Université d'Ottawa · University of Ottawa
Faculté des sciences
Mathématiques et de statistique
Faculty of Science
Mathematics and Statistics

MAT 1332 – Midterm I, June 10th, 2014
Solutions

Instructor: Dr. Jeff Musgrave

NAME____

I.D.#____

Instructions: This exam consists of 7 long answer questions on 9 pages. The marks for each question are as listed with the question itself. The total value of the exam is 35. You may use the backs of pages for additional space if necessary. You must show all work and provide clear justification to receive full points.

Permitted Calculators: TI-30X, TI-34X, Casio FX-260X, and Casio FX-300X or a similar model; Programmable and graphical calculators are **NOT** permitted.

Duration: 80 minutes

GOOD LUCK!!

NO CELL PHONES. NO BOOKS. NO NOTES.

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Student # :____________________, Total Marks: ___________ out of 30

Problem	1	2	3	4	5	6	7
Marks							

1. (6 points) Calculate the following integrals

(a) $\displaystyle\int_1^2 \frac{x-2}{x^2+2x}\,dx$

(b) $\displaystyle\int_0^{\ln\left(\frac{1}{\sqrt{3}}\right)} \frac{e^x}{1+e^{2x}}\,dx$

Solution:

(a) Since the degree of the denominator is larger than the degree of the numerator, this question requires partial fractions.

$$\frac{x-2}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2}$$
$$\frac{x-2}{x(x+2)} = \frac{A(x+2)+B(x)}{x(x+2)}$$
$$x-2 = A(x+2)+B(x)$$

Since we only have linear factors, we may use the cover-up method to solve for A and B :

set $x=-2$: $\quad -4 = B(-2) \to B = 2$
set $x=0$: $\quad -2 = A(2) \to A = -1$

Thus, we have

$$\int_1^2 \frac{x-2}{x^2+2x}\,dx = \int_1^2 \frac{-1}{x} + \frac{2}{x+2}\,dx$$
$$= -\ln(|x|) + 2\ln(|x+2|)\Big|_1^2 = -\ln(2) + 2\ln(4) - (-\ln(1) + 2\ln(3))$$
$$\approx -0.1178$$

(b) This integral requires the following substitution:

Let $u = e^x$ then $du = e^x dx$ or $\frac{du}{e^x} = dx$. We then have

$$\begin{aligned}
\int_0^{\ln\left(\frac{1}{\sqrt{3}}\right)} \frac{e^x}{1+e^{2x}}\,dx &= \int_0^{\ln\left(\frac{1}{\sqrt{3}}\right)} \frac{e^x}{1+u^2}\frac{du}{e^x} \\
&= \int_0^{\ln\left(\frac{1}{\sqrt{3}}\right)} \frac{1}{1+u^2}\,du \\
&= \arctan(u)\Big|_0^{\ln\left(\frac{1}{\sqrt{3}}\right)} \\
&= \arctan(e^x)\Big|_0^{\ln\left(\frac{1}{\sqrt{3}}\right)} \\
&= \arctan\left(e^{\ln\left(\frac{1}{\sqrt{3}}\right)}\right) - \arctan(e^0) \\
&= \arctan\left(\frac{1}{\sqrt{3}}\right) - \frac{\pi}{4} \\
&= \frac{\pi}{6} - \frac{\pi}{4} = -\frac{\pi}{12}
\end{aligned}$$

2. (4 points) Solve the separable differential equation

$$\frac{dy}{dt} = 2t\sin(t)e^{-y}$$

with initial condition $y(0) = 0$.

Solution: Since this is a separable differential equation, we separate variables and then integrate:

$$\int e^y\,dy = 2\int t\sin(t)\,dt$$

The integral of $t\sin(t)$ requires integration by parts:
Set $u = t,\ du = dt$ and $dv = \sin(t)dt,\ v = -\cos(t)$. We then have

$$\begin{aligned}\int e^y\,dy &= 2\int t\sin(t)\,dt\\ e^y &= -2t\cos(t) + 2\int(\cos(t))\,dt\\ e^y &= -2t\cos(t) + 2\sin(t) + 2C\end{aligned}$$

Using the initial condition $y(0) = 0$ we find that $C = 1/2$. Therefore,

$$\begin{aligned}e^y &= -2t\cos(t) + 2\sin(t) + 1\\ y &= \ln\big(-2t\cos(t) + 2\sin(t) + 1\big)\end{aligned}$$

3. (3 points) Evaluate the definite integral

$$\int \frac{x^3 - 2x^2 + 2x - 2}{x^2+1}\,dx.$$

Since the degree of the numerator is larger than the degree of the denominator, the first step is polynomial long division. We have

$$\frac{x^3 - 2x^2 + 2x - 2}{x^2+1} = x - 2 + \frac{x}{x^2+1}$$

Thus,

$$\begin{aligned}\int \frac{x^3 - 2x^2 + 2x - 2}{x^2+1}\,dx &= \int\left(x - 2 + \frac{x}{x^2+1}\right)dx\\ &= \frac{x^2}{2} - 2x + \frac{1}{2}\ln(x^2+1) + C\end{aligned}$$

4. (6 points) For each of the following improper integrals, determine whether or not it converges. If it converges, determine its value.

