

## Deflections

**Q1.** For the beam and loading shown in Figure 1, compute (a) the slope of the beam at C and (b) the deflection of the beam at C. Assume a constant value of  $EI = 560 \times 10^6 \text{ N}\cdot\text{mm}^2$  for the beam.

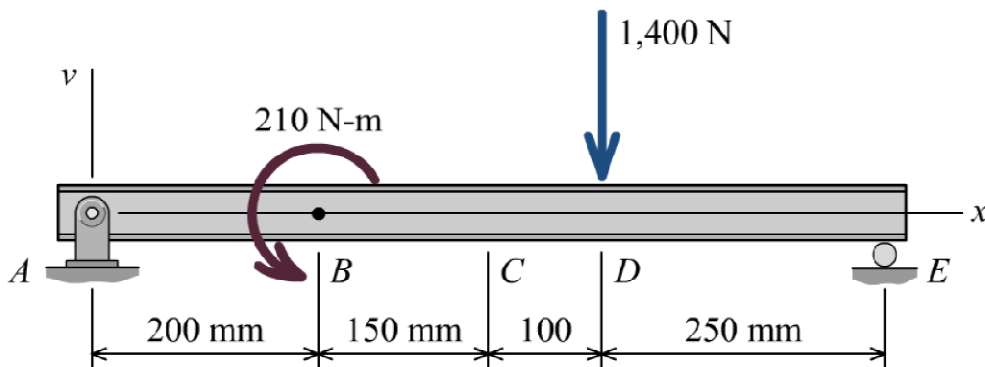


Figure 1

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### Solution

**Support reactions:** A FBD of the beam is shown to the right.

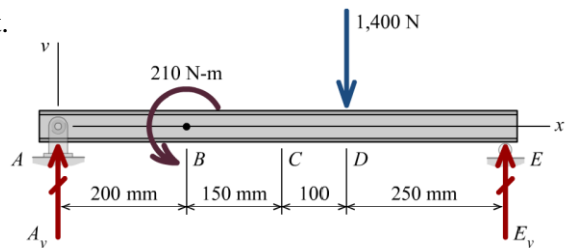
$$\sum M_A = (210 \text{ N} \cdot \text{m})(1,000 \text{ mm} / \text{m}) - (1,400 \text{ N})(450 \text{ mm})$$

$$+ E_y (700 \text{ mm}) = 0$$

$$\therefore E_y = 600 \text{ N}$$

$$\sum F_y = A_y + E_y - 1,400 \text{ N} = 0$$

$$\therefore A_y = 800 \text{ N}$$



**Load function  $w(x)$ :**

$$w(x) = 800 \text{ N} \langle x - 0 \text{ mm} \rangle^{-1} - 210,000 \text{ N} \cdot \text{mm} \langle x - 200 \text{ mm} \rangle^{-2} - 1,400 \text{ N} \langle x - 450 \text{ mm} \rangle^{-1} + 600 \text{ N} \langle x - 700 \text{ mm} \rangle^{-1}$$

**Shear-force function  $V(x)$  and bending-moment function  $M(x)$ :**

$$V(x) = 800 \text{ N} \langle x - 0 \text{ mm} \rangle^0 - 210,000 \text{ N} \cdot \text{mm} \langle x - 200 \text{ mm} \rangle^{-1} - 1,400 \text{ N} \langle x - 450 \text{ mm} \rangle^0 + 600 \text{ N} \langle x - 700 \text{ mm} \rangle^0$$

$$M(x) = 800 \text{ N} \langle x - 0 \text{ mm} \rangle^1 - 210,000 \text{ N} \cdot \text{mm} \langle x - 200 \text{ mm} \rangle^0 - 1,400 \text{ N} \langle x - 450 \text{ mm} \rangle^1 + 600 \text{ N} \langle x - 700 \text{ mm} \rangle^1$$

**Equations for beam slope and beam deflection:**

From Eq. (10.1), we can write:

$$EI \frac{d^2v}{dx^2} = M(x) = 800 \text{ N} \langle x - 0 \text{ mm} \rangle^1 - 210,000 \text{ N} \cdot \text{mm} \langle x - 200 \text{ mm} \rangle^0 - 1,400 \text{ N} \langle x - 450 \text{ mm} \rangle^1 + 600 \text{ N} \langle x - 700 \text{ mm} \rangle^1$$

