CONCORDIA UNIVERSITY
FACULTY OF ENGINEERING AND COMPUTER SCIENCE
ENGR 372/4 sec. Y FUNDAMENTALS OF CONTROL SYSTEMS

Mid-Term | Instructor: Dr. H. Hong | Date: March 9, 2000

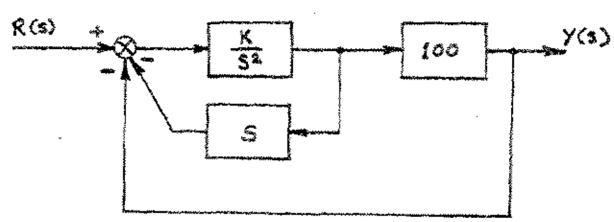
Answer all three (3) questions. Closed book exam. Total time: 1 hour 15 minutes

Problem 1.→23; Problem 2.→7; Problem 3.→5; = 35 marks total.

Problem 1. [a→1, b→7, c→3, d→2, e→1, f→3, g→3, h→3 = 23 marks]

A negative feedback control system has a block diagram as shown below.

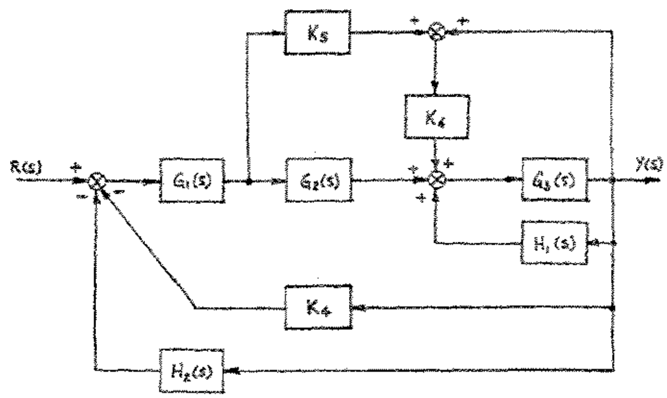
(a) Determine the closed-loop transfer function T(s) = Y(s)/R(s).
(b) For a unit step input, find the value for the gain K so that the system has 20% overshoot.
What is the natural frequency ω_n of the system?
What is the settling time Ts of the system?
(c) Sketch the closed-loop poles on the s-plane. Clearly indicate ω_n (natural frequency) and θ (angular representation of the damping ratio ζ), and their numerical values, on the figure.
(d) Using the Routh-Hurwitz stability criterion determine the range of gain K for the system to be stable?
(e) What is the "type number" of the system?
(f) For a unit step input, determine the position error constant Kp , and the steady-state error. Show all mathematical steps leading to your answer.
(g) For a unit ramp input, determine the velocity error constant Kv , and the steady-state error. Show all mathematical steps leading to your answer.
(h) For a unit acceleration input, determine the acceleration error constant Ka , and the steady-state error. Show all mathematical steps leading to your answer.



Problem 2. [a→1½, b→5½ = 7 marks]

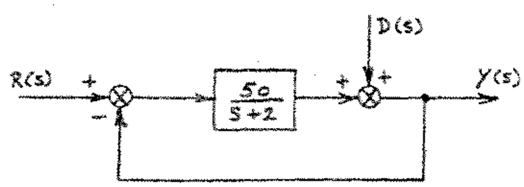
For the system represented in the block diagram below, find:

(a) the signal flow graph representation.
(b) the transfer function $T(s) = Y(s)/R(s)$, by using (only) Mason's signal-flow gain formula.

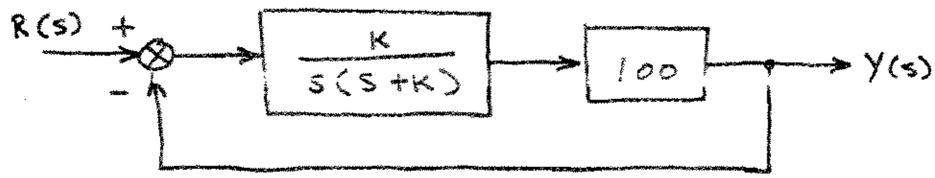


Problem 3. [5 marks]

For the system shown below find the total steady-state error due to a unit step input R(s) and a unit step disturbance D(s).



Problem # 1



$$\text{inner loop} \Rightarrow \frac{K/s^2}{1+s(K/s^2)} = \frac{K}{s^2+Ks} = \frac{K}{s(s+K)}$$

a) $$\frac{Y(s)}{R(s)} = \frac{G}{1+G} = \frac{\frac{100K}{s(s+K)}}{1+\frac{100K}{s(s+K)}} = \frac{100K}{s^2+Ks+100K}$$

