ACTU 256 / Fall 2013

Assignment 1

Due: Friday, September 20, 2013 (at the beginning of the class)

Please note:

- *Each student should submit their assignments before the beginning of class on the announced due date. Late assignments will not be accepted.*
- *Solutions must be written up carefully, showing all work, for full credit.*
- *No points will be awarded for a numerical answer that is not justified by a demonstration of the steps used.*

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1. A bank started a new savings account on September 10, 2010, using the accumulation function $a(t) = mt^2 + n$, where t is measured in years. An investment of 500 made on September 10, 2010, is worth 700 on September 10, 2012.
(a) Find the accumulated value of an investment of $1,000$ made on September 10, 2010, after three years.
(b) Find the accumulated value of an investment of $1,000$ made on September 10, 2011, after three years.
(c) Find the effective rate of interest during the second year.
(d) Find the effective rate of discount during the third year.

Solution.
September 10, 2010 (the savings account started at this time) means time $t = 0$. Therefore, September 10, 2012 means time $t = 2$.
An investment of 500 made on September 10, 2010 is worth 700 on September 10, 2012:

$$A(2) = A(0) \cdot a(2) \;\Rightarrow\; 700 = 500 \cdot (4m + n) \qquad (1)$$

The accumulation function $a(t)$ satisfies:

$$a(0) = 1 \;\Rightarrow\; n = 1.$$

Using this value of n in relation (1), it follows $m = 0.1$. So, $a(t) = 0.1t^2 + 1$.
(a). The accumulated value of an investment of $1,000$ made on September 10, 2010 after three years is:

$$A(3) = A(0) \cdot a(3) \;\Rightarrow\; A(3) = 1000 \cdot (0.1 \cdot 9 + 1) \;\Rightarrow\; A(3) = 1900.$$

(b). The accumulated value of an investment of $1,000$ made on September 10, 2011 (that is, at time $t = 1$) after three years is:

$$A(4) = 1000 \cdot \frac{a(4)}{a(1)} \quad \Rightarrow \quad A(4) = 1000 \cdot \frac{0.1 \cdot 16 + 1}{0.1 \cdot 1 + 1} = 2363.6363.$$

(c). The effective rate of interest during the second year is

$$i_2 = \frac{a(2) - a(1)}{a(1)} = \frac{(0.1 \cdot 2^2 + 1) - (0.1 \cdot 1 + 1)}{0.1 \cdot 1 + 1} = 0.2727 = 27.27^o/_o.$$

(d). The effective rate of discount during the third year is:

$$d_3 = \frac{a(3) - a(2)}{a(3)} = \frac{(0.1 \cdot 3^2 + 1) - (0.1 \cdot 2^2 + 1)}{0.1 \cdot 3^2 + 1} = 0.2631 = 26.31^o/_o.$$

2. Mary deposits $20,000$ into a bank account that pays an effective annual interest rate of $3^o/_o$ for ten years, with interest credited at the end of each year. At the beginning of the second year she makes a second deposit $2^o/_o$ greater than the previous year's deposit. If a withdrawal is made during the first 4.5 years, Mary pays a penalty of $5^o/_o$ of the withdrawal amount. Mary withdraws X at the end of each of years 4, 6 and 8. What is the amount X if Mary's bank account balance is $20,000$ at the end of ten years?

Solution. At time $t = 0$ there is a deposit of 20000. At time $t = 1$ (the beginning of the second year) there is a second deposit of $20000 \cdot (1.02) = 20400$. At time $t = 4$, Mary withdraws X and hence, she pays a penalty of $0.05X$. Mary withdraws X at time $t = 6$ and at time $t = 8$. The balance is $20,000$ at the end of ten years:

$$20000 = 20000(1.03)^{10} + 20400(1.03)^9 - X(1.05)(1.03)^6 - X(1.03)^4 - X(1.03)^2$$

Solving the above equation for X, it follows that $X = 9736.6588$.

3. The parents of three children aged 1, 3, and 6 wish to set up a trust fund that will pay $25,000$ to each child upon attainment of age 18, and $100,000$ to each child upon attainment of age 21.
(a) If the trust fund will earn nominal annual interest rate $i^{(2)} = 8^o/_o$ compounded semiannually, what amount must the parents now invest in the trust fund?

