

P1.1a $f(t) = tu(t)$

$$F(s) = \int_{0^-}^{\infty} tu(t)e^{-st} dt = \int_{0^-}^{\infty} te^{-st} dt$$

use integration by parts

$$= -\frac{t}{s} e^{-st} \Big|_{0^-}^{\infty} + \int_{0^-}^{\infty} \frac{e^{-st}}{s} dt$$

$$\begin{aligned} u &= t & dv &= e^{-st} dt \\ du &= dt & v &= -\frac{e^{-st}}{s} \end{aligned}$$

$$= \frac{e^{-st}}{s^2} (-st - 1) \Big|_{0^-}^{\infty}$$

Using l'Hopital's rule $\rightarrow F(s) \Big|_{t \rightarrow \infty} = \frac{-e^{-st}}{s^3} s \Big|_{t \rightarrow \infty} = 0$

$$\therefore F(s) = \frac{1}{s^2}$$

b) $f(t) = \cos \omega t u(t)$

$$F(s) = \int_{0^-}^{\infty} \cos \omega t e^{-st} dt$$

Integration by parts

$$= \frac{-e^{-st} \cos \omega t}{s} \Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} \frac{we^{-st}}{s} \sin \omega t dt$$

$$\begin{aligned} u &= \cos \omega t & dv &= e^{-st} dt \\ du &= -\omega \sin \omega t dt & v &= -\frac{e^{-st}}{s} \end{aligned}$$

$$= \frac{-e^{-st} \cos \omega t}{s} \Big|_{0^-}^{\infty} - \frac{\omega}{s} \left[\frac{-e^{-st}}{s} \sin \omega t + \frac{\omega}{s} \int_{0^-}^{\infty} e^{-st} \cos \omega t dt \right]$$

Integration by parts

$$\begin{aligned} u &= \sin \omega t & dv &= e^{-st} dt \\ du &= \omega \cos \omega t dt & v &= -\frac{e^{-st}}{s} \end{aligned}$$

Combining common integrals

$$\left[1 + \frac{\omega^2}{s^2} \right] \int_{0^-}^{\infty} e^{-st} \cos \omega t dt = \frac{e^{-st}}{s} \left[-\cos \omega t + \frac{\omega}{s} \sin \omega t \right]_{0^-}^{\infty} = \frac{1}{s}$$

$$\therefore F(s) = \int_{0^-}^{\infty} \cos \omega t e^{-st} dt = \frac{\frac{1}{s}}{1 + \frac{\omega^2}{s^2}} = \frac{s}{s^2 + \omega^2}$$

P1.2

$$y(t) = r(t) - r(t-2) + u(t-1) - 3u(t-4)$$

$$\mathcal{L}[y(t)] = \frac{1}{s^2} - e^{-2s} \frac{1}{s^2} + e^{-s} \frac{1}{s} - 3e^{-4s} \frac{1}{s}$$

P1.3

$$\frac{d^3 y}{dt^3} + 3 \frac{d^2 y}{dt^2} + 5 \frac{dy}{dt} + y = \frac{d^3 x}{dt^3} + 4 \frac{d^2 x}{dt^2} + 6 \frac{dx}{dt} + 8x$$

Laplace transform (ICs = 0)

$$s^3 Y(s) + 3s^2 Y(s) + 5s Y(s) + Y(s) = s^3 X(s) + 4s^2 X(s) + 6s X(s) + 8X(s)$$

$$\frac{Y(s)}{X(s)} = \frac{s^3 + 4s^2 + 6s + 8}{s^3 + 3s^2 + 5s + 1}$$

P1.4 a) $\ddot{x}(t) + 2\dot{x}(t) + 7x(t) = f(t)$
(5th edition)

