CHG 2314
Heat Transfer

Part 2a

Introduction to Conduction

- Fourier's Law
- Heat diffusion equation
- Thermal properties

Fourier’s Law

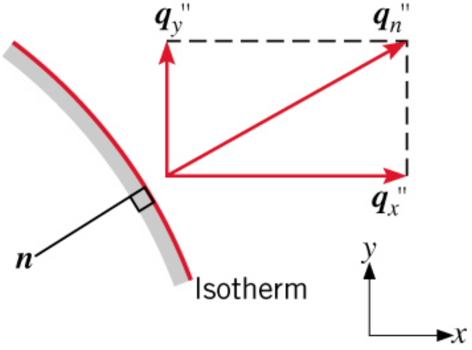
- **Fourier’s law** is a rate equation that allows determination of the conduction heat flux from knowledge of the temperature distribution in a medium
- In most general (vector) form for multidimensional conduction Fourier’s law is:

$$\vec{q}'' = -k\vec{\nabla}T$$

- Implications of Fourier’s law:
  - Heat transfer is in the direction of decreasing temperature (basis for negative sign)
  - Fourier’s law servers to define the thermal conductivity of the medium:

$$k \equiv -\vec{q}''/\vec{\nabla}T$$

  - Direction of heat transfer is perpendicular to the lines of constant temperature (isotherms)
  - Heat flux vector may be resolved into orthogonal components

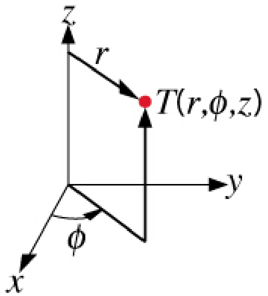


Fourier's Law in different coordinates

- **Cartesian coordinates:** *T*(*x*,*y*,*z*)

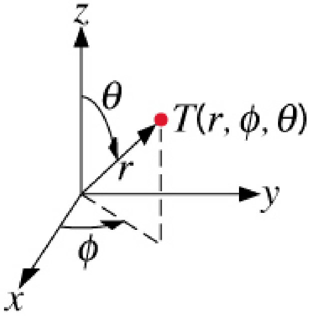
$$(2.3)\quad \vec{q}'' = \underbrace{-k\frac{\partial T}{\partial x}}_{q''_x}\vec{i} \underbrace{-k\frac{\partial T}{\partial y}}_{q''_y}\vec{j} \underbrace{-k\frac{\partial T}{\partial z}}_{q''_z}\vec{k}$$

- **Cylindrical coordinates:** *T*(*r*,ϕ,*z*)



$$(2.22)\quad \vec{q}'' = \underbrace{-k\frac{\partial T}{\partial r}}_{q''_r}\vec{i} \underbrace{-k\frac{\partial T}{r\partial \phi}}_{q''_\phi}\vec{j} \underbrace{-k\frac{\partial T}{\partial z}}_{q''_z}\vec{k}$$

- **Spherical coordinates:** *T*(*r*,ϕ,θ)



$$(2.25)\quad \vec{q}'' = \underbrace{-k\frac{\partial T}{\partial r}}_{q''_r}\vec{i} \underbrace{-k\frac{\partial T}{r\partial \theta}}_{q''_\theta}\vec{j} \underbrace{-k\frac{\partial T}{r\sin\theta\,\partial \phi}}_{q''_\phi}\vec{k}$$

Fourier’s Law and heat rate

- Cartesian coordinates (*x,y,z*) are all equivalent
  - When Fourier law is simplified to one dimension, it can be written in terms of *x, y,* or *z* coordinate
  - In Cartesian coordinates typically the area for heat conduction does not change in the direction of flow so that analysis in terms of heat flux and heat flow are the same
- Angular (cylindrical or spherical) coordinates $(\phi \text{ or } \phi, \theta)$ are not equivalent
  - When Fourier law in angular coordinates is simplified to one dimension, it becomes:

$$q'' = -k\frac{dT}{dr}$$

  - In one-dimensional heat conduction in angular coordinates, the area for heat conduction changes in the direction of flow:

