MATH 3705 * B Test 2 - Solutions. February 2012

Questions 1-4 are multiple choice. Circle the correct answer. Only the answer will be marked.

1. [**2 marks**] The equation $3(x-1)y'' + x^2y' + \frac{1}{4}y = 0$ has

(a) two regular singular points $x = 1$ and $x = 3$.

(b) one regular singular points $x = 1$ and one irregular singular point $x = 3$.

(c) one regular singular point $x = 1$.

(d) one regular singular point $x = 3$.

(e) no singular points.

Solution: The equation in standard form is $y'' + \dfrac{x^2}{3(x-1)}y' + \dfrac{1}{12(x-1)}y = 0$, $\Rightarrow$ $p(x) = \dfrac{x^2}{3(x-1)}$ and $q(x) = \dfrac{1}{12(x-1)}$ are not analytic at $x = 1$, but $(x-1)\,p(x) = \dfrac{x^2}{3}$ and $(x-1)^2\,q(x) = \dfrac{x-1}{12}$ are, $\Rightarrow$ (c).

2. [**2 marks**] The differential equation $y'' + \dfrac{2}{x+5}y' + y = 0$ has a singular point $x_0 = -5$. Then the series solution $y = \sum_{n=0}^{\infty} a_n(x-2)^n$ about $x = 2$ has the radius of convergence

(a) $R = \infty$ (b) $R \geq 2$ (c) $R \geq 5$ (d) $R \geq 7$

3. [**3 marks**] The general solution of $4x^2y'' - 8xy' + 9y = 0$ for $x \neq 0$ is

(a) $C_1|x|^{-\frac{3}{2}} + C_2|x|^{\frac{3}{2}}$ (b) $|x|^{\frac{3}{2}}(C_1 + C_2\ln|x|)$ (c) $|x|^{4+\sqrt{7}}(C_1 + C_2\ln|x|)$

(d) $|x|^4\left[C_1\cos\left(\sqrt{7}\ln|x|\right) + C_2\sin\left(\sqrt{7}\ln|x|\right)\right]$ (e) None of the above

Solution: The indicial equation is $4r(r-1) - 8r + 9 = 0$, or $4r^2 - 12r + 9 = 0$, with $r_1 = r_2 = \dfrac{12}{8} = \dfrac{3}{2}$, $\Rightarrow$ Euler Equation, case (ii) $\Rightarrow$ (b).

4. [**3 marks**] The general solution of $x^2y'' - xy' - 3y = 0$ for $x \neq 0$ is

(a) $C_1|x| + C_2|x|^{-3}$ (b) $C_1|x|^{-1} + C_2|x|^3$ (c) $|x|(C_1 + C_2\ln|3x|)$

(d) $|x|^{-1}\left[C_1\cos(3\ln|x|) + C_2\sin(3\ln|x|)\right]$ (e) None of the above

Solution: The indicial equation is $r(r-1) - r - 3 = 0$, or $r^2 - 2r - 3 = 0$, with $r_1 = -1$ $r_2 = 3$, $\Rightarrow$ Euler equation, case (i) $\Rightarrow$ (b).

Answers: c, d, b, b.

5. [8 marks] The differential equation $y'' + 2xy' + 2y = 0$ has no singular points. The power series solution near $x_0 = 0$ has the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Find the recursion relation for the coefficients a_n. (Do NOT solve it.)

Solution:

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=0}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}.$$

Substituting y, y' and y'' into the original equation yields

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 2 a_n x^n = 0.$$

Combine the series for x^n:

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} 2 a_n (n+1) x^n = 0. \qquad (*)$$

Notice that in the first series the first two terms, which correspond to $n = 0$ and $n = 1$, are zeros. So the series does not change if the summation starts with $n = 2$. Then we shift the index of summation $n \to n + 2$, and the series becomes

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n+2=2}^{\infty} (n+2)(n+2-1) a_{n+2} x^{n+2-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n.$$

Substituting the series above back to the equation $(*)$ yields

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} 2 a_n (n+1) x^n = 0,$$

which can be combined into one series as

$$\sum_{n=0}^{\infty} \left\{ (n+2)(n+1) a_{n+2} + 2 a_n (n+1) \right\} x^n = 0.$$

The above equation means that the series concerges to 0 for all x near x_0, and therefore all the coefficients in the series must be zero:

$$(n+2)(n+1) a_{n+2} + 2 a_n (n+1) = 0,$$

or

$$a_{n+2} = \frac{-2a_n}{n+2}.$$

6. [12 marks] The differential equation $xy'' - 2xy' - 2y = 0$ has a regular singular point $x_0 = 0$ and a power series solution near $x_0 = 0$.