Integrate the moment function to obtain an expression for the beam slope:

$$EI \frac{dv}{dx} = \frac{800N}{2} \langle x - 0mm \rangle^2 - 210,000N \cdot mm \langle x - 200mm \rangle^1 - \frac{1,400N}{2} \langle x - 450mm \rangle^2 + \frac{600N}{2} \langle x - 700mm \rangle^2 + C_1 \quad (a)$$

Integrate again to obtain the beam deflection function:

$$EIv = \frac{800N}{6} \langle x - 0mm \rangle^3 - \frac{210,000N \cdot mm}{2} \langle x - 200mm \rangle^2 - \frac{1,400N}{6} \langle x - 450mm \rangle^3 + \frac{600N}{6} \langle x - 700mm \rangle^3 + C_1x + C_2 \quad (b)$$

**Evaluate constants using boundary conditions:** Boundary conditions are specific values of deflection  $v$  or slope  $dv/dx$  that are known at particular locations along the beam span. For this beam, the deflection  $v$  is known at the pin support ( $x = 0$  mm) and at the roller support ( $x = 700$  mm). Substitute the boundary condition  $v = 0$  at  $x = 0$  mm into Eq. (b) to obtain:

$$C_2 = 0$$

Next, substitute the boundary condition  $v = 0$  at  $x = 700$  mm into Eq. (b) to obtain:

$$0 = \frac{800N}{6} \langle 700mm \rangle^3 - \frac{210,000N \cdot mm}{2} \langle 500mm \rangle^2 - \frac{1,400N}{6} \langle 250mm \rangle^3 + C_1(700mm)$$

$$\therefore C_1 = -22,625,000N \cdot mm^2$$

**The beam slope and elastic curve equations are now complete:**

$$EI \frac{dv}{dx} = \frac{800N}{2} \langle x - 0mm \rangle^2 - 210,000N \cdot mm \langle x - 200mm \rangle^1 - \frac{1,400N}{2} \langle x - 450mm \rangle^2 + \frac{600N}{2} \langle x - 700mm \rangle^2 - 22,625,000N \cdot mm^2$$

$$EIv = \frac{800N}{6} \langle x - 0mm \rangle^3 - \frac{210,000N \cdot mm}{2} \langle x - 200mm \rangle^2 - \frac{1,400N}{6} \langle x - 450mm \rangle^3 + \frac{600N}{6} \langle x - 700mm \rangle^3 - (22,625,000N \cdot mm^2)x$$

**(a) Beam slope at C:** The beam slope at C is:

$$EI \left. \frac{dv}{dx} \right|_C = \frac{800N}{2} (350mm)^2 - 210,000N \cdot mm(150mm) - 22,625,000N \cdot mm^2$$

$$\therefore \left. \frac{dv}{dx} \right|_C = -\frac{5.125 \times 10^6 N \cdot mm^2}{560 \times 10^6 N \cdot mm^2} = -0.009152rad = -0.00915rad$$

**(b) Beam deflection at C:** The beam deflection at C is:

$$EIv_C = \frac{800N}{6} (350mm)^3 - (210,000N \cdot mm)(150mm)^2 - (22,625,000N \cdot mm^2)(350mm)$$

$$= -4.564583 \times 10^9 N \cdot mm^3$$

$$\therefore v_C = \frac{-4.564583 \times 10^9 N \cdot mm^3}{560 \times 10^6 N \cdot mm^2} = -8.1510mm = 8.15mm \downarrow$$

**Q2.** The cantilever beam shown in Figure 2 consists of a W530 × 74 structural steel wide-flange shape [ $E = 200 \text{ GPa}$ ;  $I = 410 \times 10^6 \text{ mm}^4$ ]. Compute the deflection of the beam at C for the loading shown.

