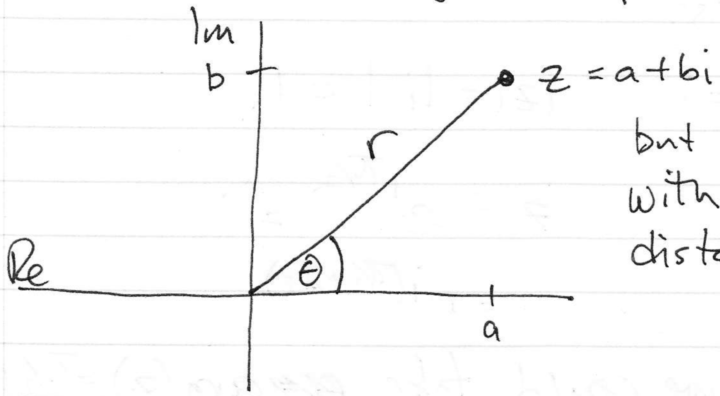
Polar Form of Complex numbers.



but z can be specified with the angle θ and distance r.

$\Downarrow$

$z = re^{i\theta}$ is the polar form of z.

This is Napier's Constant, base of natural log.

- The angle θ is called the argument of z, or $\arg(z)$, and is always given in radians, not degrees.

- trigonometry gives the dictionary between standard and polar forms:

$\frac{a}{r} = \cos\theta \rightsquigarrow a = r\cos\theta$. real part.

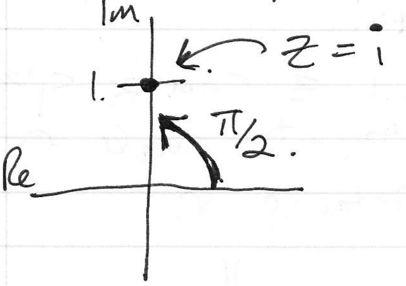
$\frac{b}{r} = \sin\theta \rightsquigarrow b = r\sin\theta$ imaginary part.

$r = |z| = \sqrt{a^2+b^2}$.

polar form ~~standard~~

$$z = re^{i\theta} = \underbrace{r(\cos\theta + i\sin\theta)}_{\text{also polar form, since uses } r \,\&\, \theta.}$$

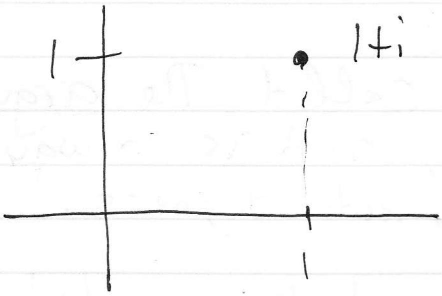
Some examples:



$|z| = |i| = 1.$

$$z = e^{i\pi/2} = i$$
$$= e^{i(\pi/2 + 2\pi)}$$

Note also, we could take $\arg(z) = \pi/2 + 2\pi$ or add as many multiples of 2π as we want.



$$r = |z| = |1+i| = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

$$\cos\theta = \frac{1}{\sqrt{2}}.$$

$$\Rightarrow \theta = \pi/4.$$

$$z = 1 + i = \sqrt{2}\, e^{i\pi/4}$$

— notice that θ is not unique here, we can add any multiple of 2π and get the same number.

— if θ is chosen so that $-\pi < \theta \le \pi$, it is the <u>principal argument</u> of z.

(negative θ means rotating clockwise from positive real axis rather than counterclockwise).

Q: Why do This?

A: Multiplication is easier & less cumbersome.

