1. (10 marks) Find and classify all critical points of the function

$$f(x, y) = x^3 - 3x + y^3 - 3y$$

$f_x = 3x^2 - 3$ $\qquad$ $f_y = 3y^2 - 3$

Must solve $\left.\begin{matrix} 3x^2 - 3 = 0 \\ 3y^2 - 3 = 0 \end{matrix}\right| \Rightarrow \begin{matrix} x^2 = 1 \\ y^2 = 1 \end{matrix}$

$\Rightarrow$ 4 critical points $(1,1), (1,-1), (-1,1), (-1,-1)$

$f_{xx} = 6x \quad f_{xy} = f_{yx} = 0, \quad f_{yy} = 6y$

$\underline{(1,1)}$: $f_{xx} f_{yy} - (f_{xy})^2 = 36xy = \underline{36 > 0}$ and $f_{xx} = 6 \cdot 1 = 6 > 0$ $\Big| \to$ local min.

$(1,-1)$: $f_{xx} f_{yy} - (f_{xy})^2 = 36xy = -36 < 0 \longrightarrow$ saddle

$(-1,1)$: $f_{xx} f_{yy} - (f_{xy})^2 = 36xy = -36 < 0 \longrightarrow$ saddle

$(-1,-1)$: $f_{xx} f_{yy} - (f_{xy})^2 = 36xy = 36 > 0$ and $f_{xx} = 6 \cdot (-1) = -6 < 0$ $\Big| \to$ local max.

2. (10 marks) Let

$$z = f(x, y) = e^{3x+y^2} \sin xy$$

If $x = x(u, v) = 2v - 3u$ and $y = y(u, v) = v^2 + uv$, compute both $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$ **using the Chain Rule**.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$\frac{\partial z}{\partial x} = 3e^{3x+y^2}\sin xy + ye^{3x+y^2}\cos xy$$

$$\frac{\partial z}{\partial y} = 2ye^{3x+y^2}\sin xy + xe^{3x+y^2}\cos xy$$

$$\frac{\partial x}{\partial u} = -3 \qquad \frac{\partial x}{\partial v} = 2 \qquad \frac{\partial y}{\partial u} = v \qquad \frac{\partial y}{\partial v} = 2v+u$$

$$\boxed{\begin{aligned}\frac{\partial z}{\partial u} = -3\Big(&3e^{3(2v-3u)+(v^2+uv)^2}\sin((2v-3u)(v^2+uv)) + \\ &(v^2+uv)e^{3(2v-3u)+(v^2+uv)^2}\cos((2v-3u)(v^2+uv))\Big) \\ + v\Big(&2(v^2+uv)e^{3(2v-3u)+(v^2+uv)^2}\sin((2v-3u)(v^2+uv)) + \\ &(2v-3u)e^{3(2v-3u)+(v^2+uv)^2}\cos((2v-3u)(v^2+uv))\Big)\end{aligned}}$$

$$\boxed{\begin{aligned}\frac{\partial z}{\partial v} = 2\Big(&3e^{3(2v-3u)+(v^2+uv)^2}\sin((2v-3u)(v^2+uv)) + \\ &(v^2+uv)e^{3(2v-3u)+(v^2+uv)^2}\cos((2v-3u)(v^2+uv))\Big) \\ + (2v+u)\Big(&2(v^2+uv)e^{3(2v-3u)+(v^2+uv)^2}\sin((2v-3u)(v^2+uv)) + \\ &(2v-3u)e^{3(2v-3u)+(v^2+uv)^2}\cos((2v-3u)(v^2+uv))\Big)\end{aligned}}$$