b) From the characteristic equation

$$s^2+Ks+100K = s^2+2\xi\omega_n s+\omega_n^2 = 0$$

$$\omega_n = \sqrt{100K} \quad \text{----- (equation 1)}$$

$$2\xi\omega_n = 2\xi\sqrt{100K} = K \quad \text{------- (equation 2)}$$

given: 20% overshoot

$$P.O. = 100\exp\left[\frac{-\xi\pi}{\sqrt{1-\xi^2}}\right]$$

$$-\frac{\xi\pi}{\sqrt{1-\xi^2}} = \ln\left(\frac{20}{100}\right) = -1.6094$$

$$1-\xi^2 = 3.8102\,\xi^2$$

$$\therefore \quad \xi = 0.4559$$

From equation 2

$$2\xi\sqrt{100K} = K$$

$$4\xi^2(100K) = K^2$$

$$\therefore\ K = 400\xi^2 = 400(0.4559)^2$$

$$K = 83.1379$$

From equation 1

$$\omega_n = \sqrt{100K} = \sqrt{100\times 83.1379}$$

$$\omega_n = 91.18 \text{ rad/sec}$$

$$\therefore\ S^2 + KS + 100K = S^2 + 83.1379S + 8313.7924$$

$$T_s = \frac{4}{\xi\omega_n} = \frac{4}{(0.4559)(91.18)}$$

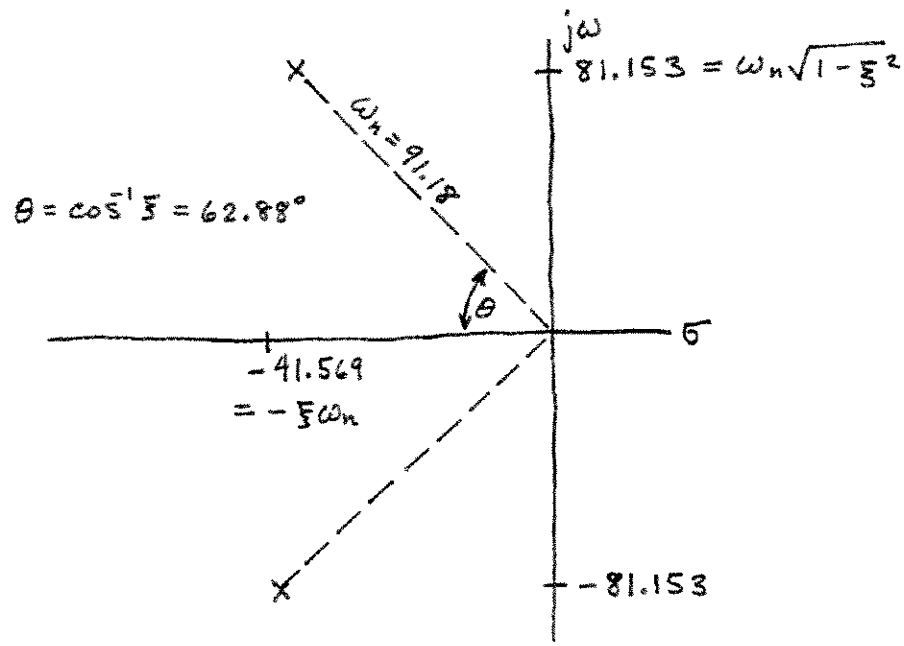
$$T_s = 0.0962 \text{ sec.}$$

c) $s^2 + 83.1379s + 8313.7924 = 0$

$$S_1, S_2 = \frac{-83.1379 \pm \sqrt{(83.1379)^2 - 4(8313.7924)}}{2}$$

$$= \frac{-83.1379 \pm 162.3061}{2}$$

$$S_1, S_2 = -41.569 \pm j81.153$$



d) From the Routh-Hurwitz Stability Criterion

$$s^2 + Ks + 100K = 0$$

$$\begin{array}{c|cc} s^2 & 1 & 100K \\ s^1 & K & 0 \\ s^0 & 100K & \end{array}$$

from row s^1, $K > 0$ for the system to be stable.

e) From the reduced block diagram, in unity feedback form, the system is "type 1".

f) $e_{ss} = \dfrac{1}{1+K_p}$ for $R(s) = \dfrac{A}{s} = \dfrac{1}{s}$

$$K_p = \lim_{s \to 0} G(s) \quad \text{where } G(s) = \frac{100K}{s(s+K)}$$

$$= \lim_{s \to 0} \left[\frac{100K}{s(s+K)} \right]$$

$$K_p = \infty$$

$$\therefore e_{ss} = \frac{1}{1+\infty} = 0$$

(3) g) $e_{ss} = \frac{1}{K_v}$ for $R(s) = \frac{A}{S^2} = \frac{1}{S^2}$

$$K_v = \lim_{s \to 0} S\,G(s) \quad \text{where } G(s) = \frac{100K}{S(S+K)}$$

$$= \lim_{s \to 0} S\left[\frac{100K}{S(S+K)}\right]$$

$$K_v = 100$$

$$\therefore\ e_{ss} = \frac{1}{100}$$

(3) h) $e_{ss} = \frac{1}{K_a}$ for $R(s) = \frac{A}{S^3} = \frac{1}{S^3}$ $(At^2/2)$

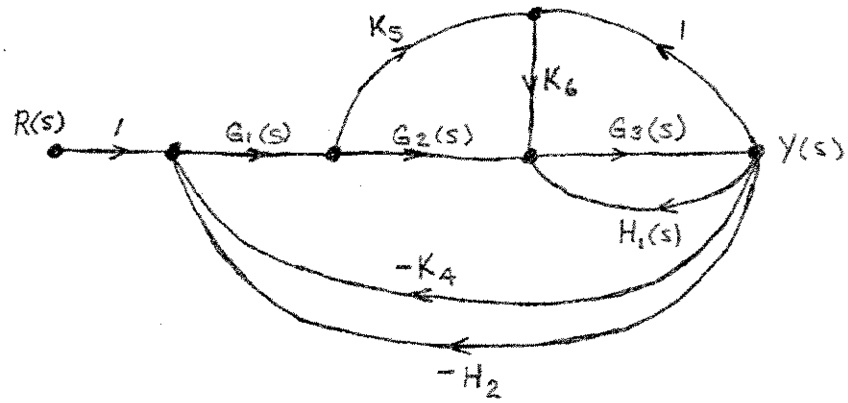
$$K_a = \lim_{s \to 0} S^2 G(s) \quad \text{where } G(s) = \frac{100K}{S(S+K)}$$