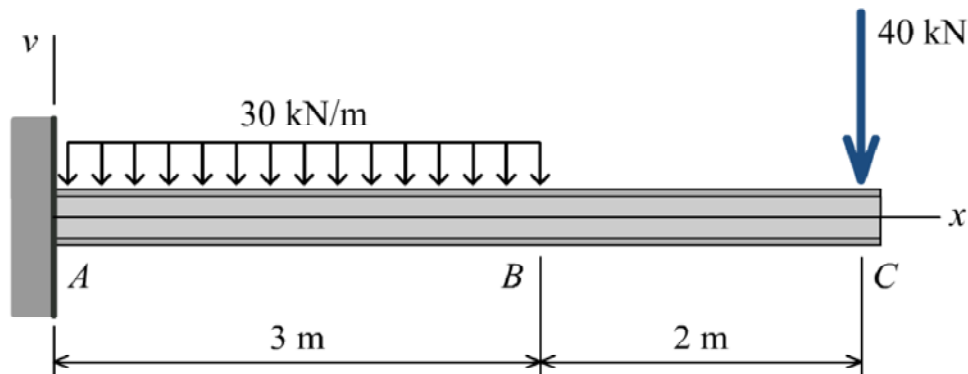


Figure 2

**Solution Support reactions:** A FBD of the beam is shown to the right.

$$\sum F_y = A_y - (30 \text{ kN/m})(3 \text{ m}) - 40 \text{ kN} = 0$$

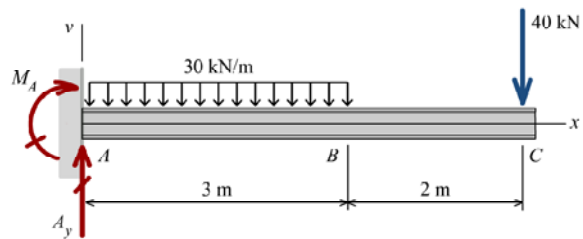
$$\therefore A_y = 130 \text{ kN}$$

$$\sum M_A = -(30 \text{ kN/m})(3 \text{ m})(1.5 \text{ m}) - (40 \text{ kN})(5 \text{ m}) - M_A = 0$$

$$\therefore M_A = -335 \text{ kN} \cdot \text{m}$$

**Load function  $w(x)$ :**

$$w(x) = 130 \text{ kN} \langle x - 0 \text{ m} \rangle^{-1} - 335 \text{ kN} \cdot \text{m} \langle x - 0 \text{ m} \rangle^{-2} - 30 \text{ kN/m} \langle x - 0 \text{ m} \rangle^0 + 30 \text{ kN} \langle x - 3 \text{ m} \rangle^0 - 40 \text{ kN} \langle x - 5 \text{ m} \rangle^{-1}$$



**Shear-force function  $V(x)$  and bending-moment function  $M(x)$ :**

$$V(x) = 130 \text{ kN} \langle x - 0 \text{ m} \rangle^0 - 335 \text{ kN} \cdot \text{m} \langle x - 0 \text{ m} \rangle^{-1} - 30 \text{ kN/m} \langle x - 0 \text{ m} \rangle^1 + 30 \text{ kN} \langle x - 3 \text{ m} \rangle^1 - 40 \text{ kN} \langle x - 5 \text{ m} \rangle^0$$

$$M(x) = 130 \text{ kN} \langle x - 0 \text{ m} \rangle^1 - 335 \text{ kN} \cdot \text{m} \langle x - 0 \text{ m} \rangle^0 - \frac{30 \text{ kN/m}}{2} \langle x - 0 \text{ m} \rangle^2 + \frac{30 \text{ kN}}{2} \langle x - 3 \text{ m} \rangle^2 - 40 \text{ kN} \langle x - 5 \text{ m} \rangle^1$$

**Equations for beam slope and beam deflection:**

From Eq. (10.1), we can write:

$$EI \frac{d^2v}{dx^2} = M(x) = 130 \text{ kN} \langle x - 0 \text{ m} \rangle^1 - 335 \text{ kN} \cdot \text{m} \langle x - 0 \text{ m} \rangle^0 - \frac{30 \text{ kN/m}}{2} \langle x - 0 \text{ m} \rangle^2 + \frac{30 \text{ kN}}{2} \langle x - 3 \text{ m} \rangle^2 - 40 \text{ kN} \langle x - 5 \text{ m} \rangle^1$$

