MACM 101 — Discrete Mathematics I

Outline Solutions to Exercises on Propositional Logic

1. **Construct a truth table for the following compound statement:** $(p \to r) \oplus (\neg q \to \neg r)$**.**

p	q	r	$(p \oplus r) \oplus (\neg q \leftrightarrow \neg r)$
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	1

2. **Determine whether the following compound statement is a tautology or contradiction**

$$(p \to (q \to r)) \to ((p \wedge q) \to r).$$

It is a tautology.
Method 1. Construct a truth table.

Method 2. Use logical equivalences:

$$\begin{array}{lll} \multicolumn{2}{l}{(p \to (q \to r)) \to ((p \wedge q) \to r)} & \\ \iff & \neg(\neg p \vee (\neg q \vee r)) \vee (\neg(p \wedge q) \vee r) & \text{expression for implications} \\ \iff & (p \wedge q \wedge \neg r) \vee (\neg p \vee \neg q \vee r) & \text{DeMorgan's law} \\ \iff & (p \wedge q \wedge \neg r) \vee \neq ((p \wedge q \wedge \neg r)) & \text{DeMorgan's law} \\ \iff & T & \text{domination law} \end{array}$$

In the last equivalence we apply the domination law to the disjunction of the formula $p \wedge q \wedge \neg r$ and its negation. Alternatively, we can apply the distributive law several times.

Method 3. Prove that the formula never takes truth value 0. The formula is an implication, therefore to make it false the premise must be true and the conclusion false. So, suppose $(p \wedge q) \to r$ is false. It again is an implication, hence $p = q = 1$ and $r = 0$. This is the only combination of truth values that can falsify the original compound statement. However, it does not, since the premise, $p \to (q \to r)$ is false when $p = q = 1$ and $r = 0$.

3. **Show that** $(p \to r) \wedge (q \to r)$ **and** $(p \wedge q) \to r)$ **are logically equivalent.**

Method 1. Construct the truth tables for both statements and compare.

Method 2. Use logical equivalences.

$$\begin{array}{lll} \multicolumn{2}{l}{(p \to r) \wedge (q \to r)} & \\ \iff & (\neg p \vee r) \wedge (\neg q \vee r) & \text{expression for implications} \\ \iff & (\neg p \wedge \neg q) \vee r & \text{distributive law} \\ \iff & \neg(p \vee q) \vee r & \text{De Morgan's law} \\ \iff & (p \wedge q) \to r) & \text{expression for implication} \end{array}$$

4. **Show that** $(s \to (p \wedge \neg r)) \wedge (p \to (r \vee q)) \wedge s$ **and** $p \wedge \neg r \wedge q$ **are not logically equivalent. Do not use truth tables.**

   It is sufficient to find one assignment of values to p, q, r, s such that the two statements get different truth values. For instance if $p = q = 1$, $r = 0$, and $s = 0$ then $(s \to (p \wedge \neg r)) \wedge (p \to (r \vee q)) \wedge s = 0$ while $p \wedge \neg r \wedge q = 1$.

   These values can be obtained, for example, as follows. We observe that the second formula does not contain s while the first one is false whenever $s = 0$. Therefore we need to find truth values of p, q, r that make the second formula true, and then set $s = 0$.

   Note: In the first formula there was either an extra opening parenthesis or a missing closing parenthesis. In particular, the formula could also be read as $(s \to (p \wedge \neg r)) \wedge ((p \to (r \vee q)) \wedge s)$. Hoewever, as is easily seen, it is logically equivalent to the correct one.

5. **Simplify the compound statement**

$$(p \vee (\neg p \wedge \neg(q \vee r))) \to (p \vee \neg(r \vee q)).$$

   Use logical equivalences:

$$\begin{array}{lll}
\multicolumn{2}{l}{(p \vee (\neg p \wedge \neg(q \vee r))) \to (p \vee \neg(r \vee q))} & \\
\iff & \neg(p \vee (\neg p \wedge \neg(q \vee r))) \vee (p \vee \neg(r \vee q)) & \text{expression for implication} \\
\iff & \neg((p \vee \neg p) \wedge (p \vee (\neg q \wedge \neg r))) \vee (p \vee (\neg q \vee \neg q)) & \text{distributive law} \\
\iff & \neg(p \vee (\neg q \wedge \neg r)) \vee (p \vee (\neg q \vee \neg q)) & \text{law of excluded middle and identity law} \\
\iff & T. & \text{law of excluded middle, see Problem 1}
\end{array}$$

6. **Prove that the Rule of Syllogism is a valid argument..**

   It suffices to prove that the corresponding expression

$$((p \to q) \wedge (q \to r)) \to (p \to r)$$

   is a tautology.

   Method 1. Construct the truth table.