$$= \lim_{s \to 0} S^2\left[\frac{100K}{S(S+K)}\right]$$

$$K_a = 0$$

$e_{ss} = \frac{1}{0} = \infty$, system cannot track a parabola input

Problem #2



$L_1 = G_3 K_6$

$L_2 = G_3 H_1$

$L_3 = -G_1 G_2 G_3 K_4$

$L_4 = -G_1 G_2 G_3 H_2$

$L_5 = -G_1 K_5 K_6 G_3 K_4$

$L_6 = -G_1 K_5 K_6 G_3 H_2$

$P_1 = G_1 G_2 G_3$

$P_2 = G_1 K_5 K_6 G_3$

$$\begin{aligned}
\Delta &= 1 - (L_1 + L_2 + L_3 + L_4 + L_5 + L_6) \\
&= 1 - G_3 K_6 - G_3 H_1 + G_1 G_2 G_3 K_4 + G_1 G_2 G_3 H_2 \\
&\quad + G_1 K_5 K_6 G_3 K_4 + G_1 K_5 K_6 G_3 H_2 \\
&= 1 - G_3 (K_6 + H_1) + G_1 G_2 G_3 (K_4 + H_2) + G_1 K_5 K_6 G_3 (K_4 + H_2)
\end{aligned}$$

$$\Delta = 1 - G_3 (K_6 + H_1) + G_1 G_3 (K_4 + H_2)(G_2 + K_5 K_6)$$

$\Delta_1 = 1 \qquad P_1 \Delta_1 = G_1 G_2 G_3$

$\Delta_2 = 1 \qquad P_2 \Delta_2 = G_1 K_6 K_5 G_3$

$$P_1 \Delta_1 + P_2 \Delta_2 = G_1 G_3 (G_2 + K_5 K_6)$$

$$\frac{Y(s)}{R(s)} = \frac{G_1 G_3 (G_2 + K_5 K_6)}{1 - G_3 (K_6 + H_1) + G_1 G_3 (K_4 + H_2)(G_2 + K_5 K_6)}$$

Problem # 3

$$E(s) = E_R(s) + E_D(s)$$

$$E_R(s) = \frac{1}{1 + \frac{50}{s+2}} \cdot R(s) = \frac{s+2}{s+52} \cdot \frac{1}{s}$$

$$E_D(s) = \frac{-1}{1 + \frac{50}{s+2}} \cdot D(s) = -\frac{s+2}{s+52} \cdot \frac{1}{s}$$

$$e_{ss} = \lim_{s \to 0} s\,E(s) = \lim_{s \to 0} s \left[E_R(s) + E_D(s) \right]$$

$$= \lim_{s \to 0} s \left[\frac{s+2}{s+52} \quad \frac{s+2}{s+52} \right] \frac{1}{s}$$

$$= 0$$

OR

$$\frac{y_R}{R} = \frac{\frac{50}{s+2}}{1 + \frac{50}{s+2}} = \frac{50}{s+52}$$

steady state $\quad y_R = \lim_{s \to 0} s \left[\frac{50}{s+52} \right] \frac{1}{s} = \frac{50}{52}$

$$\frac{y_D}{D} = \frac{1}{1 + \frac{50}{s+2}} = \frac{s+2}{s+52}$$

steady state $\quad y_D = \lim_{s \to 0} s \left[\frac{s+2}{s+52} \right] \frac{1}{s} = \frac{2}{52}$

$$y = y_R + y_D = \frac{50}{52} + \frac{2}{52} = 1$$

$$e_{ss} = R - y = 1 - 1 = 0$$

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CONCORDIA UNIVERSITY
FACULTY OF ENGINEERING AND COMPUTER SCIENCE

DEPARTMENT OF MECHANICAL AND INDUSTRIAL ENGINEERING
MECH 371/4 sec. X FUNDAMENTALS OF CONTROL SYSTEMS

Mid-Term **Instructor: H. Hong** **Date: March 4, 2003**

Answer all four (4) questions. Closed book exam. Total time: 1 hour 15 minutes

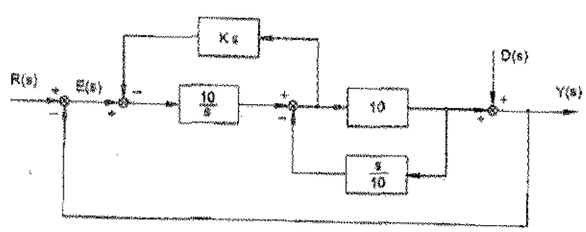
For all questions:
Clearly show all equations and mathematical steps.
Answers without mathematical steps will be given a mark of zero.

Problem 1.→ 23; Problem 2.→ 10; Problem 3.→ 10; Problem 4.→ 12 = 55 marks total.

Problem 1. [1→ 3, 2→ 2, 3→ 6, 4→ 4, 5→ 6, 6→ 2 = 23 marks]

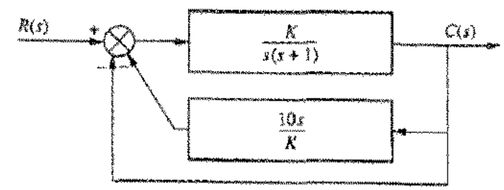
A negative feedback control system has a block diagram as shown below.

