CARTESIAN VECTORS AND THEIR ADDITION & SUBTRACTION

Today's Objectives:

Students will be able to:

a) Represent a 3-D vector in a Cartesian coordinate system.

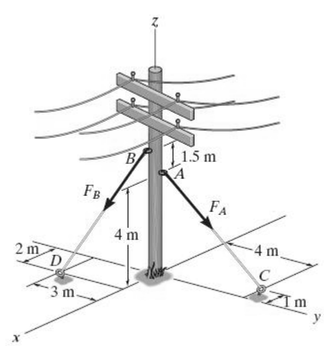
b) Find the magnitude and coordinate angles of a 3-D vector

c) Add vectors (forces) in 3-D space

In-Class Activities:

- Applications / Relevance
- A Unit Vector
- 3-D Vector Terms
- Adding Vectors
- Examples
- Attention Quiz

APPLICATIONS

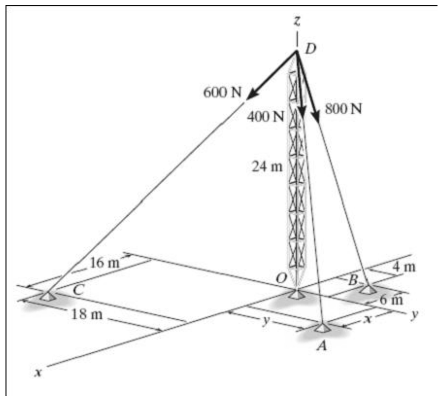


Many structures and machines involve 3-Dimensional Space.

In this case, the power pole has guy wires helping to keep it upright in high winds. How would you represent the forces in the cables using Cartesian vector form?

APPLICATIONS
(continued)

In the case of this radio tower, if you know the forces in the three cables, how would you determine the resultant force acting at D, the top of the tower?



CARTESIAN UNIT VECTORS

For a vector ***A***, with a magnitude of A, a unit vector is defined as

$\boldsymbol{u}_A = \boldsymbol{A} / A$.

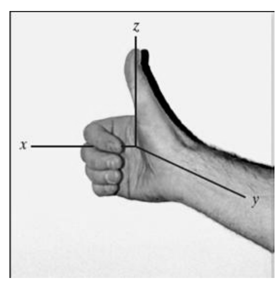
Characteristics of a unit vector **:**

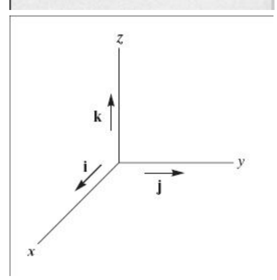
a) Its magnitude is 1.

b) It is dimensionless (has no units).

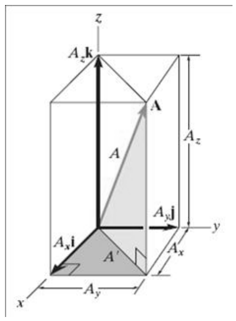
c) It points in the same direction as the original vector (***A***).

The unit vectors in the Cartesian axis system are ***i***, ***j***, and ***k***. They are unit vectors along the positive x, y, and z axes respectively.





CARTESIAN VECTOR REPRESENTATION



Consider a box with sides A_X, A_Y, and A_Z meters long.

The vector ***A*** can be defined as

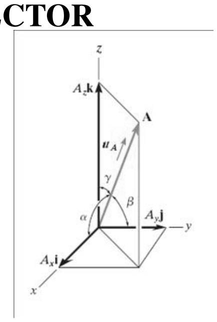
$$\boldsymbol{A} = (A_X\,\boldsymbol{i} + A_Y\,\boldsymbol{j} + A_Z\,\boldsymbol{k})\ \text{m}$$

The projection of vector ***A*** in the x-y plane is A′. The magnitude of A′ is found by using the same approach as a 2-D vector: $A' = (A_X^2 + A_Y^2)^{1/2}$.

The magnitude of the position vector ***A*** can now be obtained as

$$A = ((A')^2 + A_Z^2)^{1/2} = (A_X^2 + A_Y^2 + A_Z^2)^{1/2}$$

DIRECTION OF A CARTESIAN VECTOR



The direction or orientation of vector A is defined by the angles ά, β, and γ.

These angles are measured between the vector and the positive X, Y and Z axes, respectively. Their range of values are from 0° to 180°

Using trigonometry, "direction cosines" are found using

$$\cos\alpha = \frac{A_x}{A} \qquad \cos\beta = \frac{A_y}{A} \qquad \cos\gamma = \frac{A_z}{A}$$

These angles are not independent. They must satisfy the following equation.

$$\mathbf{\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1}$$

This result can be derived from the definition of a coordinate direction angles and the unit vector. Recall, the formula for finding the unit vector of any position vector:

$$\mathbf{u}_A = \frac{\mathbf{A}}{A} = \frac{A_x}{A}\,\mathbf{i} + \frac{A_y}{A}\,\mathbf{j} + \frac{A_z}{A}\,\mathbf{k}$$

or written another way, $u_A = \cos\alpha\,\boldsymbol{i} + \cos\beta\,\boldsymbol{j} + \cos\gamma\,\boldsymbol{k}$.

