MAT1300 Midterm Review-2014

1 Material

Sections from the book to study (9th Edition):

- Appendix A:
  - A.1: Real line and inequalities
  - A.2: Absolute Value and Distance
  - A.3: Exponents and Radicals
  - A.4: Factoring Polynomials
  - A.5: Fractions and Rationalization
- Chapter 1:
  - 1.1: Cartesian Plane and Distance
  - 1.2: Graphs of Equations
  - 1.3: Lines and Slope
  - 1.4: Functions
  - 1.5: Limits
  - 1.6: Continuity
- Chapter 2:
  - 2.1: Derivative and Slope of a Graph
  - 2.2: Rules of Differentiation
  - 2.3: Demand and Revenue Functions only
  - 2.4: Product and Quotient Rules
  - 2.5: Chain Rule
  - 2.7: Implicit Differentiation
- Chapter 4:
  - 4.1: Exponential Functions
  - 4.2: Natural Exponential Functions
  - 4.3 Derivatives of Exponential Functions
  - 4.4: Logarithmic Functions
  - 4.5 Derivatives of Logarithmic functions.

2 Topics

Key concepts from Appendix A:

1. Absolute Value
2. inequalities
3. Factoring Polynomials

Key concepts from Chapter 1:

1. Graphing equations; *finding intercepts*
2. Break-even analysis; *finding a break-even point*
3. Slope of a line; *finding equation* $y = mx + b$ *for a line*
4. Functions: domain; *finding domain of a function*
5. Composite and inverse: *finding the inverse of a function*
6. Limits: one- and two-sided; *evaluating the limit, calculating limits using rationalization and factoring*
7. Operations on limits; *calculating limits of polynomials and rational functions and functions with roots*
8. Continuity; *understand definition and apply to given functions*

Key concepts from Chapter 2:

1. Definition of derivative: *Use definition to calculate derivative of standard functions*
2. Interpretation of derivative as slope of tangent line: *finding equation of tangent line*
3. Differentiability: *understanding why certain functions are not differentiable at some points*
4. Rules for differentiation: *be able to use all the rules (and combine them as needed)*
5. Implicit differentiation: *finding derivative of implicitly defined functions*

Key concepts from Chapter 4:

1. Exponential functions; natural base e; *know graphs, calculation rules*
2. Compound interest: *know formulas for compound interest and continuous compounding and know how to apply these*
3. Logarithms; *know graphs, calculation rules; apply logarithms to solve exponential equations, e.g. in compound interest problems*
4. Derivatives of exponential and logarithmic functions.

3 Sample Problems

Problem 1. Find the tangent line of the function $f(x) = \sqrt{\frac{2x+1}{3x}}$ at the point $x = 1$.

Solution. First, we find the derivative of f: this can be done in two ways.

$$\begin{aligned} f'(x) &= \frac{1}{2} \cdot \left(\frac{2x+1}{3x}\right)^{-\frac{1}{2}} \cdot \frac{2 \cdot 3x - 3(2x+1)}{9x^2} \\ &= \frac{1}{2} \cdot \left(\frac{2x+1}{3x}\right)^{-\frac{1}{2}} \cdot \frac{-1}{3x^2} \end{aligned}$$

Alternatively first write $f(x) = \sqrt{\frac{2}{3} + \frac{1}{3x}}$, so that

$$f'(x) = \frac{1}{2} \cdot \left(\frac{2}{3} + \frac{1}{3x}\right)^{-\frac{1}{2}} \cdot \frac{-1}{3x^2}$$

Next, plug in $x = 1$ in $f'(x)$:

$$f'(1) = \frac{1}{2} \cdot \left(\frac{2}{3} + \frac{1}{3 \cdot 1}\right)^{-\frac{1}{2}} \cdot \frac{-1}{3(1^2)} = \frac{1}{2} \cdot 1 \cdot \frac{-1}{3} = -\frac{1}{6}.$$

Thus the tangent line at $x = 1$ of f has slope $-\frac{1}{6}$.