   (a) $\displaystyle\int_0^1 \frac{\ln(x)}{x}\,dx$

   (b) $\displaystyle\int_2^\infty \frac{1}{(x-2)^2}\,dx$

Solution:

(a) $\displaystyle\int_0^1 \frac{\ln(x)}{x}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{x}\ln(x)\,dx$. This integral requires substitution: set $u = \ln(x),\ du = \frac{1}{x}\,dx$ or $xdu = dx$. We then have

$$\begin{aligned}
\lim_{t\to 0^+}\int_t^1 \frac{1}{x}\ln(x)\,dx &= \lim_{t\to 0^+}\int_t^1 u\,du \\
&= \lim_{t\to 0^+} \frac{u^2}{2}\Big|_t^1 \\
&= \lim_{t\to 0^+} \frac{(\ln(x))^2}{2}\Big|_t^1 \\
&= \lim_{t\to 0^+} 0 - \frac{(\ln(t))^2}{2} \\
&= -\infty,\ \text{divergent}
\end{aligned}$$

(b)

$$\begin{aligned}
\int_2^\infty \frac{1}{(x-2)^2}\,dx &= \int_2^3 \frac{1}{(x-2)^2}\,dx + \int_3^\infty \frac{1}{(x-2)^2}\,dx \\
&= \lim_{t\to 2^+}\int_t^3 \frac{1}{(x-2)^2}\,dx + \lim_{t\to\infty}\int_3^t \frac{1}{(x-2)^2}\,dx
\end{aligned}$$

For the first integral, we have

$$\begin{aligned}
\lim_{t\to 2^+}\int_t^3 \frac{1}{(x-2)^2}\,dx &= \lim_{t\to 2^+} -(x-2)^{-1}\Big|_t^3 \\
&= \lim_{t\to 2^+}\frac{-1}{3-2} + \lim_{t\to 2^+}\frac{-1}{t-2} \\
&= -1 - \infty = -\infty,\ \text{divergent}
\end{aligned}$$

Since the first integral is divergent there is no need to evaluate the second integral, and we have $\displaystyle\int_2^\infty \frac{1}{(x-2)^2}\,dx$ is divergent.

5. (6 points) Consider the functions $g(x) = 3x^2$ and $f(x) = 5x - 2$.

   (a) Find the points of intersection of g and f.

   (b) Find the area enclosed between the two graphs between the two points of intersection found in (a).

Solution: (a) The points of intersection are given by

$$\begin{aligned} 3x^2 &= 5x - 2 \\ 3x^2 - 5x + 2 &= 0 \\ (3x - 2)(x - 1) &= 0 \end{aligned}$$

Therefore, we have $x = 2/3$, $x = 1$, and the intersection points are $(2/3, 4/3)$ and $(1, 3)$. Since $f(x) > g(x)$ for $x \in [2/3, 1]$, the area is given by

$$\begin{aligned} \int_{2/3}^{1} 5x - 2 - 3x^2 \, dx &= -x^3 + \frac{5}{2}x^2 - 2x \Big|_{2/3}^{1} \\ &= -(1)^3 + \frac{5}{2}(1)^2 - 2(1) - \Big(-(2/3)^3 + \frac{5}{2}(2/3)^2 - 2(2/3) \Big) \\ &= -0.5 - (-0.5185) = .0185. \end{aligned}$$

6. (6 points) Zombies have invaded campus! They recruit more of the undead to their ghoulish ranks at a rate

$$\frac{dz}{dt} = f(z) = -5z^3 + 40500z,$$

where t is time and z is the number of zombies.

(a) Determine all biologically meaningful equilibria of this differential equation.

(b) Determine the stability of each equilibria in (a), using the derivative test (stability theorem).

(c) Draw a phase-line diagram for $z \geq 0$. On the same coordinate system, sketch the equilibrium solutions and the solution with initial condition $z(0) = 25$.

Solution: Equilibra of the differential equation satisfy

$$\begin{aligned} 0 &= -5z^3 + 40500z \\ 0 &= -5z(z^2 - 8100) \\ 0 &= -5z(z - 90)(z + 90) \end{aligned}$$

Therefore, the equilibria are $z^* = 0$, $z^* = 90$, and $z^* = -90$. The only biologically meaningful equilibria are $z^* = 0$ and $z^* = 90$.

(b) We have $f(z) = -5z^3 + 40500z$ and $f'(z) = -15z^2 + 40500$. Thus, we have

$$\begin{aligned} f'(0) &= 40500, \text{ Therefore, } z^* = 0 \text{ is unstable} \\ f'(90) &= -15(90)^2 + 40500 = -81000, \text{ Therefore, } z^* = 90 \text{ is stable} \end{aligned}$$

(c) See the separate file for question 1 c.

7. (4 points) Determine the average value of $f(x) = xe^{5x}$ for $0 \le x \le 2$.

Solution: The average value of f is given by

$$\bar{f} = \frac{1}{2-0} \int_0^2 xe^{5x}\, dx.$$

Evaluation of this integral requires integration by parts:

set $u = x$, $du = dx$ and $dv = e^{5x}dx$, $v = \frac{e^{5x}}{5}$. Thus, we have

$$\begin{aligned}
\bar{f} = \frac{1}{2} \int_0^2 xe^{5x}\, dx &= \frac{1}{2}\left(\frac{xe^{5x}}{5}\bigg|_0^2 - \int_0^2 \frac{e^{5x}}{5}\, dx \right) \\
&= \frac{1}{2}\left(\frac{xe^{5x}}{5} - \frac{e^{5x}}{25}\bigg|_0^2 \right. \\
&= \frac{1}{2}\left(\frac{2e^{10}}{5} - \frac{e^{10}}{25} - (0 - 1/25) \right) \\
&\approx 3964.78
\end{aligned}$$

Space for extra work: DO NOT REMOVE