

Maclaurin and Taylor Series Expansions

Truncation Errors

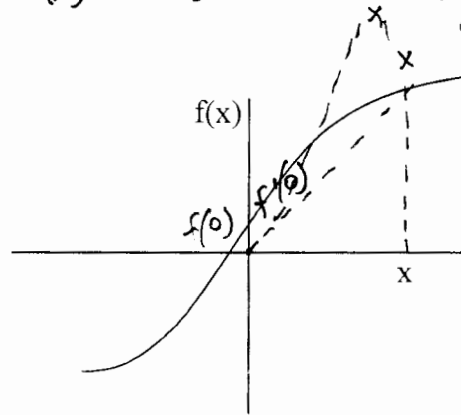
Often in numerical computing you want an analytic function to describe how something changes. Put simply, an analytic function is a mathematical formula which is smooth and easy to differentiate. It might be something like a trend of a stock price with time, or the trajectory of a missile in a game.

In this unit we look at a way of approximating analytic functions over a limited range to simple series of polynomial (powers of x) terms called **Maclaurin** and **Taylor** series.

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \dots$$

Maclaurin series.

Very often we have a function that is interesting to us in a limited range. The simplest range would be about the origin. The function might look like:



As we narrow down the range of x at the origin, the function looks more and more like a straight line, so **very, very** near the origin the function is well described by: $f(x) = A + Bx$ 1.

which is simply the equation of a straight line. This is a linear approximation.

When we include a wider range of x then it is necessary to include higher order terms that will add in the curvature. This is the basis of the series expansion in powers of x of a function over a small region:

$$f(x) = A + Bx + Cx^2 + Dx^3 + \dots \quad 2.$$

where A, B, C... are constants that can be determined from the derivatives of the function at the origin (as we will see later).

Exactly how many terms we need to include depends on the range of x and how accurately we want to approximate the function. Making that decision requires care. However, the important conclusion is that **very near** the origin, we only need a few terms, perhaps no more than:

$$f(x) = A + Bx + Cx^2 \quad 3.$$

This is a quadratic approximation.

An essential assumption if such a series is a good approximation is that each term is smaller than the one before it. Such a series is called **convergent** since each successive term adds a diminishing correction and as more terms are included the series converges to an increasingly accurate approximation.

Where do the coefficients A, B, C, ... come from?

Consider the full expansion:

$$f(x) = A + Bx + Cx^2 + Dx^3 + \dots \quad 1.$$

Now evaluate $F(x)$ at $x=0$. All the terms in x vanish because x is 0 leaving:

$$f(0) = A$$

Now take the derivative of f at $x=0$:

$$f'(x) = B + 2Cx + 3Dx^2 + \dots$$

and evaluating it at the origin:

$$f'(0) = \left. \frac{\partial F(x)}{\partial x} \right|_{x=0} = B$$

Now continuing this process, take the second derivative of F at $x = 0$:

$$f''(0) = \left. \frac{\partial^2 F(x)}{\partial x^2} \right|_{x=0} = 2C$$

Now take the third derivative of f at $x = 0$

$$f'''(0) = \left. \frac{\partial^3 F(x)}{\partial x^3} \right|_{x=0} = 3 \cdot 2 \cdot D = 3! D$$

where $3!$ means factorial 3.

We can continue this process as far as we want, and each derivative gives an expression for the next coefficient.

Using these expressions we can rewrite the original series in terms of the derivatives instead of the coefficients:

$$f(x) = f(0) + f'(0) \cdot x + \frac{f''(0)}{2!} \cdot x^2 + \frac{f'''(0)}{3!} \cdot x^3 + \dots$$

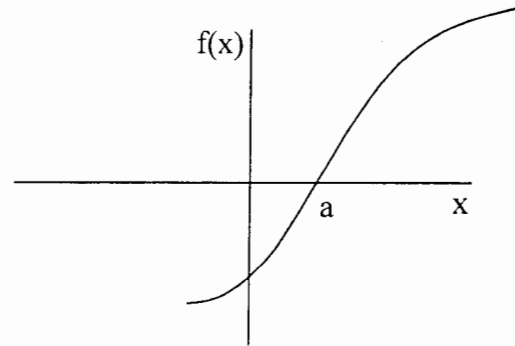
This is called a **Maclaurin series**. It can be summarised as

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0) \cdot x^k}{k!}$$

The nice feature of the series expansion is that it depends only on derivatives right at the origin. In other words, providing we know what the function is doing exactly at the origin, we can very accurately approximate its behaviour nearby.

