Math 340 Midterm
Feb 13, 2014
Duration: 80 minutes

Name: SOLUTIONS Student Number: ______________________

This exam should have 12 pages. No textbooks, calculators, or other aids are allowed. There are 5 questions in this exam, with a total worth of 60 points.

Problem 1 (10 points)

Use the two-phase simplex method to *find all maximizers* of the following linear program and *the optimal value* of its objective function. (**Remember to check if it is in standard form.**)

$$\begin{array}{lrcrcrcl} \text{maximize} & x_1 & + & 2x_2 & + & x_3 & & \\ \text{subject to} & x_1 & + & x_2 & + & x_3 & \geq & 1 \\ & 2x_1 & + & 2x_2 & + & x_3 & \leq & 4 \\ & & & x_1, & & x_2, & x_3 & \geq 0 \end{array}$$

First, bring into standard form:

$$\begin{array}{ll} \max & x_1 + 2x_2 + x_3 \\ \text{s.t.} & -x_1 - x_2 - x_3 \leq -1 \\ & 2x_1 + 2x_2 + x_3 \leq 4 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

← since this is negative, the initial dictionary is infeasible.

Phase I: $\max \; -x_0 = w$

$$\begin{array}{ll} \text{s.t.} & -x_1 - x_2 - x_3 - x_0 \leq -1 \\ & 2x_1 + 2x_2 + x_3 - x_0 \leq 4 \\ & x_0, x_1, x_2, x_3 \geq 0. \end{array}$$

Initial Phase I - dictionary

$$\begin{array}{l} x_4 = -1 + x_1 + x_2 + x_3 + x_0 \\ x_5 = 4 - 2x_1 - 2x_2 - x_3 + x_0 \\ \hline w = \qquad\qquad\qquad\quad - x_0 \end{array}$$

} Pivot to feasibility: x_0 enters, x_4 leaves }

$$\underbrace{\begin{array}{l} x_0 = 1 - x_1 - x_2 - x_3 + x_4 \\ x_5 = 5 - 3x_1 - 3x_2 - 2x_3 + x_4 \\ \hline w = -1 + x_1 + x_2 + x_3 - x_4 \end{array}}_{x_1 \text{ leaves, } x_0 \text{ enters}}$$

$$\begin{array}{l} x_1 = 1 - x_2 - x_3 + x_4 - x_0 \\ x_5 = 2 + x_3 - 2x_4 + 3x_0 \\ \hline w = \qquad\qquad\qquad - x_0 \end{array}$$

Optimal for Phase I

$w^* = 0 \Rightarrow$ original LP is feasible

<u>Phase 2:</u> Non-basic variables above (after discarding x_0): x_2, x_3, x_4

we have

$$x_1 = 1 - x_2 - x_3 + x_4$$
$$x_5 = 2 \qquad + x_3 - 2x_4$$

Next write $z = x_1 + 2x_2 + x_3$ in terms of x_2, x_3, x_4:

$$z = (1 - x_2 - x_3 + x_4) + 2x_2 + x_3 = 1 + x_2 + x_4.$$

So, the initial feasible dictionary for phase 2 is:

$$\begin{array}{l} x_1 = 1 - x_2 - x_3 + x_4 \\ x_5 = 2 \qquad + x_3 - 2x_4 \\ \hline z = 1 + x_2 \qquad + x_4 \end{array}$$

x_2 enters
x_1 leaves $\rightarrow$

$$\begin{array}{l} x_2 = 1 - x_1 - x_3 + x_4 \\ x_5 = 2 \qquad + x_3 - 2x_4 \\ \hline z = 2 - x_1 - x_3 + 2x_4 \end{array}$$

x_4 enters
x_5 leaves

$$\begin{array}{l} x_4 = 1 \qquad + \frac{1}{2}x_3 - \frac{1}{2}x_5 \\ x_2 = 2 - x_1 - \frac{1}{2}x_3 - \frac{1}{2}x_5 \\ \hline z = 4 - x_1 \qquad\qquad - x_5 \end{array}$$

