Assignment 1, due on September 19th, 2014

- *Each student should submit their assignments before the beginning of class on the announced due date. Late assignments will not be accepted.*
- *Solutions must be written up carefully, showing all work, for full credit.*
- *No points will be awarded for a numerical answer that is not justified by a demonstration of the steps used.*

One of the objectives of the assignments is to evaluate the comprehension of the calculation steps of the calculator's or computer's programs.

1. **Equation of value.** The objective is to have the following equality :

   $$\text{All payments at time } t = \text{ All debts at time } t$$

   Jim loans 8500$ to Amanda immediately, 5000$ in 3 years and 1500$ in 5 years. Amanda reimburses Jim with two equal payments of Y\$; one payment of Y at the end of the fifth year and one payment of Y at the end of the sixth year, assuming a nominal interest rate of 6%, compounded monthly.

   (a) Make a time diagram of the cash flows.

   (b) Develop the equation of value of this situation at the end of the fifth year.

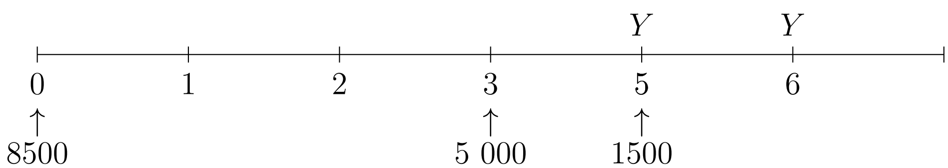
   (c) Evaluate Y.

   Solution : Equation of value. The objective is to have the following equality :

   $$\text{All payments at time } t = \text{ All debts at time } t$$

   Jim loans 8500$ to Amanda immediately, 5000$ in 3 years and 1500$ in 5 years. Amanda reimburses Jim with two equal payments of Y\$; one payment of Y at the end of the fifth year and one payment of Y at the end of the sixth year, assuming a monthly interest rate of 6%, compounded monthly.

   (a) Make a time diagram of the cash flows.



   (b) Develop the equation of value of this situation at the end of the fifth year.

$$Y + Y\left(1+\frac{0.06}{12}\right)^{-12} = 8500\left(1+\frac{0.06}{12}\right)^{5(12)} + 5000\left(1+\frac{0.06}{12}\right)^{2(12)} + 1500$$

(c) Evaluate Y.

$$Y(1+0.9419) = (11465.23 + 5635.80 + 1500) \Rightarrow Y = \frac{18601.03}{1.9419} = 9578.78\$.$$

2. Anna borrows \$60,000 to the bank at a nominal interest rate $i^{(12)} = 6\%$, (capitalized monthly). She reimburses this loan by monthly payments (at the end of each month) during 6 years. If each of the 36 first payments are of \$$R$ and each one of the last 36 payments are of \$$(R+500)$. Using annuities, determine R, and the interest that Anna paid to the bank.

Solution :

$$\begin{aligned}
60000 &= Ra_{\overline{36}|0.005} + (R+500)a_{\overline{36}|0.005} \times (1+\frac{0.06}{12})^{-3(12)} && (1)\\
&= R(32.8710) + (R+500)(32.8710) \times .8356449 && (2)\\
&= R(32.8710 + 27.4684977) + 13734.24 && (3)\\
R &= \frac{60000 - 13734.24}{60.33951} = 766.75. && (4)
\end{aligned}$$

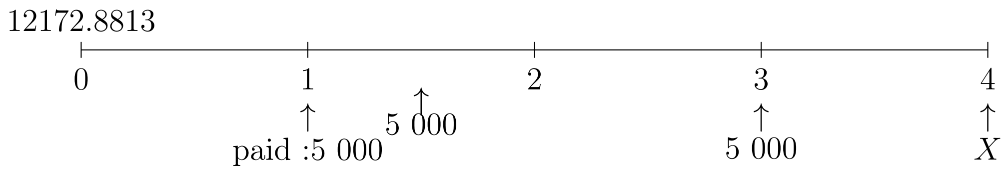
Anna will pay : 36 payments of 766.75 and 36 payments of 1266.75, for a total of 732066\$, hence $73206 - 60000 = 13206\$$ of interest.

3. Billy contracted a loan at the bank. He pays back his loan in 3 payments : the first one was \$5000 made after one year, the second one of \$5000 made one year and a half after he contracted the loan, and the last one, of \$5000, made three years after he contracted the loan. The interest rate is $i^{(2)} = 12\%$, compounded every six months.
   - a) Determine the amount of the loan
   - b) If after his first payment of \$5000, Billy renegociates his loan with the bank such that he only has one payment left of \$$X$ to do, four years after he contracted the loan, with the same interest rate, what should be the value of X ?
   - c) Consider the situation in b). If Billy decides to accumulate \$$X$ in a separate bank account, by doing 42 monthly payments of \$$P$, with a nominal interest rate of $i^{(12)} = 6\%$, compounded each month. The first payment will be one month after his payment of \$5000. Determine P, using annuities.