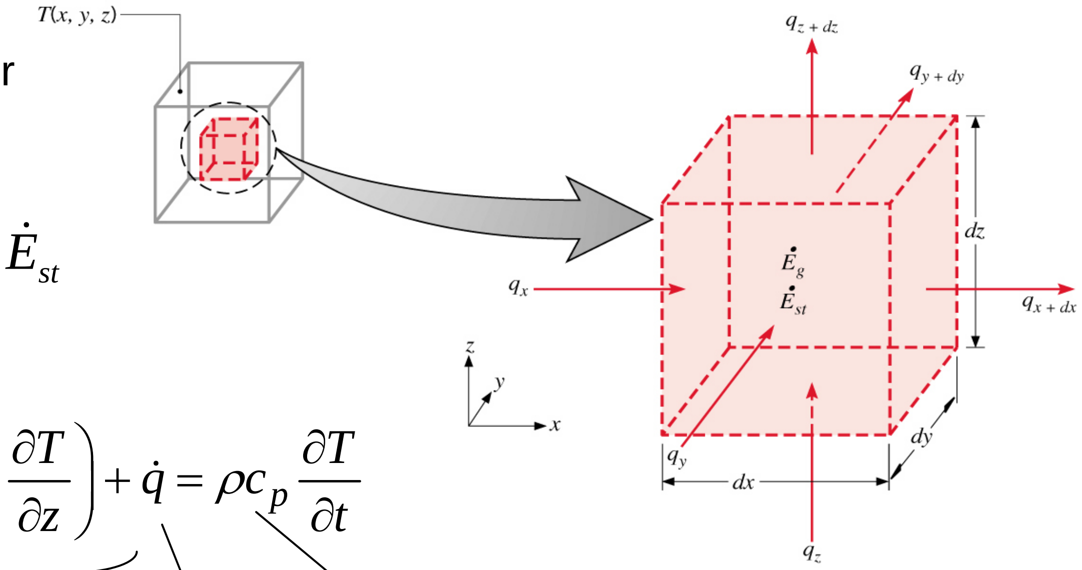
$$\text{Cylinder: } q_r = A_r q_r'' = 2\pi r L q_r'' \quad \text{or} \quad q_r' = A_r' q_r'' = 2\pi r q_r'' \quad \text{Sphere: } q_r = A_r q_r'' = 4\pi r^2 2\pi r L q_r''$$

NB: In angular coordinates $(\phi \text{ or } \phi, \theta)$, the temperature gradient is still based on temperature change over a length scale and hence has units of °C/m and not °C/deg.

Heat (Diffusion) Equation

- A differential equation whose solution provides the temperature distribution in stationary medium.
- Based on applying conservation of energy to a differential control volume through which energy transfer is exclusively by conduction.
- Cartesian coordinates:
  - Applying conservation law for closed system to the control volume:

$$(1.11) \qquad \dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = \dot{E}_{st}$$



  - Leads eventually to:

$$(2.17) \quad \underbrace{\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)} + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$

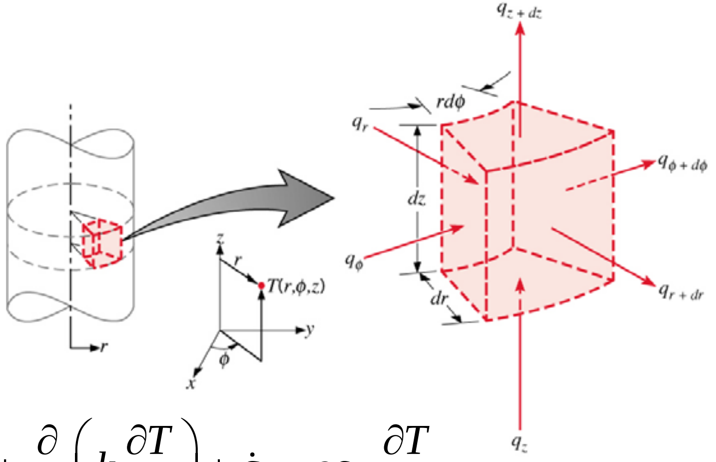
Net transfer of thermal energy into the control volume (inflow-outflow)

