Math 303 **Midterm Test 2** March 23 2012

Marks depend on quality of explanation. No aids except non-programmable calculators are permitted. Test is out of 40. There are five pages including this one.

Last Name: ____________________

First Name: ____________________ Student Number: ____________________

Table 1: Mean and Variances

Distribution	Mean	Variance
Bin (n, p)	np	$np(1-p)$
Geometric (p)	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Poisson (λ)	λ	λ
Uniform (a, b)	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exp (λ)	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$

1. Consider the branching process in which the probability that an individual has $i = 0, 1, 2$ offspring is p_i where $p_0 + p_1 + p_2 = 1$. Initially there is one individual.

   (a) (2 points) Give the expected number of individuals in generation n as a formula in terms of p_0, p_1, p_2. In particular, what is the expected number μ of individuals in generation 1?

   Solution: μ^n where $\mu = p_1 + 2p_2$.

   (b) (8 points) Let μ be the expected number of offspring in generation 1. Compute the probability that the branching process goes extinct, and express the answer as a formula in terms of p_2 and μ. Do case $\mu > 1$ first and consider other cases as time permits.

Solution: $G(s) = p_0 + p_1 s + p_2 s^2$. The extinction probability η is the smallest non-negative root of $G(s) = s$. The equation can be written in the form

$$p_0 + (p_1 - 1)s + p_2 s^2 = 0. \tag{1}$$

Since $p_0 + p_1 + p_2 = 1$ this equation has the solution $s = 1$ and can be reorganised in terms of $s - 1$ as

$$(p_1 - 1)(s - 1) + p_2(s - 1)^2 + 2p_2(s - 1) = 0. \tag{2}$$

Case (i) $p_2 = 0$, $p_1 \neq 1$: In this case the equation becomes $(p_1 - 1)(s - 1) = 0$ and since $p_1 \neq 0$ the only solution is $s = 1$. Therefore $\eta = 1$.
Case (ii) $p_2 = 0$, $p_1 = 1$: In this case the equation becomes $0 = 0$ which is solved by any s. The smallest non-negative solution is $s = 0$. Therefore $\eta = 0$.
Case (iii) otherwise: in equation (2) $s = 1$ is clearly a solution. The other solution is found by cancelling out $s - 1$ to get $(p_1 - 1) + p_2(s - 1) + 2p_2 = 0$. Therefore,

$$\eta = \min\left(1, 1 - \frac{2p_2 + p_1 - 1}{p_2}\right). \tag{3}$$

From the previous part $\mu = 2p_2 + p_1$. In terms of μ, p_2,

$$\eta = \begin{cases} 1 & \mu < 1, p_2 > 0, \\ 0 & \mu = 1, p_2 = 0, \\ 1 - \frac{\mu - 1}{p_2} & \mu > 1. \end{cases} \tag{4}$$

Remark: The general theory tells us that

$$\eta = \begin{cases} 1 & \mu < 1 \\ 1 & \mu = 1, \text{and } Z_1 = 1 \text{ with probability} < 1 \\ 0 & \mu = 1, \text{and } Z_1 = 1 \text{ with probability } 1. \end{cases} \tag{5}$$

The third case is when $p_2 = 0$ and $p_1 = 1$.

2. Let $(X_0, X_1, \dots)$ be an irreducible and ergodic Markov chain, which has finitely many states $\{1, 2, \dots, M\}$ and a stationary distribution $\pi = (\pi_1, \dots, \pi_M)$.

   (a) (2 points) Let $N_i(n)$ be the number of visits up to time n to state i. Write $N_i(n)$ in terms of indicator functions for events $X_t = i$.

   Solution: $N_i(n) = \sum_{t=0}^{n} I_{X_t=i}$.

   (b) (3 points) What is $\lim_{n\to\infty} \frac{N_i(n)}{n}$?

   Solution: $\frac{N_i(n)}{n} \to \pi_i$ because this is what our main theorem on Markov chains says. I stated it exactly in this form in my class notes; the book gives it as Remark (ii) under Theorem 4.1.

   (c) (5 points) For $f : \{1, 2, \dots, M\} \to \mathbb{R}$, prove that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{t=0}^{n} f(X_t) = \sum_{i=1}^{M} f(i)\pi_i. \tag{6}$$

   e.g. insert 1 written as a sum of indicator functions and use part (a).