2. (10 marks) Let

$$z = f(x, y) = e^{3x+y} \sin xy$$

If $x = x(u, v) = 2v - 3u$ and $y = y(u, v) = v + uv$, compute both $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ **using the Chain Rule**. Your final answers should be functions of u and v alone.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$\frac{\partial z}{\partial x} = 3e^{3x+y}\sin xy + y e^{3x+y}\cos xy$$

$$\frac{\partial z}{\partial y} = e^{3x+y}\sin xy + x e^{3x+y}\cos xy$$

$$\frac{\partial x}{\partial u} = -3 \quad \frac{\partial x}{\partial v} = 2 \quad \frac{\partial y}{\partial u} = v \quad \frac{\partial y}{\partial v} = 1+u$$

$$\boxed{\begin{aligned}\frac{\partial z}{\partial u} = -3\Big(&3e^{3(2v-3u)+v+uv}\sin((2v-3u)(v+uv)) + \\ &(v+uv)e^{3(2v-3u)+v+uv}\cos((2v-3u)(v+uv))\Big) \\ + v\Big(&e^{3(2v-3u)+v+uv}\sin((2v-3u)(v+uv)) + \\ &(2v-3u)e^{3(2v-3u)+v+uv}\cos((2v-3u)(v+uv))\Big)\end{aligned}}$$

$$\boxed{\begin{aligned}\frac{\partial z}{\partial v} = 2\Big(&3e^{3(2v-3u)+v+uv}\sin((2v-3u)(v+uv)) + \\ &(v+uv)e^{3(2v-3u)+v+uv}\cos((2v-3u)(v+uv))\Big) \\ + (1+u)\Big(&e^{3(2v-3u)+v+uv}\sin((2v-3u)(v+uv)) + \\ &(2v-3u)e^{3(2v-3u)+v+uv}\cos((2v-3u)(v+uv))\Big)\end{aligned}}$$

3. (10 marks) Find the global maximum and the global minimum of the function

$$f(x,y) = x^2 + 2y^2$$

on the set

$$x^2 + y^2 \leq 4.$$

NOTE: For part of the solution, you are required to use the method of Lagrange multipliers.

$f_x = 2x \qquad f_y = 4y \Rightarrow$ only critical point of f is $(0,0)$ WHICH IS IN THE DOMAIN

$\boxed{f(0,0) = 0}$

On the boundary of the set, i.e. $\underbrace{x^2+y^2}_{g(x,y)} = 4$, we use Lagrange multipliers. ($g(x,y)$: Constraint function)

$$\nabla f = \lambda \nabla g \Rightarrow \left.\begin{array}{l} 2x = \lambda \cdot 2x \\ 4y = \lambda \cdot 2y \\ x^2+y^2 = 4 \end{array}\right| \text{ Need to solve}$$

$2x(1-\lambda) = 0 \Rightarrow x=0$ or $\lambda = 1$

$x=0$: $\rightarrow 0^2+y^2=4 \rightarrow y = \pm 2 \longrightarrow$ $y=2 \rightarrow 4\cdot 2 = \lambda \cdot 2 \cdot 2 \rightarrow \lambda = 2$; $y=-2 \rightarrow 4\cdot(-2) = \lambda\cdot 2\cdot(-2) \rightarrow \lambda = 2$

$(x,y,\lambda) = (0,2,2)$ and $(0,-2,2)$

$\boxed{f(0,2) = 0 + 2\cdot 2^2 = 8}$

$\boxed{f(0,-2) = 0 + 2(-2)^2 = 8}$

$\lambda = 1$: $\rightarrow 4y = 1\cdot 2y \Rightarrow y = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2 \qquad (x,y,\lambda) = (2,0,1)$ and $(-2,0,1)$

$\boxed{f(2,0) = 2^2 = 4}$, $\boxed{f(-2,0) = (-2)^2 = 4}$

Global Max: $f = 8$ at $(0,2)$ and $(0,-2)$ Global min: $f = 0$ at $(0,0)$

4. (10 marks) Compute the following double integral

$$\iint_R (x^4 + 3x^2y^2)\, dA$$

where R is the rectangle $0 \leq x \leq 1$, $0 \leq y \leq 2$.

$$= \int_{x=0}^{1} \int_{y=0}^{2} (x^4 + 3x^2y^2)\, dy\, dx = \int_0^1 \left(x^4 y + x^2 y^3\right)\Big|_{y=0}^{2}\, dx$$

$$= \int_0^1 (2x^4 + 8x^2)\, dx = \frac{2x^5}{5} + \frac{8x^3}{3}\Big|_0^1 = \frac{2}{5} + \frac{8}{3} = \frac{46}{15}$$