(b) Assume that the trust fund will grow at nominal interest rate of $3^o/_o$ convertible monthly for the first four years, force of interest $\delta_t = \frac{t^2}{1+0.5t}$ for the next year, nominal rate of discount of $4^o/_o$ compounded quarterly for the next seven years and effective annual rate of discount at $5.75^o/_o$ thereafter. What amount must the parents now invest in the trust fund in order to pay $25,000$ to the child aged 6 upon attainment of age 18, and $100,000$ to the child aged 6 upon attainment of age 21.

Solution. (a) The required amount now, denoted by A, is the sum of the present values at time zero of the payments of $25,000$ to each child upon attainment of age 18, and $100,000$ to each child upon attainment of age 21:

$$A = 25000 \cdot \left[\frac{1}{\left(1+\frac{i^{(2)}}{2}\right)^{34}} + \frac{1}{\left(1+\frac{i^{(2)}}{2}\right)^{30}} + \frac{1}{\left(1+\frac{i^{(2)}}{2}\right)^{24}} \right]$$

$$+100000 \left[\frac{1}{\left(1+\frac{i^{(2)}}{2}\right)^{40}} + \frac{1}{\left(1+\frac{i^{(2)}}{2}\right)^{36}} + \frac{1}{\left(1+\frac{i^{(2)}}{2}\right)^{30}} \right] = 100077.4493,$$

where $i^{(2)} = 0.08$.
(b) For the child aged 6, the required amount now is:

$$25000 \cdot \frac{1}{\left(1+\frac{0.03}{12}\right)^{48}} \cdot e^{-\int_4^5 \frac{t^2}{1+0.5t}dt} \cdot \left(1 - \frac{0.04}{4}\right)^{28}$$

$$+100000 \cdot \frac{1}{\left(1+\frac{0.03}{12}\right)^{48}} \cdot e^{-\int_4^5 \frac{t^2}{1+0.5t}dt} \cdot \left(1 - \frac{0.04}{4}\right)^{28} \cdot (1 - 0.0575 \cdot 3) = 141.6138,$$

where

$$\int_4^5 \frac{t^2}{1+0.5t}dt = 2\int_4^5 \frac{t^2}{t+2}dt = 2\int_4^5 \left(t - 2 + \frac{4}{t+2}\right)dt$$

$$= 2\frac{t^2}{2}\Big|_4^5 - 4t\Big|_4^5 + 8\ln(t+2)\Big|_4^5 = 6.233205439.$$

4. (a) Find the nominal rate of discount convertible semiannually which is equivalent to a nominal rate of interest of $12^o/_o$ per year convertible monthly.

(b) Find the nominal rate of interest convertible daily which is equivalent to a nominal rate of discount of $5^o/_o$ per year convertible daily.

(c) Establish the following relationships:

$$\frac{i^{(m)}}{m} = \frac{\frac{d^{(m)}}{m}}{1 - \frac{d^{(m)}}{m}}; \quad \frac{d^{(m)}}{m} = \frac{\frac{i^{(m)}}{m}}{1 + \frac{i^{(m)}}{m}}; \quad \frac{i^{(m)}}{m} - \frac{d^{(m)}}{m} = \frac{i^{(m)}}{m} \cdot \frac{d^{(m)}}{m}; \quad i^{(m)} = d^{(m)}(1+i)^{\frac{1}{m}},$$

where m is a positive integer.

(d) If $i^{(m)}$ is the nominal annual interest rate and i is the equivalent simple interest rate per year, what is the relationship between i and $i^{(m)}$. Similarly, if $d^{(m)}$ is the nominal annual discount rate and d is the equivalent simple discount rate per year, what is the relationship between d and $d^{(m)}$. In both cases, m is a positive integer.

Solution. (a) We have

$$\left(1-\frac{d^{(2)}}{2}\right)^{-2} = \left(1+\frac{i^{(12)}}{12}\right)^{12}$$

$$\Rightarrow \quad d^{(2)} = 2\cdot\left[1-\left(1+\frac{i^{(12)}}{12}\right)^{-6}\right] = 2\cdot\left[1-\left(1+\frac{0.12}{12}\right)^{-6}\right] = 11.59^o/_o.$$

(b) Assuming 365 days in a year, we have

$$\left(1+\frac{i^{(365)}}{365}\right)^{365} = \left(1-\frac{d^{(365)}}{365}\right)^{-365}$$

$$\Rightarrow \quad i^{(365)} = 365\cdot\left[\left(1-\frac{d^{(365)}}{365}\right)^{-1}-1\right] = 365\cdot\left[\left(1-\frac{0.05}{365}\right)^{-1}-1\right] = 0.05000674 \sim 5^o/_o.$$