b) $\ddot{x}(t) + 15\dot{x}(t) + 56x(t) = 10f(t)$

c) $\ddot{x}(t) + 8\dot{x}(t) + 9x(t) = \dot{f}(t) + 2f(t)$

P1.4 a) $\ddot{x}(t) + 5\dot{x}(t) + 10x(t) = 7f(t)$

(6th edition) b) $\ddot{x}(t) + 21\dot{x}(t) + 110x(t) = 15f(t)$

c) $\ddot{x}(t) + 11\dot{x}(t) + 12x(t) = \dot{f}(t) + 3f(t)$

P1.5

$$a) F_1(s) = \frac{5(s+2)}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

$$A = \frac{5(s+2)}{s+1} \Big|_{s=0} = \frac{10}{1} = 10$$

$$B = \frac{5(s+2)}{s} \Big|_{s=-1} = \frac{5(-1+2)}{-1} = -5$$

$$f_1(t) = 10u(t) - 5e^{-t}u(t)$$

$$b) f_1(0^+) = \lim_{s \rightarrow \infty} s F_1(s) = \lim_{s \rightarrow \infty} \frac{5(s+2)}{s+1} = 5$$

$$c) f_1(\infty) = \lim_{s \rightarrow 0} s F_1(s) = \lim_{s \rightarrow 0} \frac{5(s+2)}{s+1} = 10$$

$$d) f_1(0^+) = 10 - 5e^0 = 5$$

$$f_1(\infty) = 10 - 5e^{\overset{0}{\infty}} = 10$$

$$a) F_2(s) = \frac{s^2 + 2s + 4}{s^2} = 1 + \frac{2}{s} + \frac{4}{s^2}$$

$$f_2(t) = \delta(t) + 2u(t) + 4r(t) = \delta(t) + 2 + 4t \text{ for } t \geq 0$$

$$b) f_2(0^-) = \lim_{s \rightarrow \infty} s F_2(s) = \lim_{s \rightarrow \infty} s + 2 + \frac{4}{s} = \infty \leftarrow$$

$$c) f_2(\infty) \rightarrow s F_2(s) = \frac{s^2 + 2s + 4}{s} \leftarrow \text{No final value due to } s$$

$$d) \lim_{t \rightarrow 0^+} f_2(t) = \lim_{t \rightarrow 0^+} \delta(t) + 2 + 4t = \boxed{2} \leftarrow \text{Do not match due to } \delta(t) \text{ at } t=0$$

$$\lim_{t \rightarrow \infty} f_2(t) = \lim_{t \rightarrow \infty} \delta(t) + 2 + 4t = \infty$$

← Confirms no final value

$$\square \text{ a) } F_3(s) = \frac{3s+7}{4s^2+24s+136} = \frac{3s+7}{4(s+3-5j)(s+3+5j)}$$

$$= \frac{c_1}{s+3-5j} + \frac{c_2}{s+3+5j}$$

$$c_1 = \lim_{s \rightarrow -3+5j} (s+3-5j)F_3(s) = \lim_{s \rightarrow -3+5j} \frac{3s+7}{4(s+3+5j)}$$

$$= \frac{-2+15j}{40j} = \frac{15+2j}{40}$$

$$= \left| \frac{15+2j}{40} \right| e^{j\phi} = \frac{\sqrt{229}}{40} e^{j\phi}$$

where $\phi = \tan^{-1}(2/15) = 0.1326$ rads

$$c_2 = \bar{c}_1$$

$$\therefore \underline{f_3(t)} = 2 |c_1| e^{-3t} \cos(5t + \phi) = \frac{\sqrt{229}}{20} e^{-3t} \cos(5t + 0.1326)$$

Alternatively, completing the square gives

$$F_3(s) = \frac{1}{4} \frac{3s+7}{(s+3)^2+25} = \frac{1}{4} \left[\frac{3(s+3)}{(s+3)^2+5^2} + \frac{(-\frac{2}{5})5}{(s+3)^2+5^2} \right]$$

$$\underline{f_3(t)} = \frac{3}{4} e^{-3t} \cos(5t)u(t) - \frac{1}{10} e^{-3t} \sin(5t)u(t)$$

$$\text{b) } f_3(0^+) = \lim_{s \rightarrow \infty} sF_3(s) = \lim_{s \rightarrow \infty} \frac{3s^2+7s}{4s^2+24s+136} = \frac{3}{4}$$

$$\text{c) } f_3(\infty) = \lim_{s \rightarrow 0} sF_3(s) = 0$$

$$\text{d) } f_3(0^+) = \frac{3}{4} e^0(1) - \frac{1}{10} e^0(0) = \frac{3}{4}$$

$$\lim_{t \rightarrow \infty} f_3(t) = \frac{3}{4}(0) - \frac{1}{10}(0) = 0$$