Thermal energy generation

Change in thermal energy storage

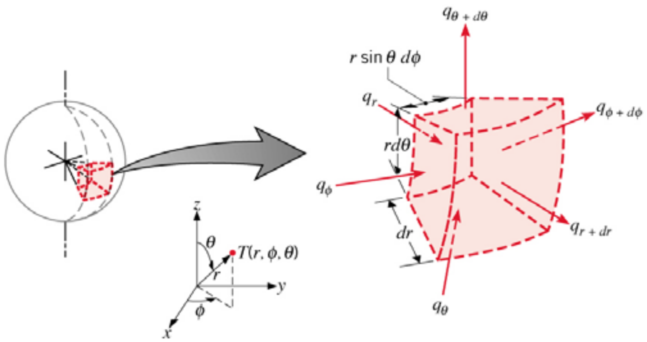
Heat (Diffusion) Equation

- Cylindrical coordinates



$$(2.24)\qquad \frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t}$$

- Cartesian coordinates:



$$(2.27)\qquad \frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta\frac{\partial T}{\partial \theta}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t}$$

Special cases of the Heat Equation in Cartesian Coordinates

- The first special case of the heat equation is the case of constant k for which Eq. (2.17) becomes:

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{q}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad (2.19)$$

- Under steady state, there is no change in the amount of energy storage, and Eq. (2.17) becomes:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=0 \qquad (2.20)$$

- For steady state, one-dimensional conduction without energy generation, Eq. (2.17) reduces to:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)=0 \qquad (2.21)$$

- Poisson Equation –steady state and constant *k*:

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=-\frac{\dot{q}}{k}$$

- Laplace’s equation – steady state, constant k, and no internal heat generation:

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=0$$

Thermal diffusivity

- Recall the heat (diffusion) equation with constant *k*

$$(2.19) \qquad \frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{q}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

- Term “diffusion” comes from the coefficient α which referred to as **thermal diffusivity**

$$\alpha=\frac{k}{\rho c_p}$$

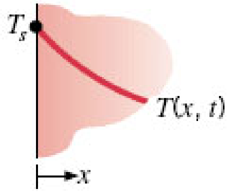
  - Thermal diffusivity is a measure of the ability of a material to ***conduct*** thermal energy relative to its ability to ***store*** thermal energy
  - The Heat (Diffusion) equation is analogous to the Navier-Stokes equation in momentum transfer and Fick’s 2nd law of diffusion in mass transfer
  - The coefficient α is called **kinematic viscosity** in momentum transfer and **diffusion coefficient** in mass transfer
  - Regardless of the transport process the units of α are the same: [m^2/s]

Boundary and initial conditions

- For transient conduction, heat equation is first order in time, requiring specification of an initial temperature distribution: $T(x,t)_{t=0} = T(x,0)$
- Since heat equation is second order in space, two boundary conditions must be specified. Some common cases:

1. Constant surface temperature

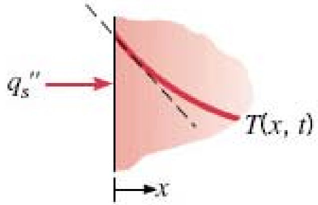
$$T(0,t) = T_s \qquad (2.29)$$



2. Constant surface heat flux

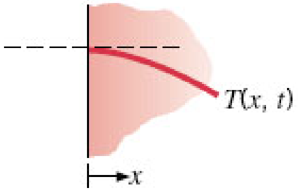
(a) Finite heat flux

$$-k\frac{\partial T}{\partial x}\bigg|_{x=0} = q_s''$$



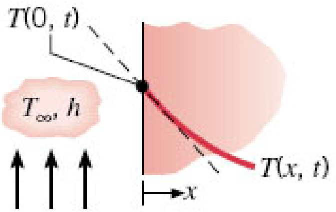
(b) Adiabatic or insu

$$\frac{\partial T}{\partial x}\bigg|_{x=0} = 0$$



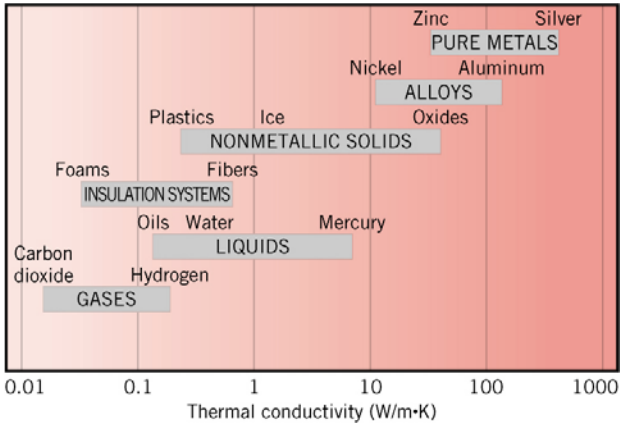
3. Convection surface condition

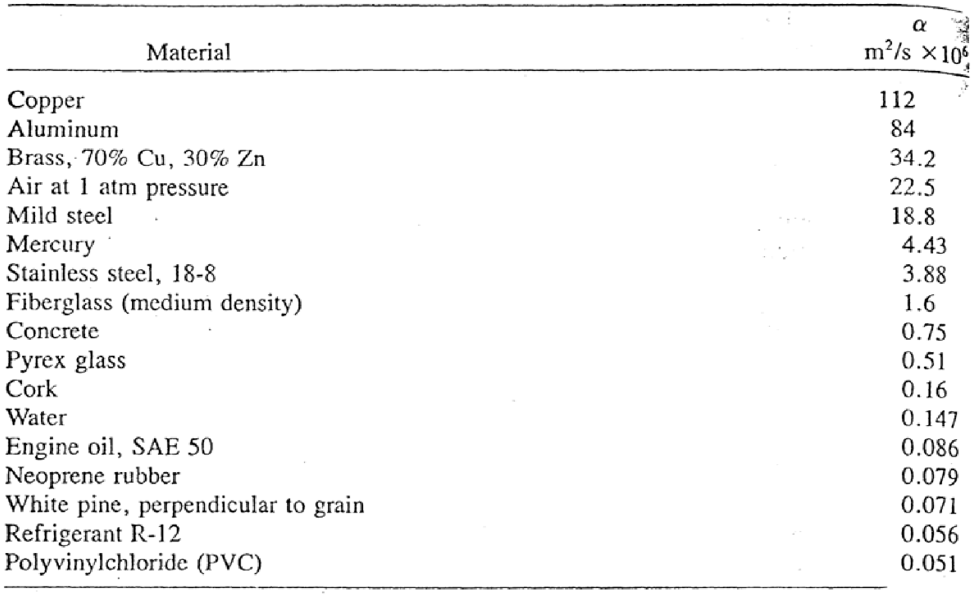
$$-k\frac{\partial T}{\partial x}\bigg|_{x=0} = h[T_\infty - T(0,t)] \qquad (2.32)$$



Thermophysical properties

- **Thermal Conductivity** *(k):* A measure of a material's ability to transfer thermal energy by conduction
- **Thermal diffusivity** (α): a measure of a material's ability to respond to changes in its thermal environment





Material	α $m^2/s \times 10^6$
Copper	112
Aluminum	84
Brass, 70% Cu, 30% Zn	34.2
Air at 1 atm pressure	22.5
Mild steel	18.8
Mercury	4.43
Stainless steel, 18-8	3.88
Fiberglass (medium density)	1.6
Concrete	0.75
Pyrex glass	0.51
Cork	0.16
Water	0.147
Engine oil, SAE 50	0.086
Neoprene rubber	0.079
White pine, perpendicular to grain	0.071
Refrigerant R-12	0.056
Polyvinylchloride (PVC)	0.051

Property Tables:

Solids:	Tables A.1 – A.3
Gases:	Table A.4
Liquids:	Tables A.5 – A.7

Thermal Conductivity

- In **Solid State** conduction may be viewed as a consequence of energy carrier (electron or phonon) motion:

$$(2.7)\quad k = \frac{1}{3}\underbrace{C}_{\substack{\text{energy carrier}\\ \text{specific heat per}\\ \text{unit volume.}}}\underbrace{\bar{c}}_{\text{average energy carrier velocity,}}\underbrace{\lambda_{mfp}}_{\substack{\text{mean free path} \rightarrow \text{average distance}\\ \text{traveled by an energy carrier before}\\ \text{a collision.}}}$$

  - Conductive solids: energy carriers are free electrons (k_e)
  - Nonconducors and semiconductors: energy carrier are photons (lattice vibration quanta)
  - Effective thermal conductivity in solids:

$$(2.8)\quad k = k_e + k_{ph}$$

    - For metals, $k_e >> k_{ph}$ and $k \approx k_e$; k_e is inversly proportional to electrical resistivity
    - For nonmetallic solids, $k_{ph} >> k_e$ and $k \approx k_{ph}$
    - For alloys, both k_e and k_{ph} are important
    - Highly crystalline solids (quartz, diamond) may have k_{ph} greater than k_e of metals

Thermal Conductivity

- In **Fluid State** (liquids and gases) conduction may be viewed as a consequence of molecular collisions
- Kinetic theory of gases:

$$k = \frac{1}{3}\underbrace{c_v}_{\text{specific heat per unit volume of fluid.}}\underbrace{\rho}_{\text{average density of fluid,}}\underbrace{\bar{c}}_{\text{mean molecular speed}}\;\underbrace{\lambda_{mfp}}_{\text{mean free path of molecules}} \tag{2.10}$$

  - Thermal conductivity of fluids is generally smaller than that of solids
  - Since liquids are denser than gases, thermal conductivity of liquids is greater than that of gases
  - For gases, the density is directly proportional while the mean free path is inversely proportional to pressure; therefore *k* is generally independent of pressure
  - As temperature increases, the mean molecular speed increases so *k* generally increases, but there are exceptions for liquids

Thermal Conductivity: Micro- and Nanoscale Effects

- The thermal conductivities provided in property tables are **bulk values**
  - When physical dimensions of a medium in any direction become very small, the collisions of energy carriers with physical boundaries may significantly affect their propagation
  - Consider to solid films made from the same material but having different thicknesses, $L_1 > L_2$:

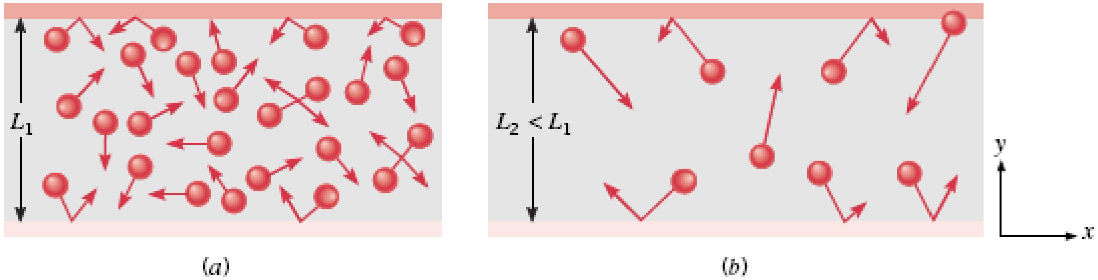


FIGURE 2.6 Electron or phonon trajectories in (*a*) a relatively thick film and (*b*) a relatively thin film with boundary effects.