Taylor Series

Suppose we are interested in a series expansion in x about a point a which is not at the origin:



We can recognise that this is the problem we have already solved – the Maclaurin series – but with the derivatives known at $x = a$ instead of $x = 0$. In the Maclaurin series x is the distance from where we wish to evaluate the series to where we know the derivatives. In the Taylor series $(x-a)$ is that same distance. The Taylor series becomes:

$$f(x) = f(a) + f'(a).(x-a) + \frac{f''(a)}{2!}.(x-a)^2 + \frac{f'''(a)}{3!}.(x-a)^3 + \dots$$

Where $f(a)$, $f'(a)$, $f''(a)$, \dots are the derivatives at $x = a$.

In compact form the Taylor series is

$$f_a(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)(x-a)^k}{k!}$$

Where $f^{(k)}(a)$ is the k^{th} derivative at point a . Setting a to 0 simply takes us back to the Maclaurin series.

The Taylor series is considered to be more general than the Maclaurin series but you can see that it is really the same thing with its coordinate origin shifted. **It is often more convenient to express a problem in terms of the Maclaurin series because the algebra is simpler.**

Truncation Errors

This type of error arises in situations where a series expansion is being used to perform a calculation and a choice is made not to include all the terms in an infinite series. The error which results from truncating at a finite number of terms is called a **truncation error**.

For example, a Taylor expansion of a function $f(x)$ about the point a is:

$$f_a(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)(x-a)^k}{k!}$$

where $f^{(k)}(a)$ denotes the k^{th} derivative evaluated at $x = a$.

Suppose the series is truncated at the n th term so that only terms less than n^{th} are retained

$$f_{\text{approx}}(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(x-a)^k}{k!}$$

The question is what is the error that is incurred?

The answer is based on the choice of a series in the first place. By choosing a series it is implicit that higher order terms are less important than lower order terms so that the series converges. If the series does not converge rapidly then it will be a poor representation of the function since many terms will need to be computed to get an accurate result. Numerical algorithms are often classified by how rapidly they converge.

To estimate the error for a converging series, the **first neglected term** will dominate and can be taken as an approximation to the error:

$$\text{error } E_n \approx \frac{f^{(n)}(a)(x-a)^n}{n!}$$

Note that because of the factor $(x-a)^n$ the error diminishes to zero as a power law in n .

Expressing a as a local origin $x-a \rightarrow x$, we can say that

$$E_n \approx x^n$$

For that reason the error is often expressed in **Big O** notation (meaning "order of") as

$$E_n \approx O(x^n)$$

converging: next in series is smaller

Example 1

Evaluate e^x at the point the $x=1$ using a Maclaurin series truncated at 5 terms.

The general derivatives are:

$$f(x) = e^x; f'(x) = e^x; f''(x) = e^x; f'''(x) = e^x; f^{(4)}(x) = e^x; \dots$$

and at the origin:

$$f(0) = e^0 = 1.0; f'(0) = e^0 = 1.0; f''(0) = e^0 = 1.0; f'''(0) = e^0 = 1.0; \dots$$

So the series is

$$e^x = 1 + x + x^2/2 + x^3/6 + x^4/24 + x^5/120 + \dots$$

$$e^x = 1 + x + x^2/2 + x^3/6 + x^4/24 + O(x^5) \quad \text{to 5 terms}$$

The 5-term series at $x = 1$ is

$$e^1 = 1 + 1 + 1/2 + 1/6 + 1/24 = 65/24 = 2.708333333$$

The exact value of e (to 7 decimal places) is 2.718281828 so the error is

$$\text{Error} = 2.718281828 - 2.708333333 = 0.009948495$$

The 6th term, that is the first neglected term = $1/120 = -0.008333333$

So the first neglected is a reasonable approximation to the error (90% of the exact error)

To get the fractional error from the **series result alone** we write

$$\text{fractional error} = \text{error/value} \sim \text{error/approximate value}$$

$$\begin{aligned} &\sim (1^{\text{st}} \text{ neglected term})/(\text{series approximation}) = (1/120)/2.708333333 \\ &= 0.008333/2.708333 = 0.0031 \text{ (to 2 significant figures)} \\ &\% \text{ fractional error} = 0.31\% \end{aligned}$$

The 5-term series approx is accurate to 0.31% !!