(1) Draw the signal flow graph for the system.
(2) Determine the closed-loop transfer function T(s) = Y(s)/R(s).
(3) For a unit step input, find the value for the gain K so that the system has 20% overshoot. What is the natural frequency ω_n of the system? What is the settling time T_S of the system?
(4) Sketch the closed-loop poles on the s-plane. Clearly indicate ω_n (natural frequency), ω_d (damped natural frequency), and θ (angular representation of the damping ratio ζ), and their numerical values, on the figure.
(5) Find the total steady-state response due to a unit step input R(s) and a unit step disturbance D(s).
(6) Determine the sensitivity of the transfer function T(s) with respect to gain K, ie: S_K^T



Problem 2. [10 marks]

For the feedback control system shown below:
(1) Determine the type number of the system.
(2) Using the concept of system type number, determine the steady state error to a unit step input.
(3) Using the concept of system type number, determine the steady state error to a unit ramp input.
(4) Using the concept of system type number, determine the steady state error to a unit parabola input.



Problem 3. [10 marks]

Using the Routh-Hurwitz test, determine the stability of the closed-loop transfer function:

$$T(s) = \frac{10}{s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56}$$

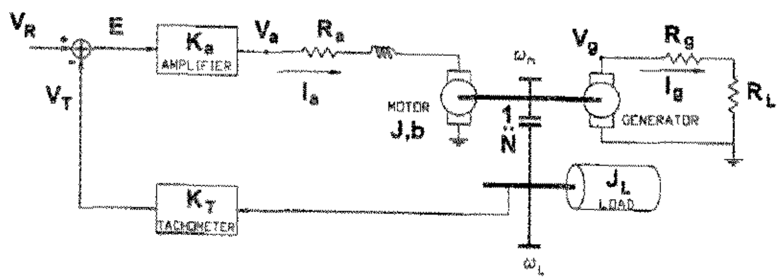
Determine, if any,
(1) the marginally stable roots, and their numerical values.
(2) the total number of unstable roots.

HINT: The negative number -4 is a vector written as $-4+0j$ or as $4e^{\pm180j}$.

Problem 4. [12 marks]

A permanent magnet DC-motor drives a rotational load through a speed reduction gear ratio. The motor also directly drives a generator that supplies current to a resistive load. The load speed is measured by a tachometer for velocity feedback control. The error between the reference and tachometer voltage drives an amplifier that supplies power to the armature of the DC-motor.

Draw the block diagram for the feedback control system shown in the schematic diagram below. Provide justification to any assumptions made.



Given equations with respect to schematic diagram:

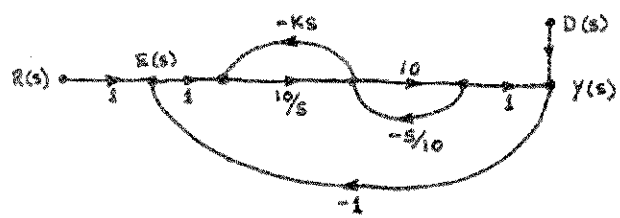
$E(s) = V_R(s) - V_T(s)$	error = reference voltage minus tachometer voltage
$V_T(s) = K_T\omega_L(s)$	tachometer voltage = tachometer gain times load speed
$V_a(s) = K_a E(s)$	amplifier voltage = amplifier gain times error voltage
$V_a(s) = R_a I_a(s) + V_b(s)$	amplifier (or armature) voltage = armature resistance times armature current plus back EMF [armature inductance neglected]
$V_b(s) = K_b\omega_m(s)$	back EMF = back EMF gain times motor speed
$T(s) = K_m I_a(s)$	motor generated torque = motor constant times armature current
$T_m(s) = T(s) - T_d(s)$	torque driving motor = motor generated torque less torque driving disturbances
$T_m(s) = J\dot{\omega}_m(s) + b\omega_m$	torque driving motor = armature inertia times motor speed plus damping times motor speed
$T_d(s) = T_{d1}(s) + T_{d2}(s)$	disturbance torque = torque driving inertia load plus torque driving generator

Equations that must be derived from schematic diagram:

$\omega_L(s) = ??$	load speed $\omega_L(s) \Rightarrow$ motor speed $\omega_m(s)$ with gear reduction. Let gear ratio be N>1
$T_{d1}(s) = ??$	load torque $T_{d1}(s) \Rightarrow$ rotational load inertia J_L; omit damping; load angular acceleration $\dot{\omega}_L(s)$; gear ratio N>1
$V_g(s) = ??$	generator voltage $V_g(s) \Rightarrow$ generator voltage gain K_g; generator (or motor) speed $\omega_m(s)$
$I_g(s) = ??$	generator current $I_g(s) \Rightarrow$ generator armature resistance R_g; resistive load R_L; generator voltage $V_g(s)$; neglect generator inductance
$T_{d2}(s) = ??$	generator torque $T_{d2}(s) \Rightarrow$ generator constant K_{GG}; generator current $I_g(s)$

Problem # 1

(1)



(2) $T(s) = \dfrac{Y(s)}{R(s)} = \dfrac{\left(\frac{10}{s}\right)(10)}{1 - \left(-[Ks]\left[\frac{10}{s}\right] - [10]\left[\frac{s}{10}\right] - \left[\frac{10}{s}\right][10]\right)}$

$T(s) = \dfrac{10^2/s}{1 + 10K + s + 10^2/s}$

$T(s) = \dfrac{100}{s^2 + s(10K+1) + 100}$

(3) $s^2 + s(10K+1) + 100 = s^2 + 2\xi\omega_n s + \omega_n^2$

$\omega_n = 10$ rad/sec

$2\xi\omega_n = 10K + 1$

$P.O. = .20 = 100 \exp\left[\dfrac{-\xi\pi}{\sqrt{1-\xi^2}}\right]$

$\ln\left[\dfrac{20}{100}\right] = -\dfrac{\xi\pi}{\sqrt{1-\xi^2}} = -1.6094$

(1/6)

