CONCORDIA UNIVERSITY

DEPARTMENT OF COMPUTER SCIENCE AND SOFTWARE ENGINEERING

COMP232 MATHEMATICS FOR COMPUTER SCIENCE

ASSIGNMENT 2 SOLUTIONS FALL 2014

1. In Problem 1(c) of Assignment 1 it was verified by truth table that

$$(p \land (\neg q \to \neg p)) \ \Rightarrow q \qquad (\star)$$

i.e., that $(p \land (\neg q \to \neg p)) \ \to q$ is a tautology.

Let LHS denote the left-hand side $(p \land (\neg q \to \neg p))$, and RHS the right-hand side q. State which of the following four approaches constitute a correct approach to prove $(\star)$, and which ones constitute a wrong approach:

(a) LHS $\Rightarrow$ RHS (b) $\neg$ LHS $\Rightarrow$ $\neg$ RHS (c) RHS $\Rightarrow$ LHS (d) $\neg$ RHS $\Rightarrow$ $\neg$ LHS

SOLUTION: The only correct approaches are (a) and (d).

Give a complete proof of one of the correct approaches to prove $(\star)$. prove $(\star)$.

PROOF:

Approach (a) is possibly the easiest approach here: Assume the LHS is True. Then $p = T$ and $(\neg q \to \neg p) = T$. Now $(\neg q \to \neg p) \equiv (p \to q)$, and since $p = T$ it follows that $q = T$, *i.e.*, the RHS is True. QED

2. In Problem 1(a) of Assignment 1 it was verified by truth table that

$$((p \lor r) \land (q \lor r)) \equiv (p \land q) \lor r \qquad (\star\star)$$

i.e., $((p \lor r) \land (q \lor r)) \leftrightarrow (p \land q) \lor r$ is a tautology.

Let LHS denote the left-hand side $(p \lor r) \land (q \lor r)$, and RHS the right-hand side $(p \land q) \lor r$. State which of the following four approaches constitute a correct approach to prove $(\star\star)$, and which ones constitute a wrong approach:

(a) LHS $\Rightarrow$ RHS and RHS $\Rightarrow$ LHS (b) $\neg$ LHS $\Rightarrow$ $\neg$ RHS and $\neg$ RHS $\Rightarrow$ $\neg$ LHS

(c) LHS $\Rightarrow$ RHS and $\neg$ RHS $\Rightarrow$ $\neg$ LHS (d) RHS $\Rightarrow$ LHS and $\neg$ LHS $\Rightarrow$ $\neg$ RHS

SOLUTION: The only correct approaches are (a) and (b).

Give a complete proof of one of the correct approaches.

SOLUTION: Approach (a) is possibly the easiest approach here. This proof consists of two parts:

(i) Assume the LHS $(p \vee r) \wedge (q \vee r)$ is True. Then both $p \vee r$ and $q \vee r$ are True. This gives two cases to be considered: (i) r is True or (ii) both p and q are True. In either case the RHS $(p \wedge q) \vee r$ is True.

(ii) Assume the RHS $(p \wedge q) \vee r$ is True. This leads to two cases: both p and q are True or r is True. In either case the LHS $(p \vee r) \wedge (q \vee r)$ is True.

3. (a) Give a proof by *cases* to show that the equation

$$5x^2 \; + \; 4y^3 \; = \; 51$$

does not have solutions $x, y \in \mathbb{Z}^+$.

SOLUTION: It is clear that we only need to consider $x \leq 3$ and $y \leq 2$. This leads to six cases: $(x, y) = (1,1),\ (1,2),\ (2,1),\ (2,2),\ (3,1),\ (3,2)$. The cases $(x, y) = (2,1),\ (2,2)$ can be excluded immediately since they would give an even sum. Thus we need only consider $(x, y) = (1,1),\ (1,2),\ (3,1),\ (3,2)$. However, these give the sums 9, 37, 49, 77, respectively, that is, not 51.

(b) For $n \in \mathbb{Z}^+$ prove by contrapositive that if $2n^3 + 3n^2 + 4n + 5$ is odd then n is even.

SOLUTION: The contrapositive is: if n is odd then $2n^3 + 3n^2 + 4n + 5$ is even. So let n be odd. Then $2n^3$ and $4n$ are even, while both $3n^2$ and 5 are odd, so the overall sum is even.

4. For each of the statements below state whether it is *True* or *False*. If *True* then give a proof. If *False* then give a counterexample.

(a) The product of any two odd integers is odd.

SOLUTION: True: The two odd integers can be written as $n_1 = 2k_1 + 1$ and $n_2 = 2k_2 + 1$, so that their product equals $(2k_1 + 1)(2k_2 + 1) = 2(2k_1k_2 + k_1 + k_2) + 1$, which is odd.

(b) The sum of any even and any odd integer is odd.