   Solution :
   - a) Determine the amount of the loan.

   $$PV = 5000\left(1+\frac{0.12}{2}\right)^{-2} + 5000\left(1+\frac{0.12}{2}\right)^{-3} + 5000\left(1+\frac{0.12}{2}\right)^{-6} = 12172.8813\$$$

   - b) If after his first payment, of \$5000, Billy renegotiates his loan with the bank such that he only has one payment left of \$$X$ to do, four years after he contracted the loan, with the same interest rate, what should be the value of X ?



$$12172.8813 = 5000(1+\frac{.12}{2})^{-2} + X(1+\frac{.12}{2})^{-8} \qquad (5)$$

$$\Rightarrow X = 12309.1279\$ \qquad (6)$$

– c) Consider the situation in b). If Billy decides to accumulate $\$X$ in a separate bank account, by doing 42 monthly payments of $\$P$, with a nominal interest rate of $i^{(12)} = 6\%$, compounded each month. The first payment will be one month after his payment of \$5000. Determine P, using annuities.

$$Ps_{\overline{42}|0.005} = 12309.1279 \Rightarrow P = 264.1073\$.$$

4. The parents of three children aged 1, 3, and 5 wish to set up a trust fund that will pay $30,000$ to each child upon attainment of age 18, and $10,000$ to each child upon attainment of age 21.
(a) If the trust fund will earn nominal annual interest rate $i^{(2)} = 8^o/_o$ compounded semiannually, what amount must the parents now invest in the trust fund ?

(b) Assume that the trust fund will grow at nominal interest rate of $3^o/_o$ convertible monthly for the first four years from now, force of interest $\delta_t = \frac{t^2}{1+0.5t}$ for the next year, nominal rate of discount of $4^o/_o$ compounded quarterly for the next eight years and effective annual rate of discount at $5.75^o/_o$ thereafter.
What amount must the parents now invest in the trust fund in order to pay $30,000$ to the child aged 5 upon attainment of age 18, and $10,000$ to the child aged 5 upon attainment of age 21.

Solution. (a) The required amount now, denoted by A, is the sum of the present values at time zero of the payments of $30,000$ to each child upon attainment of age 18, and $10,000$ to each child upon attainment of age 21 :

$$A = 30000 \cdot \left[\frac{1}{\left(1+\frac{i^{(2)}}{2}\right)^{34}} + \frac{1}{\left(1+\frac{i^{(2)}}{2}\right)^{30}} + \frac{1}{\left(1+\frac{i^{(2)}}{2}\right)^{26}} \right]$$

$$+10000 \left[\frac{1}{\left(1+\frac{i^{(2)}}{2}\right)^{40}} + \frac{1}{\left(1+\frac{i^{(2)}}{2}\right)^{36}} + \frac{1}{\left(1+\frac{i^{(2)}}{2}\right)^{32}} \right] = 35346.9568,$$

where $i^{(2)} = 0.08$.

(b) For the child aged 5, the required amount now is :

$$30000 \cdot \frac{1}{\left(1+\frac{0.03}{12}\right)^{48}} \cdot e^{-\int_4^5 \frac{t^2}{1+0.5t}dt} \cdot \left(1-\frac{0.04}{4}\right)^{32}$$

$$+10000 \cdot \frac{1}{\left(1+\frac{0.03}{12}\right)^{48}} \cdot e^{-\int_4^5 \frac{t^2}{1+0.5t}dt} \cdot \left(1-\frac{0.04}{4}\right)^{32} \cdot (1-0.0575\cdot 3) = 48.3219,$$

where

$$\int_4^5 \frac{t^2}{1+0.5t}dt = 2\int_4^5 \frac{t^2}{t+2}dt = 2\int_4^5 \left(t-2+\frac{4}{t+2}\right)dt$$

$$= 2\frac{t^2}{2}\Big|_4^5 - 4t\Big|_4^5 + 8\ln(t+2)\Big|_4^5 = 6.233205439.$$

5. (a) Find the nominal rate of discount convertible semiannually which is equivalent to a nominal rate of interest of $12^o/_o$ per year convertible monthly.

(b) Find the nominal rate of interest convertible daily which is equivalent to a nominal rate of discount of $6^o/_o$ per year convertible daily. Assume a non-leap year.

(c) Establish the following relationships :

$$\frac{i^{(m)}}{m} = \frac{\frac{d^{(m)}}{m}}{1-\frac{d^{(m)}}{m}}; \quad \frac{d^{(m)}}{m} = \frac{\frac{i^{(m)}}{m}}{1+\frac{i^{(m)}}{m}}; \quad \frac{i^{(m)}}{m} - \frac{d^{(m)}}{m} = \frac{i^{(m)}}{m}\cdot\frac{d^{(m)}}{m}; \quad i^{(m)} = d^{(m)}(1+i)^{\frac{1}{m}},$$

where m is a positive integer.