$1 - \xi^2 = 3.8104\,\xi^2$

$\therefore\ \xi = 0.4559$

$K = \dfrac{2\xi\omega_n - 1}{10} = \dfrac{(2)(0.4559)(10) - 1}{10}$

$K = 0.8119$

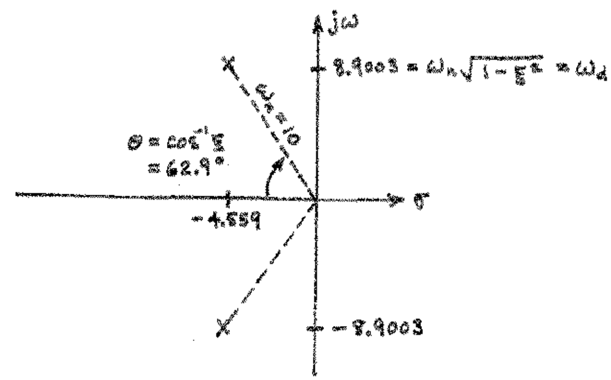
$T_s = \dfrac{4}{\xi\omega_n} = \dfrac{4}{(0.4559)(10)}$

$T_s = 0.8774$ sec.

(4) $s^2 + 9.118s + 100$

$\dfrac{-9.118 \pm \sqrt{(9.118)^2 - 4 \times 100}}{2} = \dfrac{-9.118 \pm 17.8006j}{2}$

$s_1, s_2 = -4.559 \pm 8.9003j$



(2/6)

(5) $Y(s) = T_1 R(s) + T_2 D(s)$

$$T_1 = \frac{100}{S^2 + S(10K+1) + 100}$$

$$T_2 = \frac{1\cdot(1+10K+S)}{1+10K+S+\frac{100}{S}} = \frac{S(1+10K+S)}{S^2+S(10K+1)+100}$$

for $R(s) = \frac{1}{S}$ and $D(s) = \frac{1}{S}$

$$y(\infty) = \lim_{s\to 0} S\left[\frac{100}{S^2+S(10K+1)+100}\right]\cdot\frac{1}{S} + \lim_{s\to 0} S\left[\frac{S(1+10K+S)}{S^2+S(10K+1)+100}\right]\cdot\frac{1}{S}$$

$$Y(\infty) = 1 + 0 = 1$$

(6) $S_K^T = \frac{\partial T}{\partial K}\cdot\frac{K}{T}$

$$\frac{\partial T}{\partial K} = -\frac{10S}{[S^2+S(10K+1)+100]^2}$$

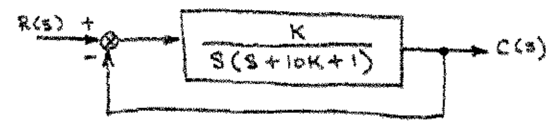
$$S_K^T = \frac{-10S}{[S^2+S(10K+1)+100]^2}\cdot K \cdot \frac{[S^2+S(10K+1)+100]}{100}$$

$$S_K^T = \frac{-KS}{10[S^2+S(10K+1)+100]}$$

(3/6)

Problem # 2

Reduce inner loop
$$\frac{\frac{K}{S(S+1)}}{1+\left(\frac{K}{S(S+1)}\right)\left(\frac{10S}{K}\right)} = \frac{K}{S(S+1)+10KS} = \frac{K}{S[S+10K+1]}$$



(1) Type 1 system, $G(s) = \frac{K}{S(S+10K+1)}$

(2) $K_p = \lim_{s\to 0} G(s) = \infty$

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0$$

(3) $K_v = \lim_{s\to 0} S\,G(s) = \frac{K}{10K+1}$

$$e_{ss} = \frac{1}{K_v} = \frac{10K+1}{K}$$

(4) $K_a = \lim_{s\to 0} S^2 G(s) = 0$

$e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$, ie: cannot track acceleration input

(4/6)

Problem #3 $\quad S^5 + 7S^4 + 6S^3 + 42S^2 + 8S + 56$

$$\begin{array}{c|ccc} S^5 & 1 & 6 & 8 \\ S^4 & 7 & 42 & 56 \\ S^3 & \frac{6\times7-1\times42}{7} = 0 & \frac{7\times8-1\times56}{7} = 0 & 0 \\ & & & 0 \end{array}$$

Since S^3 row are all zero, from S^4 row (auxilary equation):

$F(s) = 7S^4 + 42S^2 + 56$

$\frac{dF(s)}{ds} = 28S^3 + 84S \Longrightarrow$ new S^3 row

$$\begin{array}{c|ccc} S^5 & 1 & 6 & 8 \\ S^4 & 7 & 42 & 56 \\ S^3 & 28 & 84 & 0 \\ S^2 & \frac{28\times42-7\times84}{28} = 21 & \frac{28\times56-0}{28} = 56 & 0 \\ S^1 & \frac{21\times84-28\times56}{21} = 9.33 & 0 & 0 \\ & 9.33 & & \end{array}$$

(2) First column has no changes in sign, $\therefore$ no roots in RHP.