SOLUTION: True: The even integer can be written as $n_1 = 2k_1$ and the odd integer as $n_2 = 2k_2 + 1$. Then $n_1 + n_2 = 2(k_1 + k_2) + 1$, which is odd.

(c) The difference of any two odd integers is odd.

False: Counterexample: Let $n_1 = 7$ and $n_2 = 3$. Then both n_1 and n_2 are odd, while their difference $n_1 - n_2 = 4$ is even.

(d) Let a and b be integers. If $a + b$ is even, then either a or b is even.

SOLUTION: False: Counterexample: Let $a = b = 1$. Then $a + b$ is even, but both a and b are odd.

(e) For all integers a, if $a > 2$ then $a^2 - 4$ is composite.

SOLUTION: False: Counterexample: Let $a = 3$. Then $a^2 - 4 = 5$, which is prime.

5. For each of the statements below state whether it is True or False. If True then give a proof. If False then explain why, *e.g.*, by giving a counterexample.

   (a) For all positive $x, y \in \mathbb{R}$, if x is irrational and y is irrational then $x + y$ is irrational.

   SOLUTION: False: Counterexample: Let $x_1 = \frac{7}{4} - \sqrt{2}$ and $x_2 = \frac{7}{4} + \sqrt{2}$. Then both x_1 and x_2 are positive and irrational, while their sum is $\frac{7}{2}$, which is rational.

   For completeness we must also demonstrate that x_1 and x_2 are indeed irrational. This is most easily done by contradiction: Suppose $x_1 = \frac{7}{4} - \sqrt{2}$ is rational. Then $x_1 = \frac{7}{4} - \sqrt{2} = \frac{p}{q}$, for positive integers p and q. It follows that $\sqrt{2} = \frac{7}{4} - \frac{p}{q} = \frac{7q-4p}{4q}$, which is rational. But $\sqrt{2}$ is known to be irrational, and thus we have a contradiction. The proof that x_2 is irrational is very similar to that for x_1. (The fact that $\sqrt{2}$ is irrational can be demonstrated by a proof similar to the one in Problem 5(e) below.)

   (b) $\forall x, y \in \mathbb{R}$, if x is irrational and y is rational then $x - y$ is irrational.

   SOLUTION: True: We can most easily prove this by contradiction: Suppose x is irrational, y is rational, but $x - y$ is rational. Then $y = \frac{p}{q}$ and $x - y = \frac{r}{s}$, for integers p, q, r, s with $q \neq 0$ and $s \neq 0$. Thus $x = (x-y)+y = \frac{r}{s} + \frac{p}{q} = \frac{rq+ps}{sq}$, which is rational, so that we have a contradiction.

   (c) The sum of the squares of any two rational numbers is a rational number.

   SOLUTION: True: Suppose x and y are rational numbers :

$$x = \frac{p_1}{q_1} \qquad \text{and} \qquad y = \frac{p_2}{q_2} .$$

   where p_1, q_1 and p_2, q_2 are integers, with q_1 and q_2 nonzero. Then

$$x^2 + y^2 = \frac{p_1^2}{q_1^2} + \frac{p_2^2}{q_2^2} = \frac{p_1^2 q_2^2 + p_2^2 q_1^2}{q_1^2 q_2^2} .$$

   which is rational.

   (d) For all positive $x \in \mathbb{R}$, if x is irrational then $\sqrt{x}$ is irrational.

   SOLUTION: The contrapositive is:

   For all positive $x \in \mathbb{R}$, *if* $\sqrt{x}$ *is rational then* x *is rational* .

   So suppose $\sqrt{x}$ is rational : $\sqrt{x} = \frac{p}{q}$, for certain $p, q \in \mathbb{Z}$, with $q \neq 0$.

   Then $x = \frac{p^2}{q^2}$, which is rational.

   (e) $\sqrt{3}$ is irrational.

   SOLUTION: True: We prove this by contradiction: Suppose that $\sqrt{3}$ is rational. We may also assume that $\sqrt{3}$ is positive, that is, we consider only the positive root.

   Then $\sqrt{3} = \frac{p}{q}$, for positive integers p and q. By factoring out common factors we may assume that the only common divisor of p and q is 1.

Now from $\sqrt{3} = \frac{p}{q}$ it follows that $p^2 = 3q^2$. Thus $3|p^2$. It is easily seen that therefore $3|p$ (see below). Thus $p = 3k$ for some positive integer k. Hence $p^2 = (3k)^2 = 3q^2$, from which it follows that $3k^2 = q^2$. Thus also $3|q^2$. Again this implies that $3|q$. Therefore we have found that both p and q are divisible by 3, which contradicts the fact that their only common divisor is 1.