(c) Using the relationship

$$\left(1+\frac{i^{(m)}}{m}\right)^{m} = \left(1-\frac{d^{(m)}}{m}\right)^{-m}$$

leads to

$$1+\frac{i^{(m)}}{m} = \left(1-\frac{d^{(m)}}{m}\right)^{-1} \Leftrightarrow \left(1+\frac{i^{(m)}}{m}\right)\cdot\left(1-\frac{d^{(m)}}{m}\right) = 1$$

$$\Leftrightarrow \frac{i^{(m)}}{m} - \frac{d^{(m)}}{m} - \frac{i^{(m)}}{m}\cdot\frac{d^{(m)}}{m} = 0. \qquad (2)$$

Solving (2) for $\frac{i^{(m)}}{m}$, it follows:

$$\frac{i^{(m)}}{m} = \frac{\frac{d^{(m)}}{m}}{1-\frac{d^{(m)}}{m}}.$$

Solving (2) for $\frac{d^{(m)}}{m}$, it follows:

$$\frac{d^{(m)}}{m} = \frac{\frac{i^{(m)}}{m}}{1+\frac{i^{(m)}}{m}}.$$

Rearranging terms in (2) yields

$$\frac{i^{(m)}}{m} - \frac{d^{(m)}}{m} = \frac{i^{(m)}}{m} \cdot \frac{d^{(m)}}{m}.$$

By using the following two relations:

$$\frac{i^{(m)}}{m} = \frac{\frac{d^{(m)}}{m}}{1 - \frac{d^{(m)}}{m}} \quad \text{and} \quad \left(1 - \frac{d^{(m)}}{m}\right)^{-m} = 1 + i,$$

the desired result is obtained, that is,

$$i^{(m)} = d^{(m)}(1+i)^{\frac{1}{m}}.$$

(d) Under simple interest model, if $i^{(m)}$ and i are equivalent, then the accumulated value at the end of one year is:

$$1 + \frac{i^{(m)}}{m} \cdot m = 1 + i \quad \Rightarrow \quad i^{(m)} = i.$$

Under simple discount model, if $d^{(m)}$ and d are equivalent, then the present value at time zero is:

$$1 - \frac{d^{(m)}}{m} \cdot m = 1 - d \quad \Rightarrow \quad d^{(m)} = d.$$

5. On January 1, 2011, Smith deposits $1,000$ into an account earning nominal annual interest rate of $i^{(4)} = 0.04$ compounded quarterly with interest credited on the last day of March, June, September, and December. If Smith closes the account during the year, simple interest is paid on the balance from the most recent interest credit date.
(a) What is Smith's close-out balance on August 20?

(b) Suppose all four quarters in the year are considered equal, and time is measured in years. Derive expressions for Smith's accumulated amount function $A(t)$, the close-out balance at time t. Consider separately the four intervals $0 \le t \le 0.25$, $0.25 \le t \le 0.50$, $0.50 \le t \le 0.75$ and $0.75 \le t \le 1$ and draw the time diagram for each of these cases.

(c) Using part (*b*), show that if $0 \le t \le 0.25$, then it follows that $\delta_t = \delta_{t+0.25} = \delta_{t+0.50} = \delta_{t+0.75}$.
Solution. If Smith closes the account during the year, simple interest is paid on *the balance from the most recent interest credit date.*
Since $i^{(4)} = 0.04$, then $\frac{i^{(4)}}{4} = 0.01$ is $\frac{1}{4}$-year effective interest rate. Under

simple interest model, we have that $i^{(4)} = i = 0.04$.

(a) Smith's close-out balance on August 20 is:

$$1000 \cdot \left(1 + \frac{0.04}{4}\right)^2 \cdot \left(1 + \frac{51}{365} \cdot 0.04\right) = 1025.8013,$$

since Smith closes the account during the third quarter and simple interest is paid on the balance of $1000 \cdot \left(1 + \frac{0.04}{4}\right)^2$.

(b) $A(t)$ is the amount function at time t.