(1) From auxilary equation, $F(s) = S^4 + 6S + 8 = 0$

$S_1^2, S_2^2 = \frac{-6 \pm \sqrt{36-4\times8}}{2} = -3 \pm 1$

$S_1^2 = -4 = 4e^{180j}$ and $S_2^2 = -2 = 2e^{180j}$

$\therefore S_1 = 2e^{\pm 90j}$ and $S_2 = \sqrt{2}\, e^{\pm 90j}$

(5/6)

Problem #4

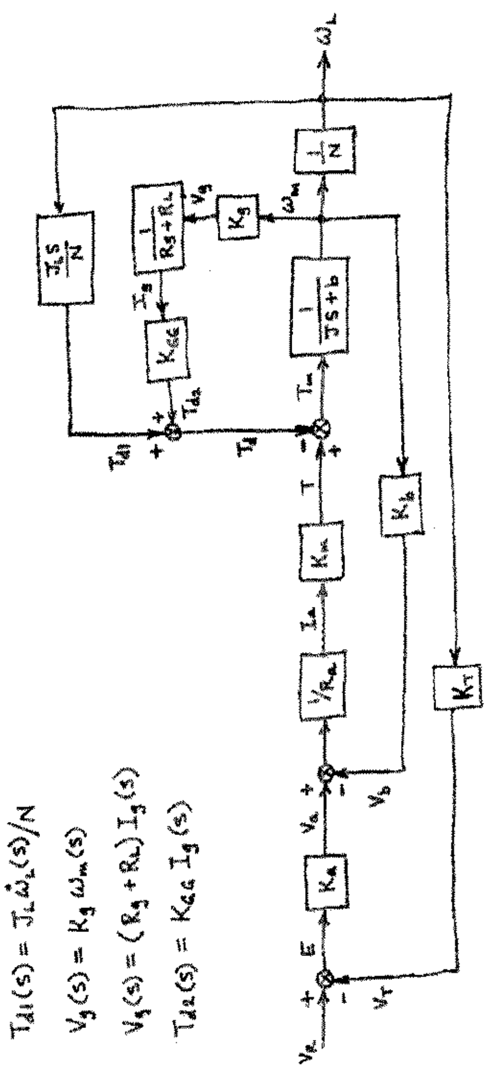
$\omega_L(s) = \omega_m(s)/N$

$T_{d1}(s) = J_L \dot{\omega}_L(s)/N$

$V_g(s) = K_g\, \omega_m(s)$

$V_g(s) = (R_g + R_L)\, I_g(s)$

$T_{d2}(s) = K_{GG}\, I_g(s)$



(6/6)

Problem #1 (a = 3, b = 5, c = 5, d1 = 4, d2 = 3 →→ 20 marks)

For the system block diagram given below:

(a) Draw the signal flow graph for the system.

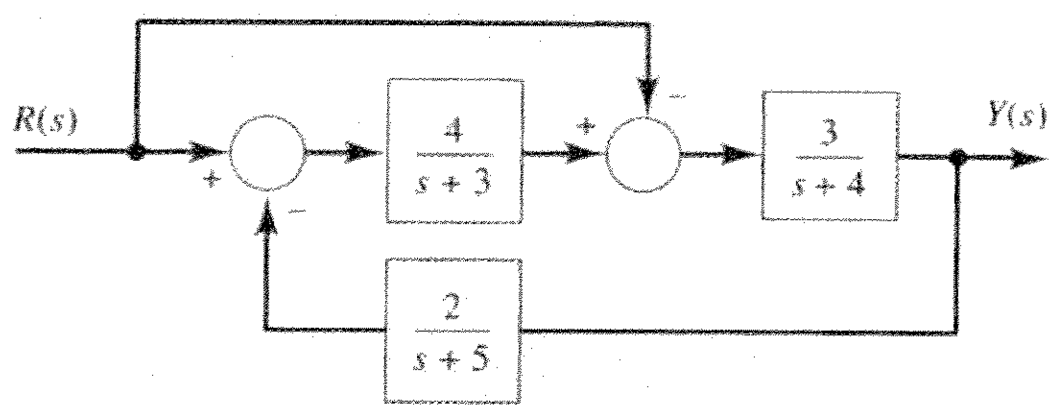
(b) Determine the overall transfer function $\frac{Y(s)}{R(s)} = G(s)$.

(c) Determine the type number of the system.

(d) Using the method of system type number as determined from part (3) of the question above, determine:

(1) The steady state error due to unit step input.

(2) The steady state error due to unit ramp input.



Problem #2 (a = 3, b = 2½ + 2½, c = 2 →→ 10 marks)