Optimal dictionary! $\boxed{z^* = 4}$

Coeff. of x_3 is 0, so:

$x_3 = t$; $x_4 = 1 + \frac{1}{2}t \geq 0$ holds $\forall t \geq 0$

$x_2 = 2 - \frac{1}{2}t \geq 0$ if $0 \leq t \leq 4$

So, the set of all optimal solutions is

$$\left\{ \left[0, 2 - \frac{t}{2}, t, 1 + \frac{t}{2}, 0\right] : 0 \leq t \leq 4 \right\}.$$

Problem 2 (10 points)

The following optimal dictionary has been obtained when solving a linear programming problem, say (P), in standard form:

$$\begin{array}{rcrcrcr} x_2 & = & 1 & - & x_1 & - & x_4 \\ x_3 & = & 2 & + & 3x_1 & + & x_4 \\ \hline z & = & 2 & & & - & 2x_4 \end{array}$$

(a) [3 pts] Find the original linear program (P).

(b) [1 pts] What is the optimal value, say z^*, of the objective function of (P)?

(c) [3 pts] State the dual (D) of (P) and find an optimal solution for (D).

(d) [1 pts] Illustrate the strong duality theorem on this example.

(e) [2 pts] Does (P) have an optimal solution x^* such that $x_2^* = 1/2$? If yes, find such a solution. If no, explain why not.

(a) (P) has two original variables : x_1, x_2
two slack variables : x_3, x_4

To get the initial dictionary : need x_3 and x_4 to be the basic variables.
So, we pivot : <u>x_4 enters, x_2 leaves</u>

$$\left.\begin{array}{rcl} x_4 & = & 1 - x_1 - x_2 \\ x_3 & = & 3 + 2x_1 - x_2 \\ \hline z & = & 2 - 2(1 - x_1 - x_2) \\ & = & 2x_1 + 2x_2 \end{array}\right\} \Rightarrow \boxed{\begin{array}{rcl} x_3 & = & 3 + 2x_1 - x_2 \\ x_4 & = & 1 - x_1 - x_2 \\ \hline z & = & 2x_1 + 2x_2 \end{array}}$$

initial dictionary of (P)

We then conclude :

$$\boxed{\begin{array}{lll} (P) & \max & 2x_1 + 2x_2 \\ & \text{s.t.} & -2x_1 + x_2 \leq 3 \\ & & x_1 + x_2 \leq 1 \\ & & x_1, x_2 \geq 0 \end{array}}$$

$\rightsquigarrow$

(b) $z^* = 2$ (read off from the given optimal dictionary)

(c) The dual LP is:

$$(D) \quad \boxed{\begin{array}{ll} \min & 3y_1 + y_2 \\ \text{s.t.} & -2y_1 + y_2 \geq 2 \\ & y_1 + y_2 \geq 1 \\ & y_1, y_2 \geq 0 \end{array}}$$

An optimal soln for (D): $y_1^* = 0$; $y_2^* = 2$ (magic coeff.)

(d) $w^* = 3y_1^* + y_2^* = 2$ which is equal to z^* (from (b)) as predicted by the strong duality theorem.

(e) The optimal solutions of (P) are given by

$$\left.\begin{array}{l} x_1 = t \ ; \ t \geq 0 \\ x_2 = 1 - t \ ; \ t \leq 1 \end{array}\right\} \quad \{(t, 1-t) : 0 \leq t \leq 1\}$$

So, set $t = \frac{1}{2}$ $\Rightarrow$ $x_1^* = \frac{1}{2}$, $x_2^* = \frac{1}{2}$ is such a solution.