ADDITION OF CARTESIAN VECTORS (Section 2.6)

Once individual vectors are written in Cartesian form, it is easy to add or subtract them. The process is essentially the same as when 2-D vectors are added.

$$\mathbf{F}_R = \Sigma\mathbf{F} = \Sigma F_x\mathbf{i} + \Sigma F_y\mathbf{j} + \Sigma F_z\mathbf{k}$$

For example, if

$\boldsymbol{A} = A_X\,\boldsymbol{i} + A_Y\,\boldsymbol{j} + A_Z\,\boldsymbol{k}$ and

$\boldsymbol{B} = B_X\,\boldsymbol{i} + B_Y\,\boldsymbol{j} + B_Z\,\boldsymbol{k}$, then

$$\boldsymbol{A} + \boldsymbol{B} = (A_X + B_X)\,\boldsymbol{i} + (A_Y + B_Y)\,\boldsymbol{j} + (A_Z + B_Z)\,\boldsymbol{k}$$

or

$$\boldsymbol{A} - \boldsymbol{B} = (A_X - B_X)\,\boldsymbol{i} + (A_Y - B_Y)\,\boldsymbol{j} + (A_Z - B_Z)\,\boldsymbol{k}\ .$$

IMPORTANT NOTES

Sometimes 3-D vector information is given as:

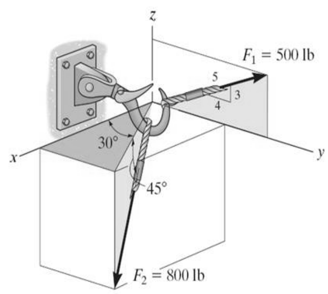
a) Magnitude and the coordinate direction angles, or,

b) Magnitude and projection angles.

You should be able to use both these types of information to change the representation of the vector into the Cartesian form, i.e.,

$\boldsymbol{F} = \{10\,\boldsymbol{i} - 20\,\boldsymbol{j} + 30\,\boldsymbol{k}\}$ N .

EXAMPLE



Given: Two forces F_1 and F_2 are applied to a hook.

Find: The resultant force in Cartesian vector form.

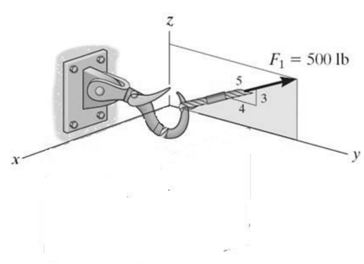
Plan:

1) Using geometry and trigonometry, write F_1 and F_2 in Cartesian vector form.

2) Then add the two forces (by adding x and y components).

Solution :

First, resolve force F_1.



$F_x = 0 = 0$ lb

$F_y = 500\ (4/5) = 400$ lb

$F_z = 500\ (3/5) = 300$ lb

Now, write F_1 in Cartesian vector form (don't forget the units!).

$\boldsymbol{F_1} = \{0\,\boldsymbol{i} + 400\,\boldsymbol{j} + 300\,\boldsymbol{k}\}$ lb

Now resolve force $\boldsymbol{F_2}$.

We are given only two direction angles, α and γ.

So we need to find the value of β.

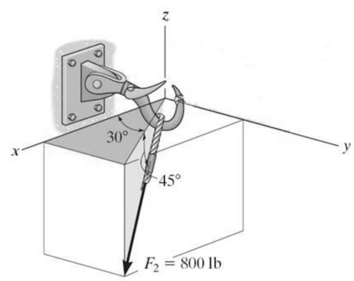
Recall that $\cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) = 1$.

Now substitute what we know:

$\cos^2(30°) + \cos^2(\beta) + \cos^2(45) = 1$.

Solving, $\beta = 75.5°$ or $104.5°$.

Since the vector is pointing in the positive direction, $\beta = 75.5°$



Now that we have the coordinate direction angles, we can find $\boldsymbol{u_G}$ and use it to determine $\boldsymbol{F_2} = 800\,\boldsymbol{u_G}$ lb.

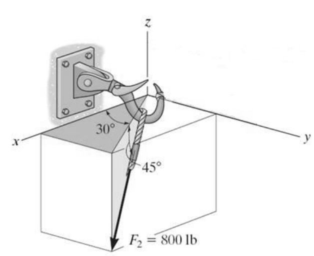
So, using $u_A = \cos\alpha\,\boldsymbol{i} + \cos\beta\,\boldsymbol{j} + \cos\gamma\,\boldsymbol{k}$.