(a) [4] Show that $r_1 = 1$ and $r_2 = 0$ are the roots of the indicial equation.
(b) [7] Find a power series solution, which corresponds to $r_1 = 1$.
(c) [1] Give the first four terms of the series solution found in part **(b)**.

Solution:

(a) Rewrite the equation in the standard form :

$$y'' - 2y' - \frac{2}{x}y = 0.$$

Here $p(x) = -2, \quad xp(x) = -2x, \quad q(x) = \frac{-2}{x}, \quad x^2 q(x) = -2x.$

$p_0 = 0,\ q_0 = 0 \ \Rightarrow\ r^2 + (p_0 - 1)r + q_0 = r^2 - r = r(r-1) = 0$ is an indicial equation. The roots are $r_1 = 1,\ r_2 = 0.$

(b) The solution $y(x)$ corresponding to $r_1 = 1$ has the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+1}, \quad \text{with} \quad y' = \sum_{n=0}^{\infty} (n+1)\, a_n x^n \quad \text{and} \quad y'' = \sum_{n=0}^{\infty} n(n+1)\, a_n x^{n-1}.$$

Substituting y, y' and y'' into the original equation yields

$$\sum_{n=0}^{\infty} n(n+1)a_n x^n - \sum_{n=0}^{\infty} 2(n+1)a_n x^{n+1} - \sum_{n=0}^{\infty} 2a_n x^{n+1} = 0.$$

After combining the series for x^{n+1} the equation becomes

$$\sum_{n=0}^{\infty} n(n+1)a_n x^n - \sum_{n=0}^{\infty} 2a_n(n+2)x^{n+1} = 0. \qquad (*)$$

Notice that in the first series the first term, which corresponds to $n = 0$, is zero. So the series does not change if the summation starts with $n = 1$. Thus, if we shift the index of summation $n \to n+1$, then the series becomes

$$\sum_{n=1}^{\infty} n(n+1)a_n x^n = \sum_{n+1=1}^{\infty} (n+1)(n+2)a_{n+1}x^{n+1} = \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+1}x^{n+1}.$$

Substituting the series above back to the equation $(*)$ yields

$$\sum_{n=0}^{\infty} (n+1)(n+2)a_{n+1}x^{n+1} - \sum_{n=0}^{\infty} 2a_n(n+2)x^{n+1} = 0,$$

which can be combined into one series as

$$\sum_{n=0}^{\infty}\{(n+1)(n+2)a_{n+1} - 2a_n(n+2)\}\, x^{n+1} = 0.$$

The above equation means that the series converges to 0 for all x near x_0. Therefore, all the coefficients in the series must be zero:

$$(n+1)(n+2)a_{n+1} - 2a_n(n+2) = 0,$$

or

$$a_{n+1} = \frac{2a_n}{n+1}.$$

Thus, we found the recurrence relation for the coefficients. Let us solve it.

$$n = 0 \;\Rightarrow a_1 = \frac{2}{1}\, a_0;$$

$$n = 1 \;\Rightarrow a_2 = \frac{2a_1}{2} = \frac{2}{2}\cdot\frac{2}{1}\, a_0;$$

$$n = 2 \;\Rightarrow a_3 = \frac{2a_2}{3} = \frac{2}{3}\cdot\frac{2}{2}\cdot\frac{2}{1}\, a_0;$$

$$n = 3 \;\Rightarrow a_4 = \frac{2a_3}{4} = \frac{2}{4}\cdot\frac{2}{3}\cdot\frac{2}{2}\cdot\frac{2}{1}\, a_0;$$

The pattern emerging for a_k is

$$a_k = \frac{2^k}{k!}\, a_0.$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=0}^{\infty} \frac{2^n\, a_0}{n!}\, x^{n+1}.$$

(c) The first four terms of the solution:

$$y = \sum_{n=0}^{\infty} \frac{2^n\, a_0}{n!} x^{n+1} \;\; = \;\; a_0(x + 2x^2 + 2x^3 + \frac{4}{3}\, x^4 + ...).$$