Integrate the moment function to obtain an expression for the beam slope:

$$EI \frac{dv}{dx} = \frac{130kN}{2} \langle x - 0m \rangle^2 - 335kN \cdot m \langle x - 0m \rangle^1 - \frac{30kN/m}{6} \langle x - 0m \rangle^3 + \frac{30kN/m}{6} \langle x - 3m \rangle^3 - \frac{40kN}{2} \langle x - 5m \rangle^2 + C_1 \quad (a)$$

Integrate again to obtain the beam deflection function:

$$EIv = \frac{130kN}{6} \langle x - 0m \rangle^3 - \frac{335kN \cdot m}{2} \langle x - 0m \rangle^2 - \frac{30kN/m}{24} \langle x - 0m \rangle^4 + \frac{30kN/m}{24} \langle x - 3m \rangle^4 - \frac{40kN}{6} \langle x - 5m \rangle^3 + C_1x + C_2 \quad (b)$$

**Evaluate constants using boundary conditions:** Boundary conditions are specific values of deflection  $v$  or slope  $dv/dx$  that are known at particular locations along the beam span. For this beam, both the slope  $dv/dx$  and the deflection  $v$  are known at the fixed support ( $x = 0$  m). Substitute the boundary condition  $dv/dx = 0$  at  $x = 0$  m into Eq. (a) to obtain:

$$C_1 = 0$$

Next, substitute the boundary condition  $v = 0$  at  $x = 0$  m into Eq. (b) to obtain:

$$C_2 = 0$$

**The beam slope and elastic curve equations are now complete:**

$$EI \frac{dv}{dx} = \frac{130kN}{2} \langle x - 0m \rangle^2 - 335kN \cdot m \langle x - 0m \rangle^1 - \frac{30kN/m}{6} \langle x - 0m \rangle^3 + \frac{30kN/m}{6} \langle x - 3m \rangle^3 - \frac{40kN}{2} \langle x - 5m \rangle^2$$

$$EIv = \frac{130kN}{6} \langle x - 0m \rangle^3 - \frac{335kN \cdot m}{2} \langle x - 0m \rangle^2 - \frac{30kN/m}{24} \langle x - 0m \rangle^4 + \frac{30kN/m}{24} \langle x - 3m \rangle^4 - \frac{40kN}{6} \langle x - 5m \rangle^3$$

**Beam deflection at C:** For the W530 × 74 structural steel wide-flange shape,  $EI = 82,000$  kN-m<sup>2</sup>. At the tip of the overhang where  $x = 5$  m, the beam deflection is:

$$EIv_C = \frac{130kN}{6} (5m)^3 - \frac{335kN \cdot m}{2} (5m)^2 - \frac{30kN/m}{24} (5m)^4 + \frac{30kN/m}{24} (2m)^4$$

$$= -2,240.416667kN \cdot m^3$$

$$\therefore v_C = \frac{2,240.416667kN \cdot m^3}{82,000kN \cdot m^2} = -0.027322m = 27.3mm \downarrow$$

**Q3.** The simply supported beam shown in Figure 3 consists of a W200 × 59 structural steel wide-flange shape [ $E = 200 \text{ GPa}$ ;  $I = 60.8 \times 10^6 \text{ mm}^4$ ]. For the loading shown, compute (a) the deflection of the beam at C and (b) the deflection of the beam at F.