$\boldsymbol{F_2} = \{800\cos(30°)\,\boldsymbol{i} + 800\cos(75.5°)\,\boldsymbol{j} - 800\cos(45°)\,\boldsymbol{k}\}$ lb

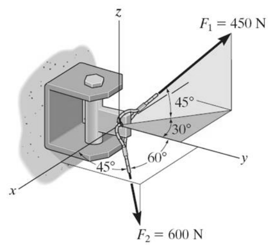
$\boldsymbol{F_2} = \{712.8\,\boldsymbol{i} + 200.3\,\boldsymbol{j} - 608.3\,\boldsymbol{k}\}$ lb

Now, $\boldsymbol{R} = \boldsymbol{F_1} + \boldsymbol{F_2}$ **or**

$\boldsymbol{R} = \{713\,\boldsymbol{i} + 600\,\boldsymbol{j} - 308\,\boldsymbol{k}\}$ lb



GROUP PROBLEM SOLVING



Given: The screw eye is subjected to two forces, $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$.

Find: The magnitude and the coordinate direction angles of the resultant force.

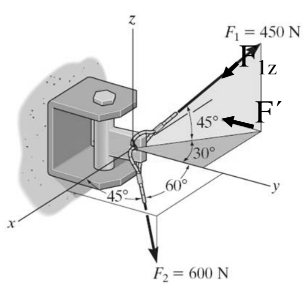
Plan:

1) Using the geometry and trigonometry, resolve and write $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ in the Cartesian vector form.

2) Add $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ to get $\boldsymbol{F_R}$.

3) Determine the magnitude and angles α, β, γ .

GROUP PROBLEM SOLVING (continued)



First resolve the force $\boldsymbol{F_1}$.

F_{1z} = 450 sin 45° = 318.2 N

F´ = 450 cos 45° = 318.2 N

F' can be further resolved as,

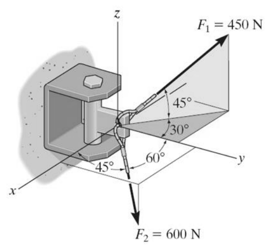
F_{1x} = – 318.2 sin 30° = – 159.1 N

F_{1y} = 318.2 cos 30° = 275.6 N

Now we can write:

$\boldsymbol{F_1} = \{- 159\, \boldsymbol{i} + 276\, \boldsymbol{j} + 318\, \boldsymbol{k}\}$ N

GROUP PROBLEM SOLVING (continued)



Now, resolve force $\boldsymbol{F_2}$.

First, we need to find the value of γ.

$\cos^2(45°) + \cos^2(30) + \cos^2(\gamma) = 1$

Solving, we get $\gamma = 120°$

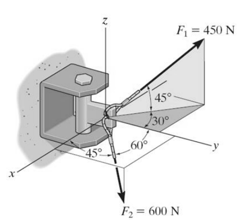
The force $\boldsymbol{F_2}$ can be represented in the Cartesian vector form as:

$\boldsymbol{F_2} = 600\{ \cos 45° \, \boldsymbol{i} + \cos 60° \, \boldsymbol{j} + \cos 120° \, \boldsymbol{k} \}$ N

$= \{ 424.3 \, \boldsymbol{i} + 300 \, \boldsymbol{j} - 300 \, \boldsymbol{k} \}$ N

$\boldsymbol{F_2} = \{ 424 \, \boldsymbol{i} + 300 \, \boldsymbol{j} - 300 \, \boldsymbol{k} \}$ N

GROUP PROBLEM SOLVING (continued)



So $\boldsymbol{F_R} = \boldsymbol{F_1} + \boldsymbol{F_2}$ and

$\boldsymbol{F_1} = \{- 159.1 \, \boldsymbol{i} + 275.6 \, \boldsymbol{j} + 318.2 \, \boldsymbol{k} \}$ N

$\boldsymbol{F_2} = \{ 424.3 \, \boldsymbol{i} + 300 \, \boldsymbol{j} - 300 \, \boldsymbol{k} \}$ N

$\boldsymbol{F_R} = \{ 265.2 \, \boldsymbol{i} + 575.6 \, \boldsymbol{j} + 18.20 \, \boldsymbol{k} \}$ N

Now find the magnitude and direction angles for the vector.

$F_R = (265.2^2 + 575.6^2 + 18.20^2)^{½} = 634.0 = 634$ N

$\alpha = \cos^{-1}(F_{Rx} / F_R) = \cos^{-1}(265.2 / 634.0) = 65.3°$

$\beta = \cos^{-1}(F_{Ry} / F_R) = \cos^{-1}(575.6 / 634.0) = 24.8°$

$\gamma = \cos^{-1}(F_{Rz} / F_R) = \cos^{-1}(18.20 / 634.0) = 88.4°$