(d) If $i^{(m)}$ is the nominal annual interest rate and i is the equivalent simple interest rate per year, what is the relationship between i and $i^{(m)}$. Similarly, if $d^{(m)}$ is the nominal annual discount rate and d is the equivalent simple discount rate per year, what is the relationship between d and $d^{(m)}$. In both cases, m is a positive integer and a simple model is assumed.

Solution. (a) We have

$$\left(1-\frac{d^{(2)}}{2}\right)^{-2} = \left(1+\frac{i^{(12)}}{12}\right)^{12}$$

$$\Rightarrow \quad d^{(2)} = 2\cdot\left[1-\left(1+\frac{i^{(12)}}{12}\right)^{-6}\right] = 2\cdot\left[1-\left(1+\frac{0.12}{12}\right)^{-6}\right] = 11.59^o/_o.$$

(b) Assuming 365 days in a year, we have

$$\left(1+\frac{i^{(365)}}{365}\right)^{365} = \left(1-\frac{d^{(365)}}{365}\right)^{-365}$$

$$\Rightarrow \quad i^{(365)} = 365\cdot\left[\left(1-\frac{d^{(365)}}{365}\right)^{-1}-1\right] = 365\cdot\left[\left(1-\frac{0.06}{365}\right)^{-1}-1\right] = 0.06000987 \sim 6^{o}/_{o}.$$

(c) Using the relationship

$$\left(1+\frac{i^{(m)}}{m}\right)^{m} = \left(1-\frac{d^{(m)}}{m}\right)^{-m}$$

leads to

$$1+\frac{i^{(m)}}{m} = \left(1-\frac{d^{(m)}}{m}\right)^{-1} \Leftrightarrow \left(1+\frac{i^{(m)}}{m}\right)\cdot\left(1-\frac{d^{(m)}}{m}\right) = 1$$

$$\Leftrightarrow \frac{i^{(m)}}{m} - \frac{d^{(m)}}{m} - \frac{i^{(m)}}{m}\cdot\frac{d^{(m)}}{m} = 0. \qquad (2)$$

Solving (2) for $\frac{i^{(m)}}{m}$, it follows :

$$\frac{i^{(m)}}{m} = \frac{\frac{d^{(m)}}{m}}{1-\frac{d^{(m)}}{m}}.$$

Solving (2) for $\frac{d^{(m)}}{m}$, it follows :

$$\frac{d^{(m)}}{m} = \frac{\frac{i^{(m)}}{m}}{1+\frac{i^{(m)}}{m}}.$$

Rearranging terms in (2) yields

$$\frac{i^{(m)}}{m} - \frac{d^{(m)}}{m} = \frac{i^{(m)}}{m}\cdot\frac{d^{(m)}}{m}.$$

By using the following two relations :

$$\frac{i^{(m)}}{m} = \frac{\frac{d^{(m)}}{m}}{1-\frac{d^{(m)}}{m}} \quad \text{and} \quad \left(1-\frac{d^{(m)}}{m}\right)^{-m} = 1+i,$$

the desired result is obtained, that is,

$$i^{(m)} = d^{(m)}(1+i)^{\frac{1}{m}}.$$

(d) Under simple interest model, if $i^{(m)}$ and i are equivalent, then the accumulated value at the end of one year is :

$$1 + \frac{i^{(m)}}{m} \cdot m = 1 + i \quad \Rightarrow \quad i^{(m)} = i.$$

Under simple discount model, if $d^{(m)}$ and d are equivalent, then the present value at time zero is :

$$1 - \frac{d^{(m)}}{m} \cdot m = 1 - d \quad \Rightarrow \quad d^{(m)} = d.$$

6. The present value of K payable in 3 years is 1450$ if the force of interest is δ, and the present value of K payable in 3 years becomes 2000$ if the force of interest is $\delta/3$. Determine the present value of K payable in 3 years, when the discount rate used is two times the discount rate induced by the force of interest δ.

   Solution : We have $Ke^{-3\delta} = 1450$ and $Ke^{-\delta} = 2000$. Hence, it is possible to express K and δ.

$$\frac{Ke^{-\delta}}{Ke^{-3\delta}} = \frac{2000}{1450} \Rightarrow e^{2\delta} = \frac{2000}{1450} \Rightarrow \delta = 16.08\%$$

   and $K = 1450e^{3(0.1608)} = 2348.94\$$. If d is the discount rate equivalent to the force of interest δ,

$$1 - d = e^{-\delta} = e^{-0.1608} \Rightarrow d = 0.1485.$$

   If the new discount rate is $d' = 2d = 29.7075307\%$, then the present value of K, that will be paid in three years with d' is

$$(1 - d')^3 K = (1 - 0.297075307)^3 2348.94 = 815.8275\$$$