Explain why:

$k_y < k_x \leq k$, where k is bulk thermal conductivity

For $L/\lambda_{mfp} < 1$:
$$k_x / k = 1 - 2\lambda_{mfp} / \left(3\pi L\right)$$
$$k_y / k = 1 - \lambda_{mfp} / \left(3L\right)$$

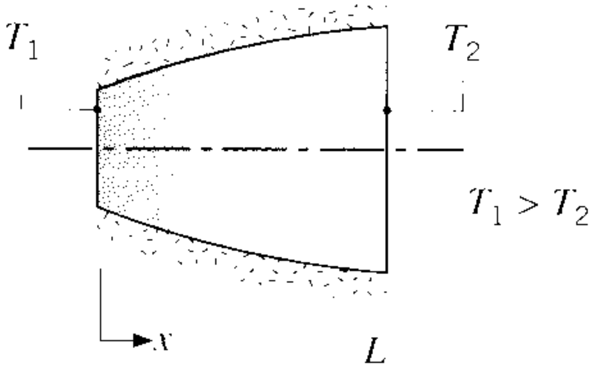
Insulation Systems

- To take advantage of the low thermal conductivity of air, insulation systems typically combine solid and gas (void space) in different geometries
  - *Conventional* – solid material (e.g., fiber, powder, flake) finely dispersed throughout an air space
  - *Cellular systems* – small voids within a rigid solid matrix, sealed from each other
  - *Refractive insulations* – multilayer parallel thin sheets or foils of high reflectivity spaced in such a way that the motion of air between the sheets is restricted
  - In high performance insulation the void space is evacuated

- Thermal conductivity of an insulation therefore represents an **effective thermal conductivity**, which depends on
  - Thermal conductivity of the solid phase
  - Thermal conductivity of the gas phase (void space)
  - Surface radiative properties of the solid phase
  - Nature and volumetric fraction of the void space

Example 1 (Problem 2.1)

Assume steady-state, one-dimensional conduction through the axisymmetric shape shown below.



Assuming constant properties and no internal heat generation, sketch the temperature distribution on *T*-*x* coordinates. Briefly explain the shape of the curve shown.

Example 2 (Problem 2.27)

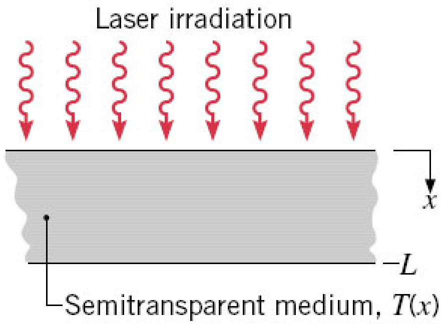
A salt-gradient solar pond is a shallow body of water that consists of three distinct fluid layers and is used to collect solar energy. The upper- and lower-most layers are well mixed and serve to maintain the upper and lower surfaces of the central layer at uniform temperatures *T1* and *T2*. Although there is bulk fluid motion in the mixed layers, there is no such motion in the central layer. Consider the conditions for which solar radiation absorption in the central layer provides non-uniform heat generation of the form:

$$\dot{q} = Ae^{-ax}$$

and the temperature distribution in the central layer is:

$$T(x) = -\frac{A}{ka^2}e^{-ax} + Bx + C$$

The quantities *A* (W/m3), *a* (1/m), *B* (K/m), and *C* (K) are known constants having the prescribed units, and *k* is the thermal conductivity, which is also constant.



1. Obtain expressions for the rate at which heat is transferred per unit area from the lower mixed layer to the central layer and from the central layer to the upper mixed layer.
2. Determine whether conditions are steady or transient.
3. Obtain expression for the rate at which thermal energy is generated at the entire, per unit surface area.

Summary – Part 2a

- Conduction rate equation – Fourier’s law
  - Defines the **thermal conductivity**, k
  - Heat flux is directional, **perpendicular** to an isothermal surface
  - Can be expressed in terms of the **3-D temperature field**, $T(x, y, z)$
  - Qualitative **temperature profiles** in solid objects

- Thermal **conductivity** and **diffusivity**
  - Conductivity depends on the state of matter, which determines mechanism for heat conduction
  - Diffusivity relates the ability to conduct heat to the ability to store heat

- The heat (diffusion) equation
  - Allows us to determine the **temperature distribution** in a medium
  - Expressed in Cartesian, cylindrical, or spherical coordinates
  - Solution of differential equation requires **boundary** and **initial conditions**
  - **Application of the solutions** and verification of their validity