MATH 1009-E **TEST 2 - solutions** October 2014

PART A: MULTIPLE CHOICE ANSWERS: a, b, b, d, d.

[5 × 2 marks each = 10 marks]

A1. $\lim_{x\to 0} \dfrac{3x^5 - 2x^3 + 7x + 1}{x^5 + 5}$ is

| (a) $\frac{1}{5}$ | (b) 3 | (c) 0 | (d) ∞ | (e) None of the above. |

A2. $\lim_{x\to 3} \dfrac{9 - x^2}{x - 3}$ is

(a) 6 | (b) -6 | (c) 0 (d) ∞ (e) None of the above.

A3. $\lim_{x\to -\infty} \dfrac{7x^4 + x^3 + 3^2x - 3}{x^7 + 3x^2 + 1}$ is

(a) 7 | (b) 0 | (c) -3 (d) ∞ (e) None of the above.

A4. $\lim_{x\to \infty} \dfrac{3x^5 + x^3 - 4x^2 - 11}{x^2 + x + 1}$ is

(a) 3 (b) 0 (c) -11 | (d) ∞ | (e) None of the above.

A5. The function $g(x) = \dfrac{x - 3}{x^4 + 1}$ is discontinuous at

(a) $x = \pm 1$ (b) $x = 3$ (c) $x = -1$ | (d) continuous everywhere |

(e) None of the above.

PART B: LONG STYLE QUESTIONS.

[**7 marks**] **B1**. Let $f(x) = \begin{cases} 3, & x < 1, \\ -x, & x \geq 1. \end{cases}$

[**1**] **(a)** Sketch the graph of f.

[**5**] **(b)** Find the following limits:

(i) $\lim_{x \to 1+} f(x) = -1$; **(ii)** $\lim_{x \to 1-} f(x) = 3$;

(iii) $\lim_{x \to 1} f(x) =$ DNE (Does Not Exist);

(iv) $\lim_{x \to -15} f(x) = 3$; **(v)** $\lim_{x \to 6} f(x) = -6$;

[**1**] **(c)** Is the function f continuous at $x = 1$? Explain.

Since the left-hand-side limit (i) is not the same as the righ-hand-side limit (ii), the limit does not exist at $x = 1$. Thus, the function is discontinuous at $x = 1$. The graph has a gap at $x = 1$.

[**8 marks**] **B2**. Use the Intermediate Value Theorem to show that the equation has at least one solution in the designated interval. Explain why the theorem is applicable.

[**4**] **(a)** $x^3 + 2x^2 - 5 = 0$, in $(1, 2)$

[**4**] **(b)** $4x - 1 = 2^x$, in $(0, 1)$.

Marking guideline : 1 mark - for stating that the function is continuous and therefore the IVT is applicable, 3 - for the rest.

Solution: **(a)** $f(x) = x^3 + 2x^2 - 5$ is a polynomial, therefore, it is continuous on the interval $[1, 2]$. Thus, we can apply the IVT on that interval.

$$f(1) = (1)^3 + 2 \cdot 1^2 - 5 = -2 < 0; \quad f(2) = 2^3 + 2 \cdot 2^2 - 5 = 8 + 8 - 5 = 11 > 0.$$

Since the values of the function have opposite signs at the endpoints of the interval, then there is at least one point on the interval, at which this function assumes the value of zero.

(b) In the context of this problem, $f(x) = 4x - 1 - 2^x$. As a difference of two continuous function, it is continuous on the interval $[0, 1]$,

$$f(0) = 4 \cdot 0 - 1 - 2^0 = -2 < 0; \quad f(1) = 4 \cdot 1 - 1 - 2^1 = 4 - 3 = 1 > 0.$$

Since the values of the function have opposite signs at the endpoints of the interval, then there is at least one point on the interval, at which this function assumes the value of zero.

[5 marks] B3. The average rate of change of a function f over the interval $[x, x+h]$ is given by

$$\frac{f(x+h)-f(x)}{h}.$$

Suppose that $f(x) = \frac{2}{x}$.

[3.5] (a) What is the average rate of change of f over each of the intervals $[2,5]$ and $[2,3]$?

[1.5] (b) What is the instantaneous rate of change of f at $x = 1$? (You are allowed to use the rules of differentiation.)

Solution:

(a) For the interval $[2,5]$ we have $x = 2$, $h = 3$:

$$\frac{f(5)-f(2)}{3} = \frac{\frac{2}{5}-\frac{2}{2}}{3} = \frac{-0.6}{3} = -0.2,$$

and for the interval $[2,3]$ we have $x = 2$, $h = 1$:

$$\frac{f(3)-f(2)}{1} = \frac{\frac{2}{3}-\frac{2}{2}}{1} = -\frac{1}{3}.$$

(b) The derivative $f'(1)$ gives the instantaneous rate of change:

$$f'(x) = 2\cdot(-1)x^{-2} = -\frac{2}{x^2}, \qquad f'(1) = -\frac{2}{1^2} = -2.$$

[10 marks] B4. Find the derivative of the following functions using the appropriate rules of differentiation:

[3] (a) $f(x) = \dfrac{4x^2}{2x^3-7}$. (Simplify the numerator.)

[2] (b) $g(x) = 3^{1-4x}$.

[2] (c) $h(x) = 4(x^5+2x^2-1)^{1/4}$.

[3] (d) $k(x) = \log_5(2x^3+1)$.

Solution:

(a) $$f'(x) = \frac{(4x^2)'(2x^3-7)-4x^2(2x^3-7)'}{(2x^3-7)^2} = \frac{8x(2x^3-7)-4x^2\cdot 6x^2}{(2x^3-7)^2} = \frac{-8x^4-56x}{(2x^3-7)^2}.$$

(b) $g'(x) = 3^{1-4x}\cdot \ln 3 \cdot (1-4x)' = (-4\ln 3)\,3^{1-4x}$.

(c) $h'(x) = 4\cdot\frac{1}{4}(x^5+2x^2-1)^{-3/4}\cdot(x^5+2x^2-1)' = (x^5+2x^2-1)^{-3/4}(5x^4+4x)$.

(d) $$k'(x) = \frac{1}{(2x^3+1)\ln 5}\cdot(2x^3+1)' = \frac{6x^2}{(2x^3+1)\ln 5}.$$