Take two complex numbers, $z = r_1 e^{i\theta_1}$, $w = r_2 e^{i\theta_2}$

Then: $zw = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2})$

$$= (r_1 r_2) e^{i\theta_1} e^{i\theta_2}$$

$$= (r_1 r_2) e^{i\theta_1 + i\theta_2} \quad \text{(exponent rule)}$$

$$= (r_1 r_2) e^{i(\theta_1 + \theta_2)}$$

(multiply absolute values and add arguments)

Inverses are easier too: if $z = re^{i\theta}$, then

$$z^{-1} = \frac{1}{z} = \frac{1}{r} e^{-i\theta}.$$

(Then, $z\frac{1}{z} = re^{i\theta}\frac{1}{r}e^{-i\theta} = \frac{r}{r}e^{i(\theta-\theta)}$

$$= 1 \cdot e^{0}$$

$$= 1. \quad)$$

Example: Find polar form of $\dfrac{1+i}{1-\sqrt{3}i}$

$$1+i \longrightarrow z_1 = \sqrt{2}\, e^{i\pi/4} = \sqrt{2}\left(\cos \pi/4 + i \sin \pi/4\right).$$

$$1-\sqrt{3}i \longrightarrow z_2 = 2e^{-i\pi/3} = 2\left(\cos -\pi/3 + i \sin(-\pi/3)\right).$$

$$\left(\cos\theta = \frac{1}{2} \Rightarrow \theta = \pm\pi/3\right)$$

need to choose right sign here, since $1-\sqrt{3}i$ lies in 4th quadrant.

$$\left|\frac{1+i}{1-\sqrt{3}i}\right| = \frac{|z_1|}{|z_2|} = \frac{\sqrt{2}}{2}$$

$$\arg\left(\frac{1+i}{1-\sqrt{3}i}\right) = \pi/4 - \left(-\frac{\pi}{3}\right) = \frac{3\pi}{12} + \frac{4\pi}{12} = \frac{7\pi}{12}.$$

$$\therefore \frac{1+i}{1-\sqrt{3}i} = \frac{\sqrt{2}}{2} e^{i\,7\pi/12}$$

$$= \frac{\sqrt{2}}{2}\left(\cos\frac{7\pi}{12} + i \sin 7\pi/12\right).$$

- That's all for complex numbers for now.

— Vectors in $\mathbb{R}^n$

$\mathbb{R}$: real #s, scalors $\rightsquigarrow$ real line.
$\mathbb{R}^2 = \{(x,y) \mid x,y \in \mathbb{R}\}$ $\rightsquigarrow$ plane.
$\mathbb{R}^3 = \{(x,y,z) \mid x,y,z \in \mathbb{R}\}$ $\rightsquigarrow$ 3-space.
$\vdots$
$(n>0)$. $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}\}$ $\rightsquigarrow$ n-space.

<u>Notation</u>: $\vec{v} = (v_1, \ldots, v_n) = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

matrix form, or column vector.

All the basic rules from $\mathbb{R}^2$ and $\mathbb{R}^3$ extend to $\mathbb{R}^n$:

1) $\vec{x} = (x_1, \ldots, x_n) = (y_1, \ldots, y_n) = \vec{y}$ if and only if

$x_1 = y_1, \ldots, x_n = y_n$.

2) $(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$.

3) for $c \in \mathbb{R}$, $c(x_1, \ldots, x_n) = (cx_1, \ldots, cx_n)$.
(in particular, $-(x_1, \ldots, x_n) = (-x_1, \ldots, -x_n)$.)

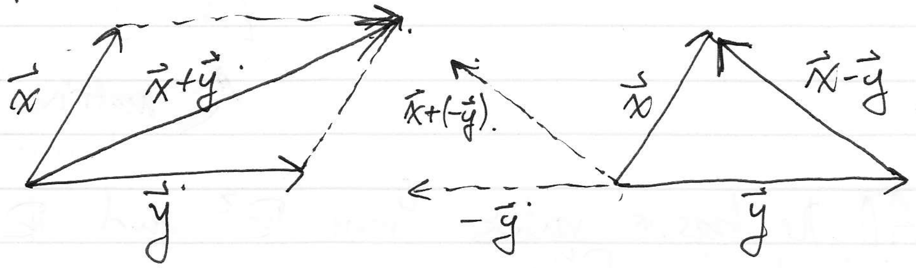
4) $\vec{0} = \underbrace{(0, 0, \ldots, 0)}_{n \text{ times}}$

- All the usual rules of arithmetic from $\mathbb{R}$ apply when adding. but note: there is no way to multiply vectors!