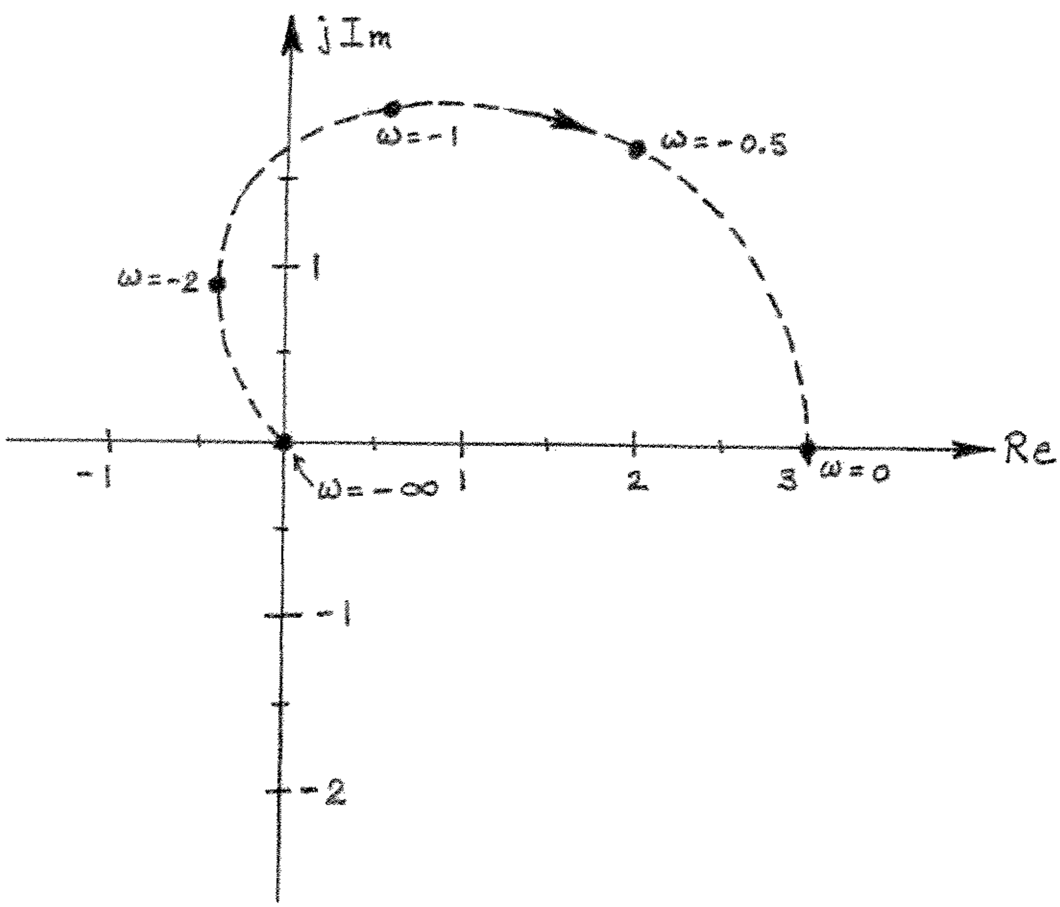
The open-loop transfer function of a system is given by:

$$GH(s) = \frac{6}{(s+1)(s+2)}$$

(a) The figure below gives a partial sketch of the Nyquist plot for the system, from $\omega = 0$ to $\omega = -\infty$. Complete the drawing for the sketch of the Nyquist plot. Explain your procedure.

(b) From the Nyquist plot, estimate the gain margin and the phase margin. Show your procedures clearly.

(c) Is the closed-loop system stable or unstable? Explain your answer clearly.



Problem #3 (Part A: a = 3, b = 2 ⟶ 5 marks)
(Part B: a = 10, b = 3½ + 3½, c = 3 ⟶ 20 marks)

(A) A feedback control system has the following characteristic equation:

$$p(s) = s^4 + 3s^3 + s^2 + 3s + 2$$

(a) Using the Routh-Hurwitz criterion, determine the stability of the system.
(b) Is the system stable or unstable? Explain your answer.

(B) The open-loop transfer function for a unity feedback system is:

$$G(s) = \frac{K(s+1)}{s(s-3)}$$

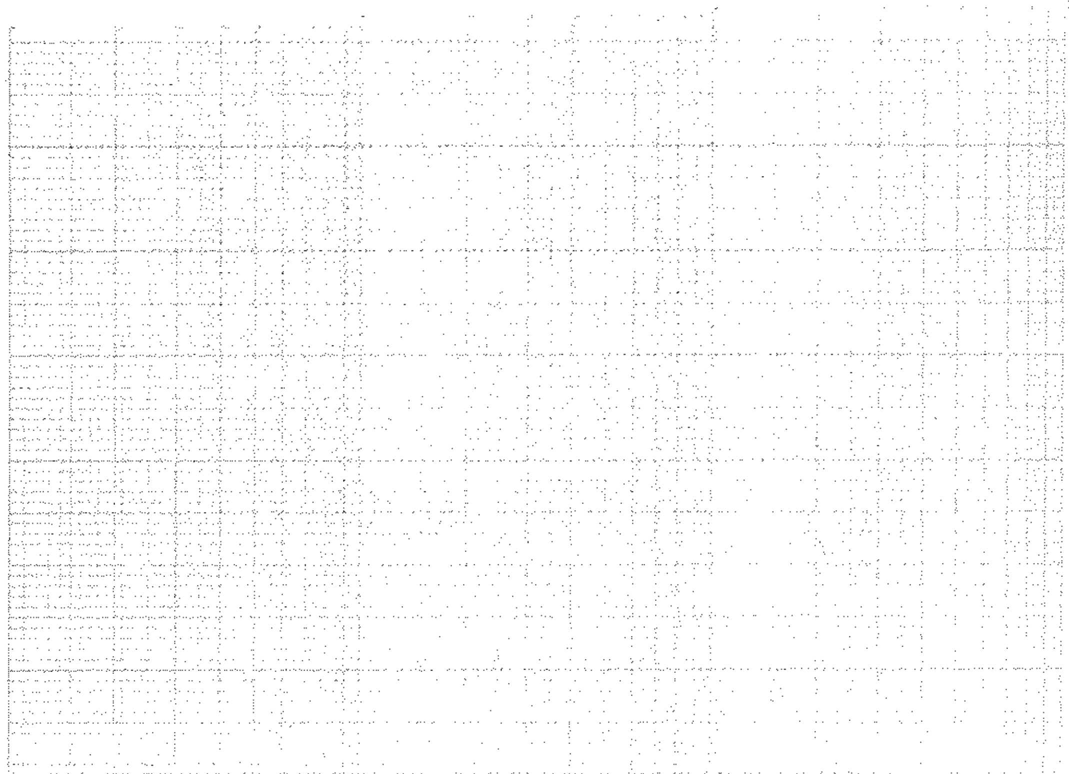
(a) Draw the root-locus for the closed-loop system, showing all relevant steps and details.
(b) Show that the closed-loop poles $s = -2 \pm j\sqrt{3}$ lie on the locus.
Determine the gain K at these poles.
(c) Derive the sensitivity equation of *G(s)* with respect to the gain *K*. ie: derive S_K^G .