- If $0 \leq t \leq 0.25$, then $A(t) = 1000 \cdot (1 + 0.04 \cdot t)$, since Smith closes the account during the first quarter and simple interest is paid on the balance of 1000.
- If $0.25 \leq t \leq 0.50$, then $A(t) = 1000 \cdot \left(1 + \frac{0.04}{4}\right)^1 \cdot (1 + 0.04 \cdot (t - 0.25))$, since Smith closes the account during the second quarter and simple interest is paid on the balance of $1000 \cdot \left(1 + \frac{0.04}{4}\right)^1$.
- If $0.50 \leq t \leq 0.75$, then $A(t) = 1000 \cdot \left(1 + \frac{0.04}{4}\right)^2 \cdot (1 + 0.04 \cdot (t - 0.50))$, since Smith closes the account during the third quarter and simple interest is paid on the balance of $1000 \cdot \left(1 + \frac{0.04}{4}\right)^2$.
- If $0.75 \leq t \leq 1$, then $A(t) = 1000 \cdot \left(1 + \frac{0.04}{4}\right)^3 \cdot (1 + 0.04 \cdot (t - 0.75))$, since Smith closes the account during the fourth quarter and simple interest is paid on the balance of $1000 \cdot \left(1 + \frac{0.04}{4}\right)^3$.

Consequently,

$$A(t) = \begin{cases} 1000 \cdot (1 + 0.04 \cdot t) & \text{for } 0 \leq t \leq 0.25 \\ 1000 \cdot (1.01) \cdot (1 + 0.04 \cdot (t - 0.25)) & \text{for } 0.25 \leq t \leq 0.50 \\ 1000 \cdot (1.01)^2 \cdot (1 + 0.04 \cdot (t - 0.50)) & \text{for } 0.50 \leq t \leq 0.75 \\ 1000 \cdot (1.01)^3 \cdot (1 + 0.04 \cdot (t - 0.75)) & \text{for } 0.75 \leq t \leq 1. \end{cases}$$

(b) For $0 \leq t \leq 0.25$, we have:

- $\delta_t = \frac{A'(t)}{A(t)} = \frac{[1000 \cdot (1+0.04 \cdot t)]'}{1000 \cdot (1+0.04 \cdot t)} = \frac{0.04}{1+0.04 \cdot t}$.
- $0.25 \leq t + 0.25 \leq 0.50$, and if we denote $t_1 = t + 0.25$, we have

  $$\delta_{t_1} = \frac{A'(t_1)}{A(t_1)} = \frac{[1000 \cdot (1.01) \cdot (1 + 0.04 \cdot (t_1 - 0.25))]'}{1000 \cdot (1.01) \cdot (1 + 0.04 \cdot (t_1 - 0.25))} = \frac{0.04}{1 + 0.04 \cdot (t_1 - 0.25)}$$

  and hence,

  $$\delta_{t+0.25} = \frac{0.04}{1 + 0.04 \cdot (t + 0.25 - 0.25)} = \frac{0.04}{1 + 0.04 \cdot t} = \delta_t.$$

- $0.50 \leq t + 0.50 \leq 0.75$, and if we denote $t_2 = t + 0.50$, we have

$$\delta_{t_2} = \frac{A'(t_2)}{A(t_2)} = \frac{[1000 \cdot (1.01)^2 \cdot (1 + 0.04 \cdot (t_2 - 0.50))]'}{1000 \cdot (1.01)^2 \cdot (1 + 0.04 \cdot (t_2 - 0.50))} = \frac{0.04}{1 + 0.04 \cdot (t_2 - 0.50)}$$

and hence,

$$\delta_{t+0.50} = \frac{0.04}{1 + 0.04 \cdot (t + 0.50 - 0.50)} = \frac{0.04}{1 + 0.04 \cdot t} = \delta_t.$$

- $0.75 \leq t + 0.75 \leq 1$, and if we denote $t_3 = t + 0.75$, we have

$$\delta_{t_3} = \frac{A'(t_3)}{A(t_3)} = \frac{[1000 \cdot (1.01)^3 \cdot (1 + 0.04 \cdot (t_3 - 0.75))]'}{1000 \cdot (1.01)^3 \cdot (1 + 0.04 \cdot (t_3 - 0.75))} = \frac{0.04}{1 + 0.04 \cdot (t_3 - 0.75)}$$

and hence,

$$\delta_{t+0.75} = \frac{0.04}{1 + 0.04 \cdot (t + 0.75 - 0.75)} = \frac{0.04}{1 + 0.04 \cdot t} = \delta_t.$$