   Solution:

$$\frac{1}{n}\sum_{t=0}^{n} f(X_t) = \frac{1}{n}\sum_{t=0}^{n} \underbrace{\sum_{i=1}^{M} I_{X_t=i}}_{\text{this equals one}} f(X_t) = \sum_{i=1}^{M} \frac{1}{n}\sum_{t=0}^{n} I_{X_t=i} f(X_t) = \sum_{i=1}^{M} \frac{1}{n}\sum_{t=0}^{n} I_{X_t=i} f(i) \tag{7}$$

$$= \sum_{i=1}^{M} \left(\frac{1}{n}\sum_{t=0}^{n} I_{X_t=i}\right) f(i) = \sum_{i=1}^{M} \frac{N_i(n)}{n} f(i) \overset{\text{part (b)}}{\to} \sum_{i=1}^{M} \pi_i f(i) \tag{8}$$

3. (a) (6 points) Give two standard definitions of a Poisson process $N(t)$ of rate r, briefly explaining terms in the definitions.

Solution: The Poisson process with rate r is a nonnegative integer valued process $N(t)$ such that

1. $N(0) = 0$
2. $N(t_2) - N(s_2)$ and $N(t_1) - N(s_1)$ are independent random variables when the intervals (s_1, t_1) and (s_2, t_2) are disjoint.
3. $\mathbb{P}(N(t+s) - N(s) = n) = \frac{(rt)^n}{n!} e^{-rt}$.

The second definition is the same except that axiom 3 becomes

$$\text{axiom 3': } \quad \mathbb{P}(N(s+h) - N(s) = n) = \begin{cases} 1 - rh - o(h) & n = 0, \\ rh + o(h) & n = 1, \\ o(h) & n > 1. \end{cases} \tag{9}$$

$o(h)$ denotes a function $f(h)$ that does not depend on s such that $f(h)/h \to 0$ as $h \to 0$.
Remarks:

1. Since probability sums to one you can leave out case $\mathbb{P}(N(s+h) - N(s) = 0) = 1 - rh - o(h)$ in axiom 3'. I put it in because most of us have some trouble getting used to the idea that in short time intervals events are rare.
2. I called axiom 2 "independent increments". In axiom 3 and 3', the right hand side does not depend on s. I called this "stationarity".

(b) (4 points) For $i = 1, 2$ let $N_i(t)$ be independent Poisson processes with rates r_i. Let $N(t) = N_1(t) + N_2(t)$. Show that $N(t)$ is a Poisson process with rate $r_1 + r_2$ by checking that $N(t)$ satisfies one of your definitions. If short of time check the part involving $N(s+h) - N(s) = 0$.

Solution: Use the second definition. Axioms 1 and 2 hold for $N(t)$ because they hold for $N_i(t)$. Check axiom 3': Using axiom 3' for N_1 and N_2 and the independence of N_1 and N_2,

$$\mathbb{P}\Big(N(s+h) - N(s) = 0\Big) = \mathbb{P}\Big(N_1(s+h) - N_1(s) = 0, N_2(s+h) - N_2(s) = 0\Big) \tag{10}$$

$$\overset{\text{independence}}{=} \mathbb{P}\Big(N_1(s+h) - N_1(s) = 0\Big)\mathbb{P}\Big(N_2(s+h) - N_2(s) = 0\Big) \tag{11}$$

$$\overset{\text{axiom 3'}}{=} \Big(1 - r_1 h - o(h)\Big)\Big(1 - r_2 h - o(h)\Big) = 1 - (r_1 + r_2)h - o(h). \tag{12}$$

The other cases $= 1$ and > 1 are checked in the same way and are easier.
Remark: A lot of people used Definition 5.1. This is good if you correctly state the theorem about the sum of two independent Poisson random variables with parameters λ_i being Poisson with parameter $\lambda_1 + \lambda_2$, but otherwise there is some red ink because I am suspicious that it is a just a leap of faith rather than profound knowledge. Quite a few people wrote

$$\mathbb{P}(N(t+s) - N(s) = n) = \mathbb{P}(N_1(t+s) - N_1(s) = n)\mathbb{P}(N_2(t+s) - N_2(s) = n) = \frac{(rt)^n}{n!} e^{-rt} \tag{13}$$

but if you want to go by this route then correct is

$$\mathbb{P}(N(t+s)-N(s)=n)=\sum_{n_1=0}^{n}\mathbb{P}(N_1(t+s)-N_1(s)=n_1)\mathbb{P}(N_2(t+s)-N_2(s)=n-n_1) \tag{14}$$

$$= \ldots \text{figure it out; involves } \binom{n}{n_1} \tag{15}$$

However, I recommend you try to appreciate how axiom 3' is used above. To teach yourself try checking that

$$\mathbb{P}\Big(N(s+h)-N(s)=1\Big)=(r_1+r_2)h+o(h). \tag{16}$$

A lot of people are writing things like

$$N(s+t)-N(s)=N(t) \tag{17}$$

This is true IN DISTRIBUTION, but the random variables are not equal, so prepare for a question on the final about what this distinction means!!

4. Light bulbs in a building are replaced as soon as they fail. The number $N(t)$ that fail up to time t (measured in days) is a Poisson process of rate r. Let S_n denote the time of the nth failure and let $T_n = S_n - S_{n-1}$. Find:

   (a) (2 points) the probability that there are no failures in a given seven days, say March 1-7.