Problem 3 (10 points)

Consider the LP: Maximize $2x_1 + x_2$ subject to $x_2 \leq 2, x_1 + x_2 \leq 3, x_1 - x_2 \leq 1$, and, $x_1, x_2 \geq 0$. Write the slack variables for this linear program, and write the dual linear program and dial slack variables. Having done that, check that $x_1 = 2, x_2 = 1$ is an optimal solution by using the complementary slackness theorem.

(P)
$$\begin{aligned} \max \quad & 2x_1 + x_2 \\ \text{s.t.} \quad & x_2 \leq 2 \\ & x_1 + x_2 \leq 3 \\ & x_1 - x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned} \Rightarrow \boxed{\begin{aligned} &\text{Slack variables for (P)} \\ &x_3 = 2 - x_2 \\ &x_4 = 3 - x_1 - x_2 \\ &x_5 = 1 - x_1 + x_2 \end{aligned}} \quad (*)$$

(D)
$$\begin{aligned} \min \quad & 2y_1 + 3y_2 + y_3 \\ \text{s.t.} \quad & y_2 + y_3 \geq 2 \\ & y_1 + y_2 - y_3 \geq 1 \\ & y_1, y_2, y_3 \geq 0 \end{aligned} \qquad \boxed{\begin{aligned} &\text{Slack var. for (D)} \\ &y_4 = y_2 + y_3 - 2 \\ &y_5 = y_1 + y_2 - y_3 - 1 \end{aligned}} \quad (**)$$

given $x_1^* = 2$; $x_2^* = 1$:

from $(*)$: $x_3^* = 1$; $x_4^* = 0$; $x_5^* = 0$. Using CS, this implies $\boxed{y_1^* = 0}$

Also: $x_1^* \neq 0 \Rightarrow \boxed{y_4^* = 0}$; $x_2^* \neq 0 \Rightarrow \boxed{y_5^* = 0}$

Then, from $(**)$:

$$\left.\begin{aligned} y_2^* + y_3^* &= 2 \\ y_2^* - y_3^* &= 1 \end{aligned}\right\} \Rightarrow y_2^* = 3/2 \ ; \ y_3^* = 1/2 \qquad (\text{and } y_1^* = 0)$$

Check if dual-feasible: (i) non-negativity ✓

(ii) plug in: $y_2^* + y_3^* = 2 \geq 2$ ✓; $y_1^* + y_2^* - y_3^* = 1 \geq 1$ ✓

So, $y^* = \begin{bmatrix} 0 \\ 3/2 \\ 1/2 \end{bmatrix}$ is dual-feasible. We conclude that the given solution is indeed optimal.

Problem 4 (10 + 2 points)

Suppose we have a standard form primal linear program

$$\text{maximize } \mathbf{c}^{\mathsf{T}}\mathbf{x} \text{ subject to } A\mathbf{x} \leq \mathbf{b}, \quad \mathbf{x} \geq 0,$$

with 5 original variables $x_1, \ldots, x_5$ and 4 inequality constraints (in addition to the non-negativity constraints $x_i \geq 0$). The primal objective is

$$\text{maximize } 2x_1 + 3x_2 + 4x_5$$

and the dual objective is

$$\text{minimize } -y_1 + y_2 - 2y_3 + y_4.$$

This is all the information we have regarding the underlying optimization problem. A possibly inaccurate linear program solver produces the following candidates for primal/dual feasible solutions.

Primal Solution Candidate $(x_1, \ldots, x_5)$	Dual Solution Candidate $(y_1, \ldots, y_4)$
$(0, 0, 0, 0, 0)$	$(-1, 0, 0, 1)$
$(1, 0, 1, 2, 0)$	$(1, 4, 0, 10)$
$(1, 0, 1, 1, 0)$	$(2, 1, 0, 1)$
$(0, \frac{2}{3}, 1, 2, \frac{1}{2})$	$(1, 5, 1, 2)$

(a) [2 pts] Specify precisely what we know about the vectors $\mathbf{b}$ and $\mathbf{c}$, and the matrix A (such as its dimension, the values of its entries etc.).

