Extra Exercises for MAT 2141

1. The set $B = \{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}\}$ is a basis of $\mathbf{R}^2$. Find explicit formulae for the basis $\{f_1, f_2\}$ dual to B, i.e. find $a, b, c, d \in \mathbf{R}$ such that $f_1(\begin{bmatrix} x \\ y \end{bmatrix}) = ax + by$ and $f_2(\begin{bmatrix} x \\ y \end{bmatrix}) = cx + dy$.

2. Equip $\mathcal{P}_3(\mathbf{R})$ with the inner product defined by

$$\langle f,\, g\rangle = \int_{-1}^{1} t^2 f(t)g(t)dx$$

a) Find an orthogonal basis for the subspace $U = \mathcal{P}_2(\mathbf{R})$.

b) Find the orthogonal projection of $p(t) = t^3$ onto the subspace U.

c) Find the smallest element of the set $\{||t^3 - (a + bt + ct^2)|| \mid a, b, c \in \mathbf{R}\}$

3. Let

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

a) Find the eigenvalues of A.

b) Find an orthogonal basis for each eigenspace of A.

c) Find an orthogonal matrix P such that $P^{-1}AP$ is diagonal, and give this diagonal matrix.

d) Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be defined by $T(v) = A.v$. Find a basis B of $\mathbf{R}^3$ such that the matrix $M_B(T)$ is diagonal, and give this diagonal matrix.

4. Let

$$A = \begin{bmatrix} 5 & -2 & 2 \\ -2 & 2 & 4 \\ 2 & 4 & 2 \end{bmatrix}.$$

a) Find the eigenvalues of A.

b) Find an orthogonal basis for each eigenspace of A.

c) Find an orthogonal matrix P such that $P^{-1}AP$ is diagonal, and give this diagonal matrix.

d) Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be defined by $T(v) = A.v$. Find a basis B of $\mathbf{R}^3$ such that the matrix $M_B(T)$ is diagonal, and give this diagonal matrix.

5. Let $f, g, h : \mathbf{R} \to \mathbf{R}$ be twice-differentiable functions. Prove that if the matrix

$$\begin{bmatrix} f(0) & g(0) & h(0) \\ f'(0) & g'(0) & h'(0) \\ f''(0) & g''(0) & h''(0) \end{bmatrix}$$

is invertible, then f, g and h are linearly independent in $\mathbf{F}(\mathbf{R})$.