Now we know that the tangent line has to have the form $y = -\frac{1}{6}x + b$. We need to find b.
Plug in $x = 1$ into f:

$$f(1) = \sqrt{\frac{2 \cdot 1 + 1}{3 \cdot 1}} = \sqrt{\frac{3}{3}} = 1.$$

Thus the tangent line has to pass through the point $(1, 1)$. Now use this to find b:

$$y = -\frac{1}{6}x + b \Rightarrow 1 = -\frac{1}{6} \cdot 1 + b \Rightarrow b = \frac{7}{6}.$$

Therefore the tangent line has equation $y = -\frac{1}{6}x + \frac{7}{6}$.

Problem 2. Using the definition of derivative, find the derivative of $f(x) = (2x-1)^2 - 5$.

Solution. We calculate

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{[(2(x + \Delta x) - 1)^2 - 5] - [(2x-1)^2 - 5]}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{[(2x + 2\Delta x - 1)^2 - 5] - [(2x-1)^2 - 5]}{\Delta x} \end{aligned}$$

Now $(2x-1)^2 - 5 = 4x^2 - 4x + 1 - 5 = 4x^2 - 4x - 4$, while

$$(2x + 2\Delta x - 1)^2 - 5 = 4x^2 + 8x\Delta x - 4x + 4(\Delta x)^2 - 4\Delta x - 4$$

So

$$[(2x + 2\Delta x - 1)^2 - 5] - [(2x-1)^2 - 5] = 8x\Delta x + 4(\Delta x)^2 - 4\Delta x$$

and thus

$$\begin{aligned}
f'(x) &= \lim_{\Delta x\to 0}\frac{[(2x+2\Delta x-1)^2-5]-[(2x-1)^2-5]}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{8x\Delta x+4(\Delta x)^2-4\Delta x}{\Delta x}\\
&= \lim_{\Delta x\to 0} 8x+4\Delta x-4\\
&= 8x-4
\end{aligned}$$

Alternatively (and perhaps more efficiently) we could have first used that $f(x)=4x^2-4x-4$ and then use the definition on that function:

$$\begin{aligned}
f'(x) &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{[4(x+\Delta x)^2-4(x+\Delta x)-4]-[4x^2-4x-4]}{\Delta x}
\end{aligned}$$

Etcetera...

Problem 3. You deposit \$4,000 in a bank account. The bank pays 4% interest, compounded twice per year.

1. How much money is in your account after 3 years?
2. How long does it take before your balance is \$10,000?

Solution. For all these questions we use the formula for compound interest

$$A(t)=P(1+\frac{r}{n})^{nt}=4000(1+\frac{0.04}{2})^{2t}=4000(1.02)^{2t}.$$

1. Calculate $A(3)=4000(1.02)^{2\cdot 3}\approx \$4504$.
2. On the one hand we know $A(t)=4000(1.02)^{2t}$, on the other hand we want to know for which t we have $A(t)=10000$. Thus

$$\begin{aligned}
10000 &= 4000(1.02)^{2t}\\
2.5 &= (1.02)^{2t}\\
\ln(2.5) &= \ln(1.02^{2t})\\
\ln(2.5) &= 2t\ln(1.02)\\
t &= \frac{\ln 2.5}{2\ln 1.02}.
\end{aligned}$$

Problem 4. Find the slope of the tangent line of the graph of $\sqrt{xy}+x^2y^2=2$ at the point $(1,1)$.

Solution. We use implicit differentation and take derivatives on both sides, giving:

$$\begin{aligned}
[\sqrt{xy}+x^2y^2]' &= 1'\\
\frac{1}{2}(xy)^{-\frac{1}{2}}(xy)'+[2xy^2+x^2 2yy'] &= 0\\
\frac{y+xy'}{2\sqrt{xy}}+2xy^2+2x^2yy' &= 0
\end{aligned}$$

Fortunately we don't have to solve for y'. Just plug in $x = 1, y = 1$:

$$\begin{aligned}
\frac{1 + 1y'}{2\sqrt{1 \cdot 1}} + 2 \cdot 1 \cdot 1^2 + 2 \cdot 1^2 \cdot 1y' &= 0 \\
\frac{1 + y'}{2} + 2 + 2y' &= 0 \\
\frac{1}{2} + \frac{y'}{2} + 2 + 2y' &= 0 \\
\frac{5}{2}y' &= -\frac{5}{2} \\
y' &= -1
\end{aligned}$$