Geometrically, the above rules have meaning:

1) two vectors are equal if and only if they have the same direction and magnitude

2) parallelogram rule.



3) Scalar multiplication scales the vector by $|c|$, and reverses direction if $c<0$.
Two vectors are parallel $\Leftrightarrow$ they are scalar multiples of each other.

4). The zero vector has no magnitude (nor any direction).

- Linear Combinations

- uses both arithmetic operations available in $\mathbb{R}^n$
- we will be interested in what vectors we can get from a given collection of vectors via these operations.

Def^n: A linear combination ~~is~~ of a set of vectors $\vec{v}_1, \ldots, \vec{v}_m$ is any

$$k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_m\vec{v}_m,$$

where $k_1, \ldots, k_m \in \mathbb{R}$.

Examples: 1) The linear combinations of a set of one vector, $\{\vec{v}\}$, are just the scalar multiples of that vector, $\{c\vec{v} \mid c \in \mathbb{R}\}$.

2) $\vec{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\vec{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$3\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 4\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \\ 9 \end{bmatrix}$$ is a linear combination.

but $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is not, since if it were, then

$$\begin{bmatrix}0\\1\\0\end{bmatrix} = a\begin{bmatrix}1\\2\\3\end{bmatrix} + b\begin{bmatrix}1\\0\\0\end{bmatrix}$$ for some $a, b \in \mathbb{R}$.

$$\Leftrightarrow \left.\begin{array}{l} 0 = a + b \\ 1 = 2a \Rightarrow a = \frac{1}{2} \\ 0 = 3a \Rightarrow a = 0 \end{array}\right\} \times.$$

<u>Properties of + & scalar multiplication</u>

$\vec{w}, \vec{u}, \vec{v} \in \mathbb{R}^n$, $c, d \in \mathbb{R}$

- $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
- $\vec{u} + \vec{0} = \vec{u}$
- $\vec{u} + (-\vec{u}) = \vec{0}$
- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
- $(c+d)\vec{u} = c\vec{u} + d\vec{u}$
- $(cd)\vec{u} = c(d\vec{u})$
- $1\vec{u} = \vec{u}$.

— <u>Dot Product</u>
aka "Inner Product".

- ~~prod~~ provides a link between algebra & geometry of vector spaces.

recall the dot product from $\mathbb{R}^3$:
given $\vec{x} = (x_1, x_2, x_3)$, $\vec{y} = (y_1, y_2, y_3)$

$$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + x_3 y_3 \longleftarrow \text{this is a scalar, not a vector.}$$

The dot product is related to length/norm:

$$\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2} = \sqrt{\vec{x} \cdot \vec{x}}$$

remember also that

$$\|\vec{x} - \vec{y}\| = \text{distance between } \vec{x} \text{ and } \vec{y}.$$

Clearly there's nothing special about the number of entries/coordinates here, and we can define the dot product for $\mathbb{R}^n$:

$$\vec{x} = (x_1, \dots, x_n), \quad \vec{y} = (y_1, \dots, y_n).$$

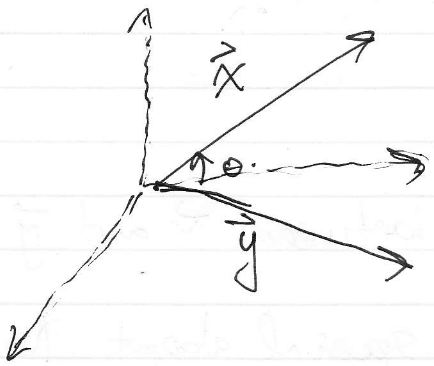
$$\begin{cases} \vec{x} \cdot \vec{y} = x_1 y_1 + \dots + x_n y_n \\ \|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}}. \end{cases}$$

<u>FACT</u>: $\|\vec{x}\| = 0$ if and only if $\vec{x} = \vec{0}$.