   Solution: $\mathbb{P}\big(N(s+7) - N(s) = 0\big) = e^{-7r}$

   (b) (2 points) the probability that $T_2 > 2T_1$.

   Solution: $\mathbb{P}\left(T_2 > 2T_1\right)$ is

$$\int_0^\infty \int_0^\infty I_{t_2>2t_1} re^{-rt_2} re^{-rt_1}\, dt_2\, dt_1 = \int_0^\infty \left(\int_0^\infty I_{t_2>2t_1} re^{-rt_2}\, dt_2 \right) re^{-rt_1}\, dt_1 \tag{18}$$

$$= \int_0^\infty e^{-2rt_1} re^{-rt_1}\, dt_1 = \frac{1}{3}. \tag{19}$$

   (c) (2 points) the probability that the third failure occurs after four days.

   Solution:

$$\mathbb{P}\Big(S_3 > 4\Big) = \mathbb{P}\Big(N(4) < 3\Big) \tag{20}$$

$$= \mathbb{P}\Big(N(4) = 0\Big) + \mathbb{P}\Big(N(4) = 1\Big) + \mathbb{P}\Big(N(4) = 2\Big) = e^{-4r} + \frac{4r}{1!}e^{-4r} + \frac{(4r)^2}{2!}e^{-4r}. \tag{21}$$

   (d) (2 points) the probability that the third failure occurs after six days given that exactly one fails in the first two days.

   Solution:

$$\mathbb{P}\Big(S_3 > 6 \Big| N(2) = 1\Big) = \mathbb{P}\Big(S_2 > 4\Big) = \mathbb{P}\Big(N(4) < 2\Big) \tag{22}$$

$$= \mathbb{P}\Big(N(4) = 0\Big) + \mathbb{P}\Big(N(4) = 1\Big) + = e^{-4r} + \frac{4r}{1!}e^{-4r}. \tag{23}$$

   You may find the following lengthy version easier to understand

$$\mathbb{P}\Big(S_3 > 6 \Big| N(2) = 1\Big) = \mathbb{P}\Big(N(6) = 0,1,2 \Big| N(2) = 1\Big) = \frac{\mathbb{P}\Big(N(6) = 0,1,2 \text{ and } N(2) = 1\Big)}{\mathbb{P}\Big(N(2) = 1\Big)} \tag{24}$$

$$= \frac{\mathbb{P}\Big(N(6) - N(2) = 0,1 \text{ and } N(2) = 1\Big)}{\mathbb{P}\Big(N(2) = 1\Big)} \tag{25}$$

$$\overset{\text{independent increments}}{=} \frac{\mathbb{P}\Big(N(6) - N(2) = 0,1\Big)\mathbb{P}\Big(N(2) = 1\Big)}{\mathbb{P}\Big(N(2) = 1\Big)} \tag{26}$$

$$\overset{\text{stationary}}{=} \mathbb{P}\Big(N(6-2) = 0,1\Big) = \mathbb{P}\Big(N(4) = 0,1\Big) = e^{-4r} + \frac{4r}{1!}e^{-4r}. \tag{27}$$

(e) (2 points) the probability that none fail from March 1-7 if 5 fail in the period March 1 – 28.

Solution: The five failures are uniformly and independently distributed over the 28 days so the answer is $\left(\frac{3}{4}\right)^5$. Some of you cleverly came to the same conclusion by proving that the five failures are uniformly and independently distributed as follows:

$$\mathbb{P}\Big(N(7) = 0 \Big| N(28) = 5\Big) = \frac{\mathbb{P}\big(N(7) = 0, N(28) = 5\big)}{\mathbb{P}\big(N(28) = 5\big)} = \frac{\mathbb{P}\big(N(7) = 0, N(28) - N(7) = 5\big)}{\mathbb{P}\big(N(28) = 5\big)} \tag{28}$$

$$= \frac{\mathbb{P}\big(N(7) = 0, N(28) - N(7) = 5\big)}{\mathbb{P}\big(N(28) = 5\big)} \tag{29}$$

$$\overset{\text{independent increments}}{=} \frac{\mathbb{P}\big(N(7) = 0\big)\mathbb{P}\big(N(28) - N(7) = 5\big)}{\mathbb{P}\big(N(28) = 5\big)} \tag{30}$$

$$\overset{\text{stationary}}{=} \frac{\mathbb{P}\big(N(7) = 0\big)\mathbb{P}\big(N(21) = 5\big)}{\mathbb{P}\big(N(28) = 5\big)} = \frac{\big(e^{-7r}\big)\big((21r)^5/5!e^{-21r}\big)}{(28r)^5/5!e^{-28r}} = \frac{3^5}{4^5} \tag{31}$$