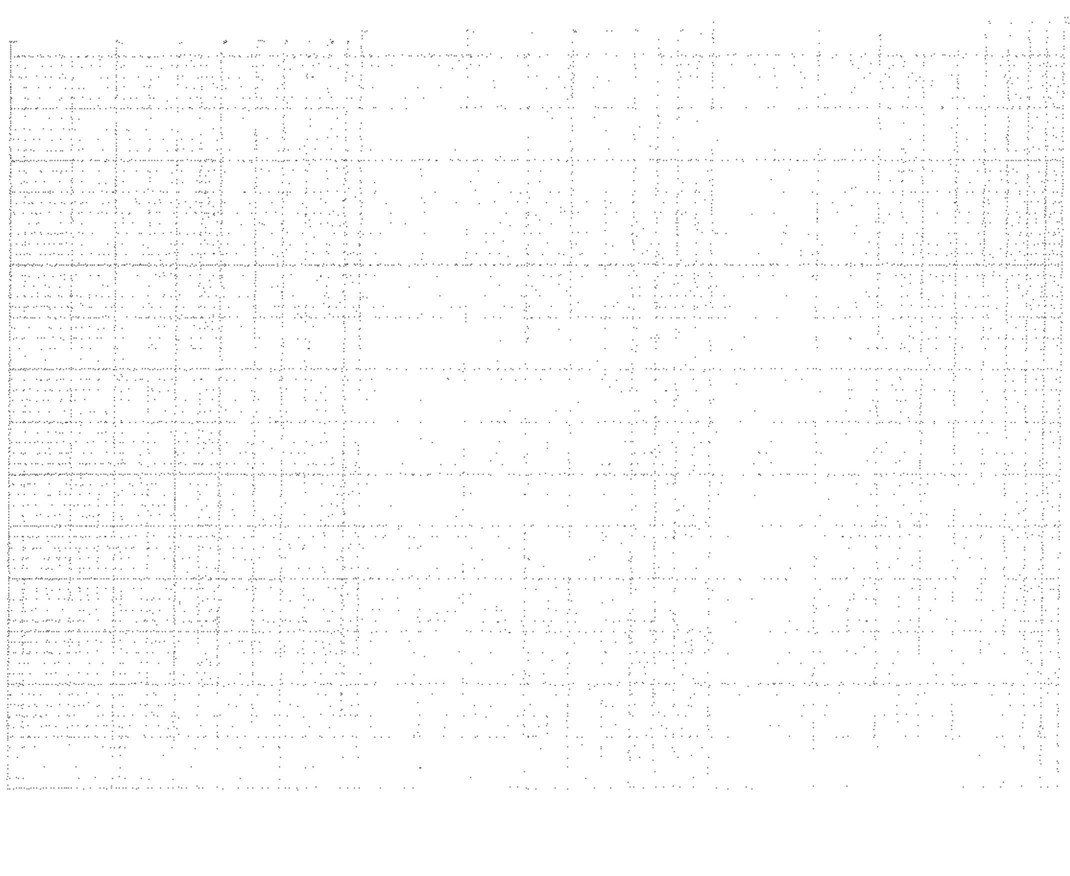
Problem #4 (a = 10, b1 = 2+2, b2 = 2, c1 = 3, c2 = 1 ⟶ 20 marks)

A unity feedback system has a plant transfer function given by:

$$G_p = \frac{3200}{s(s+4)(s+8)}$$

(a) On the semi-log paper provided, plot the asymptotic Bode diagram. Show all relevant steps. Indicate the relevant slopes on the diagram.
(b) (1) From the asymptotic Bode diagram, determine the gain margin and the phase margin. Show these values on the diagram clearly.
(2) Is the closed loop system stable or unstable? Explain your answer clearly.
(c) Introduce a constant gain K controller (ie: proportional controller) to obtain a phase margin of 45 degrees.
(1) What is the numerical value of this constant gain K controller?
(2) Draw the block diagram for the complete closed-loop system including the controller.





Problem #5 (a = 5, b = 10, c = 2, d = 8 →→ 25 marks)

A unity negative feedback control system has the plant transfer function given by:

$$G_p = \frac{10}{s(s+1)}$$

Given information:

percent overshoot $P.O. = 100e^{-\zeta\pi/\sqrt{1-\zeta^2}}$

time to peak $T_p = \dfrac{\pi}{\omega_d} = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}}$

where ω_n, ω_d and ζ is the system natural frequency, system damped natural frequency, and system damping ratio, respectively.

(a) Using the root locus method, design a proportional controller so that the system transient response due to a step input *r(t)* has exactly 4.3 % overshoot.

(b) Using the root locus method, design a PD controller so that the system transient response due to a step input *r(t)* has exactly 4.3 % overshoot and time to peak $T_P = 1.047 sec$.

(c) Discuss which system designed, either (a) or (b), has better performance due to a step input.

(d) For the two systems found in parts (a) and (b), determine the steady-state error due to a unit step input and due to a unit ramp input.