Also generalize ~~the~~

$$\vec{x} \cdot \vec{y} = 0 \iff \vec{x} \text{ and } \vec{y} \text{ are } \underline{\text{perpendicular}} \text{ or } (\underline{\text{orthogonal}})$$

from ~~the MA431 case~~ $\mathbb{R}^2$ and $\mathbb{R}^3$



$$\vec{x} \cdot \vec{y} = \|x\| \|y\| \cos\theta.$$

We will say $\vec{x}, \vec{y} \in \mathbb{R}^n$ are <u>orthogonal</u> (or <u>perpendicular</u>) if $\vec{x} \cdot \vec{y} = 0$.

— <u>Angles between vectors in $\mathbb{R}^n$</u>

— <u>Cauchy-Schwartz Inequality</u>

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\| \quad \text{for any } \vec{x}, \vec{y} \in \mathbb{R}^n$$

This implies

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$$

— The triangle inequality.

This lets us define the angle between $\vec{x}$ and $\vec{y}$ to be:

1.) $0 \leq \theta \leq \pi$ ← makes the angle <u>unique</u>.

2.) $\cos\theta = \underbrace{\dfrac{\vec{x}\cdot\vec{y}}{\|\vec{x}\|\|\vec{y}\|}}$

Cauchy-Schwartz implies that this will always be between -1 and 1, so we'll always be able to find θ.

This agrees with the earlier definition of orthogonality, since if the angle is $90°$ (or $\pi/2$), then $\cos\theta = 0$, which it would have to be since $\vec{x}\cdot\vec{y} = 0$.

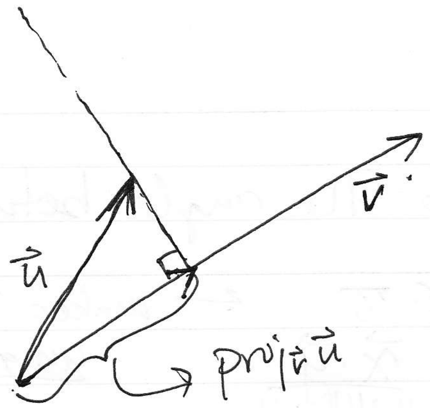
<u>note</u>: by the definition, the angle b/t two vectors is never more than $180°$.

— <u>Orthogonal Projection</u>.

<u>Idea</u>: The orthogonal projection of one vector ~~along~~ $\vec{u}$ onto another, $\vec{v}$ is the unique vector such that:

– $\text{proj}_{\vec{v}}(\vec{u})$ is parallel to $\vec{v}$ (i.e, a scalar multiple).

– $\vec{u} - \text{proj}_{\vec{v}}(\vec{u})$ is orthogonal to $\vec{v}$ (so $(\vec{u} - \text{proj}_{\vec{v}}(\vec{u}))\cdot\vec{v} = 0$).



'shadow of $\vec{u}$ on $\vec{v}$'.

The formula is:

$$\mathrm{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v}$$

– taking the orthogonal projection breaks $\vec{u}$ into two pieces, one parallel to $\vec{v}$, and one other orthogonal to $\vec{v}$ (the two extremes), and has applications in error correction for Universal Mobile Telecommunications Systems.

– <u>Note</u>: you can tell $\mathrm{proj}_{\vec{v}}(\vec{u})$ is parallel to $\vec{v}$ just from the formula: it's a scalar multiple of $\vec{v}$.

And

$$\begin{aligned}(\vec{u} - \mathrm{proj}_{\vec{v}}(\vec{u}))\cdot\vec{v} &= \left(\vec{u} - \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v}\right)\cdot\vec{v} \\ &= \vec{u}\cdot\vec{v} - \frac{(\vec{u}\cdot\vec{v})}{\|\vec{v}\|^2}\underbrace{\vec{v}\cdot\vec{v}}_{=\|\vec{v}\|^2} \\ &= \vec{u}\cdot\vec{v} - \vec{u}\cdot\vec{v} \\ &= 0\end{aligned}$$

so $\vec{u} - \mathrm{proj}_{\vec{v}}(\vec{u})$ is orthogonal to $\vec{v}$.