(b) [4 pts] For each pair of solution candidates, determine whether they **can** possibly be primal feasible and dual feasible, respectively, at the same time. State clearly your argument for each case and cite the relevant theorems.

(c) [1 pts] Which of these pairs **could be** primal/dual optimal? In each case, determine the corresponding optimal value of the primal objective function. Again, provide an argument.

(d) [5 pts] Now, consider the following pair of candidates:

Primal Solution Candidate $(x_1, \ldots, x_5)$	Dual Solution Candidate $(y_1, \ldots, y_4)$
$(1, 0, 1, 1, 1/2)$	$(0, 4, 0, 0)$

Find a linear program with the above primal and dual objective functions such that the given pair is primal/dual optimal for it.

(a) * 4 inequality constraints & 5 original variables

$\Rightarrow$ A is 4×5; we have no other info on A.

* $c^T = [2, 3, 0, 0, 4]$

* $b^T = [-1, 1, -2, 1]$.

(b) Pair 1: $(-1, 0, 0, 1)$ is not dual-feasible (non-neg. const.)

Pair 2: primal obj. $z = 2$ } weak duality thm is satisfied
dual obj $w = 13$

this pair can be primal/dual feasible.

Pair 3: $z = 2$; $w = 0$
weak duality thm says this cannot be a primal/dual feasible pair.

Pair 4: $z = 4$; $w = 4$ Could be ~~dual~~ primal/dual feasible

(c) Pair 4 is the only pair that can be primal/dual optimal, with $z^* = 4$.

(d) we need to ensure that given solutions ~~satisfy~~ ~~(?) they~~ are primal/dual feasible, respectively, as $z(1, 0, 1, 1, 1/2) = w(0, 4, 0, 0) = 4$, so by strong duality thm, it is sufficient to ~~co~~ show feasibility.

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{15} \\ \vdots & & & \vdots \\ a_{41} & \cdots & & a_{45} \end{bmatrix}$.

$x_1 \; x_2 \; x_3 \; x_4 \; x_5 \;\vdots\; x_6 \; x_7 \; x_8 \; x_9$
$y_5 \; y_6 \; y_7 \; y_8 \; y_9 \;\vdots\; y_1 \; y_2 \; y_3 \; y_4$

By CS, circled variables are equal to 0.

Primal feasibility:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 + a_{15}x_5 \le -1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{25}x_5 \le 1$$
$$a_{31}x_1 + a_{32}x_2 + \cdots + a_{35}x_5 \le -2$$
$$a_{41}x_1 + a_{42}x_2 + \cdots + a_{45}x_5 \le 1.$$

Note also that $x_2 = 0$ (as given) and $x_7 = 0$ (by complementary slackness) which means the second inequality is actually an equality. We get:

$$\boxed{\begin{aligned} a_{11}x_1 + a_{13}x_3 + a_{14}x_4 + a_{15}x_5 &\le -1 \\ a_{21}x_1 + a_{23}x_3 + a_{24}x_4 + a_{25}x_5 &= 1 \\ a_{31}x_1 + a_{33}x_3 + a_{34}x_4 + a_{35}x_5 &\le -2 \\ a_{41}x_1 + a_{43}x_2 + a_{44}x_4 + a_{45}x_5 &\le 1. \end{aligned}}$$

need to satisfy these! $(*)$

Dual feasibility: Similar to above, and observing $y_5 = y_7 = y_8 = y_9 = 0$, we get

$$\left.\begin{aligned} a_{11}y_1 + a_{21}y_2 + a_{31}y_3 + a_{41}y_4 &\overset{\text{since } y_5=0}{=} 2 \\ a_{12}y_1 + a_{22}y_2 + a_{32}y_3 + a_{42}y_4 &\ge 3 \\ a_{13}y_1 + a_{23}y_2 + a_{33}y_3 + a_{43}y_4 &= 0 \\ a_{14}y_1 + a_{24}y_2 + a_{34}y_3 + a_{44}y_4 &= 0 \\ a_{15}y_1 + a_{25}y_2 + a_{35}y_3 + a_{45}y_4 &= 4 \end{aligned}\right\} \Longrightarrow$$