OR

$$= \int_{y=0}^{2} \int_{x=0}^{1} (x^4 + 3x^2y^2)\, dx\, dy = \int_0^2 \left(\frac{x^5}{5} + x^3y^2\right)\Big|_0^1\, dy$$

$$= \int_0^2 \left(\frac{1}{5} + y^2\right) dy = \frac{1}{5}y + \frac{y^3}{3}\Big|_0^2 = \frac{2}{5} + \frac{8}{3} = \frac{46}{15}$$

1. (10 marks) Find and classify all critical points of the function

$$f(x,y) = x^3 - 6x + y^3 - 6y$$

1 $\boxed{f_x = 3x^2 - 6}$ $\boxed{f_y = 3y^2 - 6}$ 1

Must solve $\left.\begin{array}{l} 3x^2 - 6 = 0 \\ 3y^2 - 6 = 0 \end{array}\right| \to \begin{array}{l} x^2 = 2 \\ y^2 = 2 \end{array}$ 0.5

$\Longrightarrow$ 4 critical points $(\sqrt{2}, \sqrt{2})$, $(\sqrt{2}, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$, $(-\sqrt{2}, -\sqrt{2})$ 0.5 0.5 0.5 0.5

$f_{xx} = 6x$ 0.5 $f_{xy} = f_{yx} = 0$, 0.5 $f_{yy} = 6y$ 0.5

$(\sqrt{2}, \sqrt{2})$: $f_{xx} f_{yy} - (f_{xy})^2 = 36xy = 72 > 0$ and $f_{xx} = 6 \cdot \sqrt{2} > 0$ 0.5 $\longmapsto$ local min 0.5

$(\sqrt{2}, -\sqrt{2})$: $f_{xx} f_{yy} - (f_{xy})^2 = 36xy = -72 < 0$ 0.5 $\longrightarrow$ saddle 0.5

$(-\sqrt{2}, \sqrt{2})$: $f_{xx} f_{yy} - (f_{xy})^2 = 36xy = -72 < 0$ 0.5 $\longrightarrow$ saddle 0.5

$(-\sqrt{2}, -\sqrt{2})$: $f_{xx} f_{yy} - (f_{xy})^2 = 36xy = 72 > 0$ 0.5 and $f_{xx} = -6 \cdot \sqrt{2} < 0$ $\longmapsto$ local max. 0.5

For each pt. they forget, they lose all marks for that particular pt. (i.e. 1.5)

−5 If they do NOT use chain rule (i.e. first substitute, then diff.)

2. (10 marks) Let

$$z = f(x,y) = e^{x+3y}\sin xy$$

If $x = x(u,v) = 4u - 3v$ and $y = y(u,v) = u + uv$, compute both $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$ **using the Chain Rule**. Your final answers should be functions of u and v alone.

2 $\dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial u}$

2 $\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial v}$

$\dfrac{\partial z}{\partial x} = e^{x+3y}\sin xy + y e^{x+3y}\cos xy$ 1

$\dfrac{\partial z}{\partial y} = 3e^{x+3y}\sin xy + x e^{x+3y}\cos xy$ 1

$\dfrac{\partial x}{\partial u} = 4$ 0.5 $\quad \dfrac{\partial x}{\partial v} = -3$ 0.5 $\quad \dfrac{\partial y}{\partial u} = 1+v$ 0.5 $\quad \dfrac{\partial y}{\partial v} = u$ 0.5

$$\frac{\partial z}{\partial u} = 4\Big(e^{4u-3v+3(u+uv)}\sin((4u-3v)(u+uv)) + (u+uv)e^{4u-3v+3(u+uv)}\cos((4u-3v)(u+uv))\Big)$$

