) and (2), we get

$$108\left(\frac{5}{3}\right) x^{\frac{2}{3}} y^{-\frac{2}{3}} \lambda = 2\left(5 \cdot \frac{2}{3} x^{-\frac{1}{3}} y^{\frac{1}{3}}\right) \lambda$$

or $$27 x^{\frac{2}{3}} y^{-\frac{2}{3}} = x^{-\frac{1}{3}} y^{\frac{1}{3}} \Rightarrow 27x = y. \quad (4)$$

Using (4), (3) becomes $600 = 5x^{\frac{2}{3}}(27x)^{\frac{1}{3}} = 5x^{\frac{2}{3}} \cdot 3x^{\frac{1}{3}}$

$= 15x \Rightarrow x = 40$ and $y = 27x = 27(40) = 1080$.

6. $f(x) = \int_1^x \frac{1}{2+t^3}\,dt$

$$f'(x) = \frac{1}{2+x^3} \quad \text{and} \quad f''(x) = -\frac{3x^2}{(2+x^3)^2}.$$

Thus, $f'(2) = \frac{1}{2+2^3} = \frac{1}{10}$ and $f''(2) = -\frac{3\cdot 2^2}{(2+2^3)^2} = -\frac{3}{25}$

Hence, the equation of the tangent line to the graph $y = f'(x)$ at $x=2$ is given by

$$y - f'(2) = f''(2)(x-2)$$

$$y - \frac{1}{10} = -\frac{3}{25}(x-2) \quad \text{or} \quad y = -\frac{3}{25}x + \frac{17}{50}.$$

Solution - P.