(using $y_1 = y_3 = y_4 = 0$) $y_2 = 4$

$$a_{21}\cdot 4 = 2$$
$$a_{22}\cdot 4 \ge 3$$
$$a_{23}\cdot 4 = 0$$
$$a_{24}\cdot 4 = 0$$
$$a_{25}\cdot 4 = 4$$

So, $\boxed{a_{21} = \frac{1}{2}\ ;\ a_{22} = 1 \text{ (choice) };\ a_{23} = 0\ ;\ a_{24} = 0\ ;\ a_{25} = 1}$ (1)

Plug in to $(*)$ together with $x_1 = 1$, $x_3 = 1$, $x_4 = 1$, $x_5 = 1/2$.

$a_{11} + a_{13} + a_{14} + \frac{1}{2} a_{15} \leq -1$

$\frac{1}{2} + \frac{1}{2} = 1$ ✓

$a_{31} + a_{33} + a_{34} + \frac{1}{2} a_{35} \leq -2$

$a_{41} + a_{43} + a_{44} + \frac{1}{2} a_{45} \leq 1.$

Then, set
$$\boxed{\begin{array}{ll} a_{11} = -1 \ ; & a_{13} = a_{14} = a_{15} = 0 \\ a_{31} = -2 \ ; & a_{33} = a_{34} = a_{35} = 0 \\ a_{41} = 1 \ ; & a_{43} = a_{44} = a_{45} = 0 \end{array}}$$
2

1 & 2 specifies values of certain a_{ij}. The rest are free, so we set them to 0. We finally get:

$$\max \ \cancel{2x_1 + 3x_2 + 4x_5} \ [2, 3, 0, 0, 4]\, x$$

$$\text{s.t.} \quad \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} x \leq \begin{bmatrix} -1 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$

$$x \geq 0$$

is an LP with the given primal/dual pair as an optimal solution.

Problem 5 (10 points)

(a) [3 pts] Let (P) be a linear program in standard form. Define precisely what it means for (P) to be unbounded.

(b) [3 pts] Can (P) be unbounded if any feasible solution $x = [x_1 \ \ldots \ x_n]$ of (P) satisfies $|x_j| \leq 1000$ for all $j = 1, \ldots, n$. Justify your answer.

(c) [4pts] Suppose, this time, that the feasible set for (P) is unbounded, i.e., for any number $L > 0$, there exists a feasible solution x with at least one coordinate larger than L. Does this mean (P) is unbounded? If yes, give a proof. If no, construct a bounded (P) with an unbounded feasible set.

(a) Let (P) be given as max $c^T x$ s.t. $Ax \leq b$, $x \geq 0$.
We say that (P) is unbounded if for every $M > 0$, there exists a feasible solution x_M such that
$z(x_M) = c^T x_M > M$.

(b) No. Let $z = c^T x$ be the objective of (P). Then

$$|z| \leq |c_1| \cdot |x_1| + |c_2| \cdot |x_2| + \cdots + |c_n||x_n|$$
$$\leq 1000 \left(|c_1| + |c_2| + \cdots + |c_n| \right)$$

So, for any $M > 1000 \left(|c_1| + \cdots + |c_n| \right)$, there is no feasible solution for which $c^T x > M$.

(c) Here is a counter-example:

$$\begin{aligned} \max \ & \cancel{[illegible]} -x_1 - x_2 = z \\ \text{s.t.} \ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

The feasible set is ~~[illegible]~~ $\{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0\}$ which is clearly unbounded. On the other hand $z^* = 0$. So the problem is not unbounded.