1

$$+ (1+v)\Big(3e^{4u-3v+3(u+uv)}\sin((4u-3v)(u+uv)) + (4u-3v)e^{4u-3v+3(u+uv)}\cos((4u-3v)(u+uv))\Big)$$

$$\frac{\partial z}{\partial v} = -3\Big(e^{4u-3v+3(u+uv)}\sin((4u-3v)(u+uv)) + (u+uv)e^{4u-3v+3(u+uv)}\cos((4u-3v)(u+uv))\Big)$$

1

$$+u\Big(3e^{4u-3v+3(u+uv)}\sin((4u-3v)(u+uv)) + (4u-3v)e^{4u-3v+3(u+uv)}\cos((4u-3v)(u+uv))\Big)$$

for each
−0.5 if they leave x's and y's

If parametrize bound.. −2

3. (10 marks) Find the global maximum and the global minimum of the function

$$f(x, y) = x^2 + 3y^2$$

on the set

$$x^2 + y^2 \leq 4.$$

NOTE: For part of the solution, you are required to use the method of Lagrange multipliers.

$f_x = 2x$ 0.5 $f_y = 6y$ 0.5 $\Rightarrow$ only critical point of f is $(0,0)$ WHICH IS IN THE DOMAIN 0.5

$\boxed{f(0,0)=0}$ 0.5

On the boundary of the set, i.e. $\underbrace{x^2+y^2}_{g(x,y)\text{ constraint function}}=4$, we use Lagrange multipliers

$\nabla f = \lambda \nabla g \Rightarrow$ 3 $\boxed{\begin{array}{l} 2x = \lambda \cdot 2x \\ 6y = \lambda \cdot 2y \\ x^2+y^2=4 \end{array}}$ Need to solve

$2x(1-\lambda)=0 \Rightarrow x=0$ or $\lambda=1$

$x=0$: $0^2+y^2=4 \rightarrow y=\pm 2 \rightarrow$ $y=2 \rightarrow 6\cdot 2 = \lambda\cdot 2\cdot 2 \rightarrow \lambda=3$
$y=-2 \rightarrow 6\cdot(-2) = \lambda\cdot 2\cdot(-2) \rightarrow \lambda=3$

$(x,y,\lambda)=(0,2,3)$ and $(0,-2,3)$ 1.5

$\boxed{f(0,2) = 0 + 3\cdot 2^2 = 12}$

$\boxed{f(0,-2) = 0 + 3(-2)^2 = 12}$

$\lambda=1$: $\rightarrow 6y = 1\cdot 2y \Rightarrow y=0 \Rightarrow x^2=4 \Rightarrow x=\pm 2$ $(x,y,\lambda)=(2,0,1)$ and $(-2,0,1)$

$\boxed{f(2,0)=2^2=4}$, $\boxed{f(-2,0)=(-2)^2=4}$ 1.5

Global max: $f=12$ at $(0,2)$ and $(0,-2)$ 1

Global min: $f=0$ at $(0,0)$. 1

4. (10 marks) Compute the following double integral

$$\iint_R (x^5 + 3x^2y^2)\, dA$$

where R is the rectangle $0 \le x \le 1$, $0 \le y \le 2$.

$$= \int_{x=0}^{1} \int_{y=0}^{2} (x^5 + 3x^2y^2)\, dy\, dx = \int_0^1 \left(x^5 y + x^2 y^3\right)\Big|_{y=0}^{2} dx$$

← bounds in same order 2

2

$$= \int_0^1 (2x^5 + 8x^2)\, dx = \frac{2x^6}{6} + \frac{8x^3}{3}\Big|_0^1 = \frac{2}{6} + \frac{8}{3} = 3$$

2 2 2

OR

$$= \int_{y=0}^{2} \int_{x=0}^{1} (x^5 + 3x^2y^2)\, dx\, dy = \int_0^2 \left(\frac{x^6}{6} + x^3y^2\Big|_0^1\right) dy =$$

$$\int_0^2 \left(\frac{1}{6} + y^2\right) dy = \frac{y}{6} + \frac{y^3}{3}\Big|_0^2 = \frac{2}{6} + \frac{8}{3} = 3$$