Example 2.

Evaluate e^{-x} at the point the $x=1$ using a MacLaurin series truncated at 5 terms.

The general derivatives are:

$$f(x) = e^{-x}; f'(x) = -e^{-x}; f''(x) = e^{-x}; f'''(x) = -e^{-x}; f^{(4)}(x) = e^{-x}; \dots$$

and at the origin:

$$f(0) = e^{-0} = 1.0; f'(0) = -e^{-0} = -1.0; f''(0) = e^{-0} = 1.0; f'''(0) = -e^{-0} = -1.0; \dots$$

$$e^{-x} = 1 - x + x^2/2 - x^3/6 + x^4/24 - x^5/120 + \dots$$

$$e^{-x} = 1 - x + x^2/2 - x^3/6 + x^4/24 + O(x^5) \quad \text{to 5 terms}$$

The 5-term series at $x = 1$ is $e^{-1} = 1 - 1 + 1/2 - 1/6 + 1/24 = 9/24 = 0.375$

The 6th term that is the first neglected term = $-1/120 = -0.008333333$

(The exact value of e^{-1} is $1/e = 1/(2.718281828) = 0.367879441$)

To get the fractional error from the series result alone we write

$$\text{fractional error} = \text{error/value} \sim \text{error/approximate value}$$

$$\begin{aligned} &= (1^{\text{st}} \text{ neglected term})/(\text{series approximation}) = (1/120)/0.375 \\ &= 0.008333/0.375 = 0.0222 \\ &\% \text{ fractional error} = 2.22\% \end{aligned}$$

Example 3

Evaluate $\sin(x)$ at the point the $x=0.5$ using a MacLaurin series truncated after the term in x^3 .

The general derivatives are:

$$f(x) = \sin(x); f'(x) = \cos(x); f''(x) = -\sin(x);$$

$$f'''(x) = -\cos(x); f^{(4)}(x) = \sin(x); f^{(5)}(x) = \cos(x); \dots$$

and at the origin:

$$f(0) = \sin(0) = 0.0; f'(0) = \cos(0) = 1.0; f''(0) = -\sin(0) = 0.0;$$

$$f'''(0) = -\cos(0) = -1.0; f^{(4)}(0) = \sin(0) = 0.0; f^{(5)}(0) = \cos(0) = 1.0;$$

$$\begin{aligned} \sin(x) &= 0.0 + x + 0.0(x^2/2) - x^3/6 + 0.0(x^4/24) + \dots \text{ up to the term in } x^5 \\ &= x - x^3/6 + x^5/120 + \dots \end{aligned}$$

Terminating the series after the term in x^3 gives

$$\sin(x) = x - x^3/6$$

and at $x = 0.5$

$$\sin(0.5) = 0.5 - (0.5)^3/6 = 0.5 - 1/48 = 23/48 = 0.479166 \text{ to 6 significant figures.}$$

The exact value = 0.479426 so the exact error = 0.479426 - 0.479166 = 0.00026

The term in x^5 that is the first neglected term = $(0.5)^5/120 = 0.00026042$ which is therefore a good approximation to the error

To get the fractional error from the series result alone we write

$$\text{fractional error} = \text{error/value} \sim \text{error/approximate value}$$

$$\begin{aligned} &= (1^{\text{st}} \text{ neglected term})/(\text{series approximation}) = (2.6042 \cdot 10^{-4})/0.479166 \\ &= 5.434 \cdot 10^{-4} \end{aligned}$$

$$\% \text{ fractional error} = 0.054\%$$