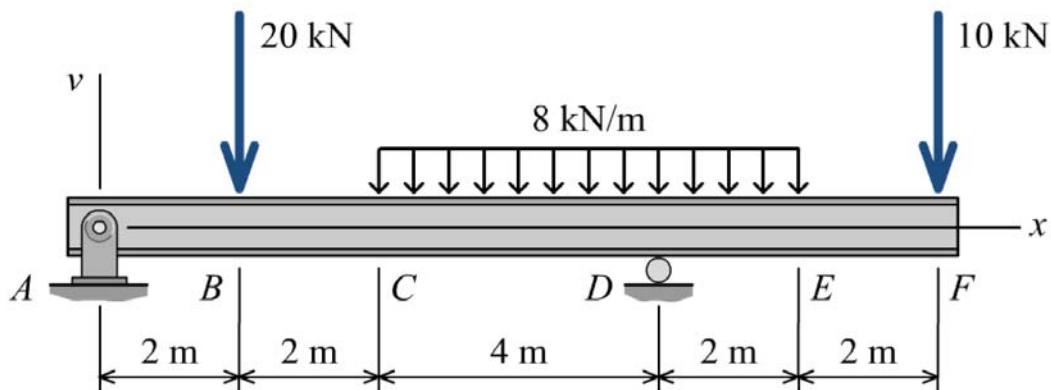


Figure 3

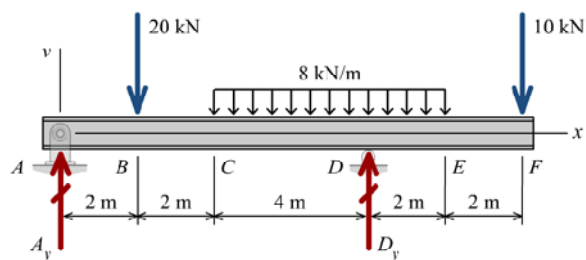
**Solution Support reactions:** A FBD of the beam is shown to the right.

$$\sum M_A = -(20 \text{ kN})(2 \text{ m}) - (8 \text{ kN/m})(6 \text{ m})(7 \text{ m}) - (10 \text{ kN})(12 \text{ m}) + D_y(8 \text{ m}) = 0$$

$$\therefore D_y = 62 \text{ kN}$$

$$\sum F_y = A_y + D_y - 20 \text{ kN} - (8 \text{ kN/m})(6 \text{ m}) - 10 \text{ kN} = 0$$

$$\therefore A_y = 16 \text{ kN}$$



**Load function  $w(x)$ :**

$$w(x) = 16 \text{ kN} \langle x - 0 \text{ m} \rangle^{-1} - 20 \text{ kN} \langle x - 2 \text{ m} \rangle^{-1} - 8 \text{ kN/m} \langle x - 4 \text{ m} \rangle^0 + 62 \text{ kN} \langle x - 8 \text{ m} \rangle^{-1} + 8 \text{ kN} \langle x - 10 \text{ m} \rangle^0 - 10 \text{ kN} \langle x - 12 \text{ m} \rangle^{-1}$$

**Shear-force function  $V(x)$  and bending-moment function  $M(x)$ :**

$$V(x) = 16 \text{ kN} \langle x - 0 \text{ m} \rangle^0 - 20 \text{ kN} \cdot m \langle x - 2 \text{ m} \rangle^0 - 8 \text{ kN/m} \langle x - 4 \text{ m} \rangle^1 + 62 \text{ kN} \langle x - 8 \text{ m} \rangle^0 + 8 \text{ kN} \langle x - 10 \text{ m} \rangle^1 - 10 \text{ kN} \langle x - 12 \text{ m} \rangle^0$$

$$M(x) = 16 \text{ kN} \langle x - 0 \text{ m} \rangle^1 - 20 \text{ kN} \langle x - 2 \text{ m} \rangle^1 - \frac{8 \text{ kN/m}}{2} \langle x - 4 \text{ m} \rangle^2 + 62 \text{ kN} \langle x - 8 \text{ m} \rangle^1 + \frac{8 \text{ kN}}{2} \langle x - 10 \text{ m} \rangle^2 - 10 \text{ kN} \langle x - 12 \text{ m} \rangle^1$$

### Equations for beam slope and beam deflection:

From Eq. (10.1), we can write:

$$EI \frac{d^2v}{dx^2} = M(x) = 16kN \langle x - 0m \rangle^1 - 20kN \langle x - 2m \rangle^1 - \frac{8kN/m}{2} \langle x - 4m \rangle^2 + 62kN \langle x - 8m \rangle^1 + \frac{8kN/m}{2} \langle x - 10m \rangle^2 - 10kN \langle x - 12m \rangle^1$$