For completeness we prove a fact that was used two times in the above proof, namely that $3|p^2$ implies $3|p$. This is most easily done by proving the contrapositive: If p is not divisible by 3 then p^2 is not divisible by 3. So assume p is not divisible by 3. Then p can be written as $p = 3k + 1$ or $p = 3k + 2$. In the first case this gives $p^2 = (3k + 1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$, which is not divisible by 3. In the second case we have $p^2 = (3k + 2)^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$, which is not divisible by 3 either.

(f) $\log_5(2)$ is irrational.

SOLUTION: True: This is most easily proved by contradiction: Suppose $\log_5(2)$ is rational. (We know that $\log_5(2)$ is positive.) Then $\log_5(2) = \frac{p}{q}$ for positive integers p and q. By definition of the logarithm function this means that $2 = 5^{p/q}$, or equivalently, $2^q = 5^p$. Now 2^q is even, while 5^p is odd, so they cannot be equal. Hence we have a contradiction. (That 2^q is even is clear, since $2|2^q$. Also is is clear that 5^p is odd, for if it were even then it would be divisible by 2, which is clearly not the case.)

6. Give a proof of each of the following:

(a) If the integers $1, 2, 3, \cdots, 7,$ are placed around a circle, in any order, then there exist two adjacent integers that have a sum greater than or equal to 9 .

SOLUTION: Using a direct proof: The number 7 must be placed somewhere. The smallest neighbors 7 can possibly have are 1 and 2. Thus the sum of 7 and the biggest of its neighbors is at least 9 .

(b) If the integers $1, 2, 3, \cdots, 16$ are placed around a circle, in any order, then there exist three integers in consecutive locations around the circle that have a sum greater than or equal to 27.

SOLUTION: By contradiction: Suppose there is an order to place the integers, such that *all* 16 groups of three consecutive integers have a sum less than 27, that is, less than or equal to 26. Excluding the number 1, which must be placed somewhere, say in location 1, let us look at the 5 groups of 3 consecutive integers starting at locations 2, 5, 8, 11, and 14, respectively. Since all 16 possible groups must have a sum less than or equal to 26, it follows that the sum of each of the five groups mentioned above is also at most 26. The total sum of all integers around the circle is then less than or equal to $1 + 5 \times 26 = 131$. However, we know that this sum must equal $1 + 2 + 3 + \cdots + 16 = 16 \times 17/2 = 136$, so that we have a contradiction.

7. For each of the following, determine whether it is valid or invalid. If valid then give a proof. If invalid then give a counter example.

   (a) $(A \cap B) \cup (C \cap D) \;=\; (A \cap D) \cup (C \cap B)$

   SOLUTION: INVALID: Let $A = \{1\}$, $B = \{2\}$, $C = \{2\}$, and $D = \{1\}$. Then $(A \cap B) \cup (C \cap D)$ is empty, while $(A \cap D) \cup (C \cap B) = \{1, 2\}$, so that these two sets are unequal for this choice of the sets A, B, C and D.

   (b) $A - (B \cup C) = (A - B) \cap (A - C)$

   SOLUTION: VALID:

$$\begin{aligned} x \in A - (B \cup C) &\iff x \in A \;\wedge\; x \notin (B \cup C) \\ &\iff x \in A \;\wedge\; \neg(x \in B \vee x \in C) \\ &\iff x \in A \;\wedge\; (x \notin B \wedge x \notin C) \\ &\iff x \in A \;\wedge\; x \notin B \;\wedge\; x \in A \;\wedge\; x \notin C \\ &\iff x \in (A - B) \;\wedge\; x \in (A - C) \\ &\iff x \in (A - B) \cap (A - C). \end{aligned}$$

   (c) $B \cap C \subseteq A \;\Rightarrow\; (C - A) \cap (B - A)$ is empty

   SOLUTION: VALID: We can prove this by contradiction: Suppose $B \cap C \subseteq A$, but $(C - A) \cap (B - A)$ is not empty. Then there exists an element $x \in (C - A) \cap (B - A)$. Thus this x satisfies $x \in C$, $x \in B$ and $x \notin A$. But this contradicts the assumption that $B \cap C \subseteq A$.

   (d) $(A \cup B) - (A \cap B) = A \;\Rightarrow\; B$ is empty

   SOLUTION: VALID: We prove this by contradiction: Suppose $(A \cup B) - (A \cap B) = A$, but B is not empty. Then there exists an element $x \in B$. There are two possible cases:

      i. x is also an element of the set A. Then $x \in A \cup B$ and $x \in A \cap B$. Hence $x \notin (A \cup B) - (A \cap B)$. Since $(A \cup B) - (A \cap B) = A$ it follows that $x \notin A$ which is a contradiction.

      ii. x is not an element of the set A. Then $x \in A \cup B$ and $x \notin A \cap B$. Hence $x \in (A \cup B) - (A \cap B)$. Since $(A \cup B) - (A \cap B) = A$ it follows that $x \in A$ which is a contradiction.