   Method 2. Use logical equivalences to show that this statement is equivalent to 1.

$$\begin{array}{lll}
\multicolumn{2}{l}{((p \to q) \wedge (q \to r)) \to (p \to r)} & \\
\iff & \neg((\neg p \vee q) \wedge (\neg q \vee r)) \vee (\neg p \vee r) & \text{expression for implication} \\
\iff & (p \wedge \neg q) \vee (q \wedge \neg r)) \vee \neg p \vee r & \text{De Morgan's law + double negation law} \\
\iff & ((p \vee \neg p) \wedge (\neg q \vee \neg p)) \vee (q \wedge \neg r) \vee r & \text{distributive law} \\
\iff & (\neg q \vee \neg p) \vee (q \wedge \neg r) \vee r & \text{law of excluded middle + identity law} \\
\iff & \neg q \vee \neg p \vee ((q \vee r) \wedge (\neg r \vee r) & \text{distributive law} \\
\iff & \neg q \vee \neg p \vee (q \vee r) & \text{law of excluded middle + identity law} \\
\iff & \neg q \vee \neg p \vee q \vee r & \text{associative law} \\
\iff & T \vee r \vee \neg p & \text{law of excluded middle} \\
\iff & \text{T} & \text{domination law}
\end{array}$$

   Method 3. Suppose that the premises $(p \to q)$ and $(q \to r)$ are true. If $p \to r$ is false then, by definition of implication, we know that p is true and q is false. Then, since $p \to q$ is true, by definition of implication, q must be true. If q is true then, as $q \to r$ is true, we conclude that r must be true. But this contradicts our assumption that $p \to r$ is false. Q.E.D.

7. **Each of two rooms (room I and room II) contains either a lady or a tiger. If a room contains a lady, the sign on its door is true. If it contains a tiger, the sign is false. The signs are**

I	II
THERE IS A LADY IN THE OTHER ROOM	**IT MAKE NO DIFFERENS WHICH ROOM TO PICK**

Which rooms contain ladies?

Let the primitive statements be:

p, 'the first room contains a lady'
q, 'the second room contain a lady'

Then the sign on the first door says that q, and the sign on the second door claims $p \leftrightarrow q$. We know that a sign is true if and only if the corresponding room contains a lady. Therefore the statements $p \leftrightarrow q$ and $q \leftrightarrow (p \leftrightarrow q)$ are true.

By checking all possible truth values of p, q we find that both statements above are true only if $p = q = 1$, that is, both rooms contain ladies.

8. **What relevant conclusion can be drawn from the following premises:**
"I am dreaming or hallucinating."
"If I am dreaming, I am snoring."
"If I am hallucinating, I see elephants running down the road."
"I am not snoring."

The relevant conclusions are: "I am hallucinating" and "I see elephants running down the road."
Let the primitive statements be:

d, 'I am dreaming'
h, 'I am hallucinating'
s, 'I am snoring'
e, 'I see elephants running down the road'

Then the premises are translated as: $d \vee h$, $d \to s$, $h \to e$, $\neg s$.

And the conclusion: e.

Steps	Reason
1. $d \to s$	premise
2. $\neg s$	premise
3. $\neg d$	modus tollens to Steps 1 and 2
4. $d \vee h$	premise
5. h	rule of disjunctive syllogism
6. $h \to e$	premise
7. e	modus ponens to Steps 5 and 6.

9. **Write the following argument in symbolic form. Then establish the validity of the argument or give a counterexample to show that it is invalid.**

> **If it does not rain or it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on. If the sailing race is held, then the trophy will be awarded. The trophy was not awarded. Therefore, it rained.**

Let the primitive statements be:

r, 'it rains'
f, 'it is foggy'

s, 'sailing race will be held'
ℓ, 'lifesaving demonstration will go on'
t, 'the trophy is awarded'

Then the premises are translated as: $(\neg r \vee \neg f) \to (s \wedge \ell)$, $s \to t$, $\neg t$.

And the conclusion: r.

Steps	Reason
1. $s \to t$	premise
2. $\neg t$	premise
3. $\neg s$	Modus Tollens to 1 and 2
4. $\neg s \vee \neg \ell$	rule of amplification to 3
5. $(\neg r \vee \neg f) \to (s \wedge \ell)$	premise
6. $\neg(s \wedge \ell) \to \neg(\neg r \vee \neg f)$	contrapositive
7. $(\neg s \vee \neg \ell) \to (r \wedge f)$	De Morgan's law
8. $r \wedge f$	Modus Ponens to 4 and 7
9. r	rule of simplification

10. **Using rules of inference and logic equivalences give the reasons for the steps verifying the following argument.**
Premises: $(t \vee q) \to (p \wedge \neg r)$**,** $(u \vee \neg s) \to t$**,** $(p \wedge \neg r) \to q$**,** $\neg s \vee t \vee q$**,** $\neg t \wedge u$**,** $p \vee \neg q$**.**
Conclusion: p**.**

Steps	Reasons
1) $\neg t \wedge u$	premise
2) $\neg t$	rule of simplification
3) $(u \vee \neg s) \to t$	premise
4) $\neg t \to \neg(u \vee \neg s)$	contrapositive
5) $\neg t \to (\neg u \wedge s)$	De Morgan's law
6) $\neg u \wedge s$	Modus Ponens to 2 and 5
7) s	rule of simplification
8) $(t \vee q) \to (p \wedge \neg r)$	premise
9) $\neg s \vee t \vee q$	premise
10) $s \to (t \vee q)$	expression for implication
11) $s \to (p \wedge \neg r)$	rule of syllogism to 10 and 8
12) $p \wedge \neg r$	Modus Ponens to 7 and 11
13) $(p \wedge \neg r) \to q$	premise
14) q	Modus Ponens to 12 and 13
15) $p \vee \neg q$	premise
16) p	rule of disjunctive syllogism (or rule of resolution)