Integrate the moment function to obtain an expression for the beam slope:

$$EI \frac{dv}{dx} = \frac{16kN}{2} \langle x - 0m \rangle^2 - \frac{20kN}{2} \langle x - 2m \rangle^2 - \frac{8kN/m}{6} \langle x - 4m \rangle^3 + \frac{62kN}{2} \langle x - 8m \rangle^2 + \frac{8kN/m}{6} \langle x - 10m \rangle^3 - \frac{10kN}{2} \langle x - 12m \rangle^2 + C_1$$

Integrate again to obtain the beam deflection function:

$$EIv = \frac{16kN}{6} \langle x - 0m \rangle^3 - \frac{20kN}{6} \langle x - 2m \rangle^3 - \frac{8kN/m}{24} \langle x - 4m \rangle^4 + \frac{62kN}{6} \langle x - 8m \rangle^3 + \frac{8kN/m}{24} \langle x - 10m \rangle^4 - \frac{10kN}{6} \langle x - 12m \rangle^3 + C_1x + C_2$$

**Evaluate constants using boundary conditions:** Boundary conditions are specific values of deflection  $v$  or slope  $dv/dx$  that are known at particular locations along the beam span. For this beam, the deflection  $v$  is known at the pin support ( $x = 0$  m) and at the roller support ( $x = 8$  m). Substitute the boundary condition  $v = 0$  at  $x = 0$  m into Eq. (b) to obtain:

$$C_2 = 0$$

Next, substitute the boundary condition  $v = 0$  at  $x = 8$  m into Eq. (b) to obtain:

$$0 = \frac{16kN}{6} (8m)^3 - \frac{20kN}{6} (6m)^3 - \frac{8kN/m}{24} (4m)^4 + C_1(8m)$$

$$\therefore C_1 = -70kN \cdot m^2$$

**The beam slope and elastic curve equations are now complete:**

$$EI \frac{dv}{dx} = \frac{16kN}{2} \langle x - 0m \rangle^2 - \frac{20kN}{2} \langle x - 2m \rangle^2 - \frac{8kN/m}{6} \langle x - 4m \rangle^3 + \frac{62kN}{2} \langle x - 8m \rangle^2 + \frac{8kN/m}{6} \langle x - 10m \rangle^3 - \frac{10kN}{2} \langle x - 12m \rangle^2 - 70kN \cdot m^2$$

$$EIv = \frac{16kN}{6} \langle x - 0m \rangle^3 - \frac{20kN}{6} \langle x - 2m \rangle^3 - \frac{8kN/m}{24} \langle x - 4m \rangle^4 + \frac{62kN}{6} \langle x - 8m \rangle^3 + \frac{8kN/m}{24} \langle x - 10m \rangle^4 - \frac{10kN}{6} \langle x - 12m \rangle^3 - (70kN \cdot m^2)x$$

**(a) Beam deflection at C:** For the W200 × 59 structural steel wide-flange shape,  $EI = 12,160$  kN-m<sup>2</sup>. At C where  $x = 4$  m, the beam deflection is:

$$EIv_C = \frac{16kN}{6} (4m)^3 - \frac{20kN}{6} (2m)^3 - (70kN \cdot m^2)(4m) = -136kN \cdot m^3$$

$$\therefore v_C = \frac{-136kN \cdot m^3}{12,160kN \cdot m^2} = -0.011184m = 11.18mm \downarrow$$

(b) **Beam deflection at  $F$ :** At  $F$  where  $x = 12$  m, the beam deflection is:

$$\begin{aligned}EIv_F &= \frac{16kN}{6}(12m)^3 - \frac{20kN}{6}(10m)^3 - \frac{8kN/m}{24}(8m)^4 + \frac{62kN}{6}(4m)^3 + \frac{8kN/m}{24}(2m)^4 - (70kN \cdot m^2)(12m) \\ &= -264kN \cdot m^3 \\ \therefore v_F &= \frac{-264kN \cdot m^3}{12,160kN \cdot m^2} = -0.021711m = 21.7mm \downarrow\end{aligned}$$