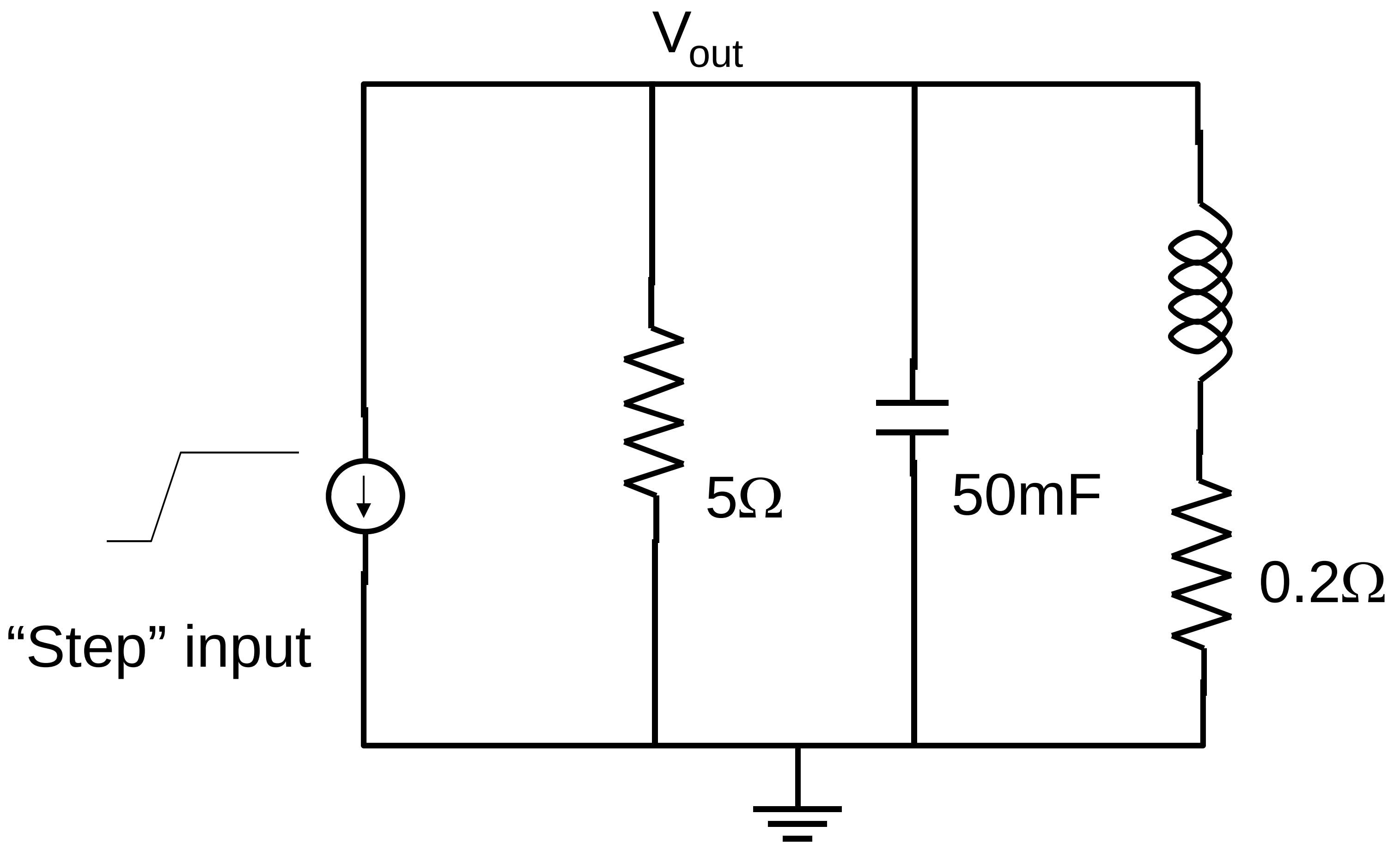
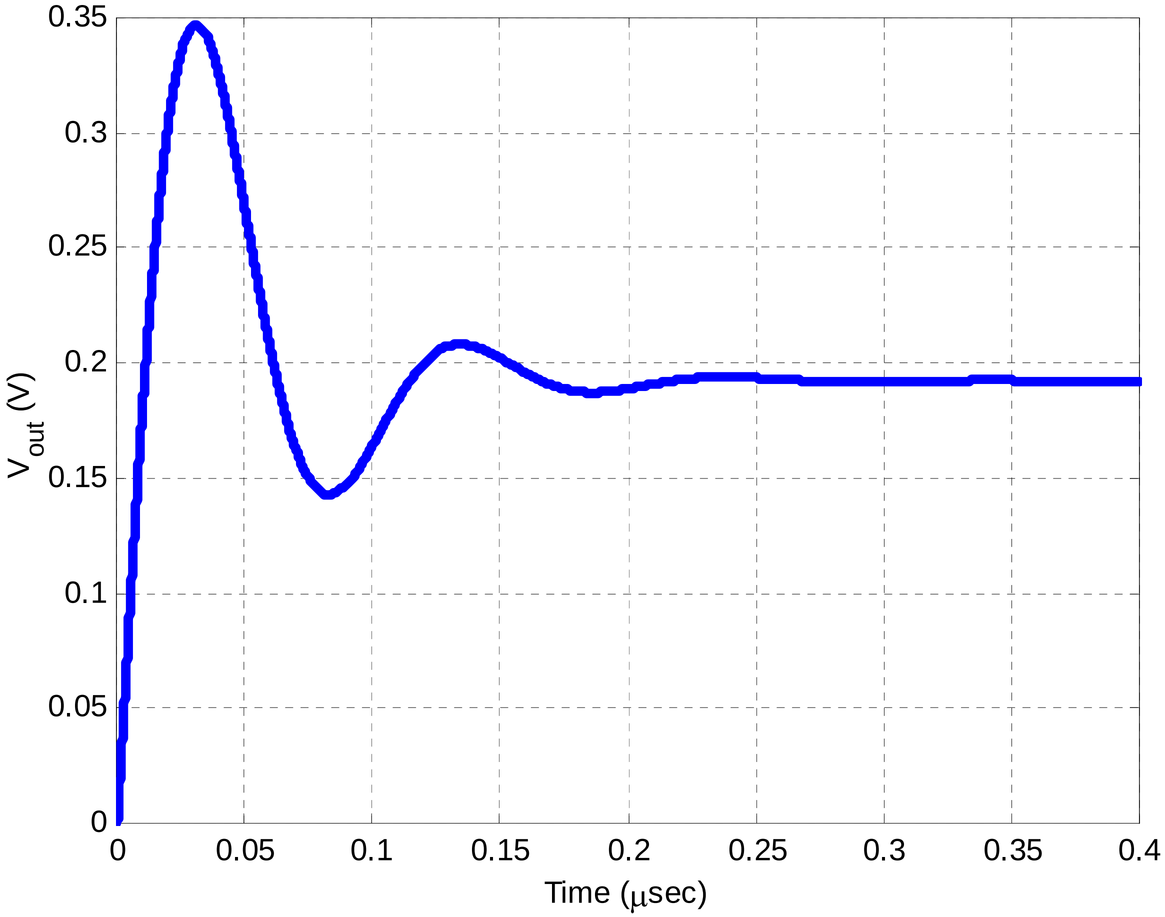
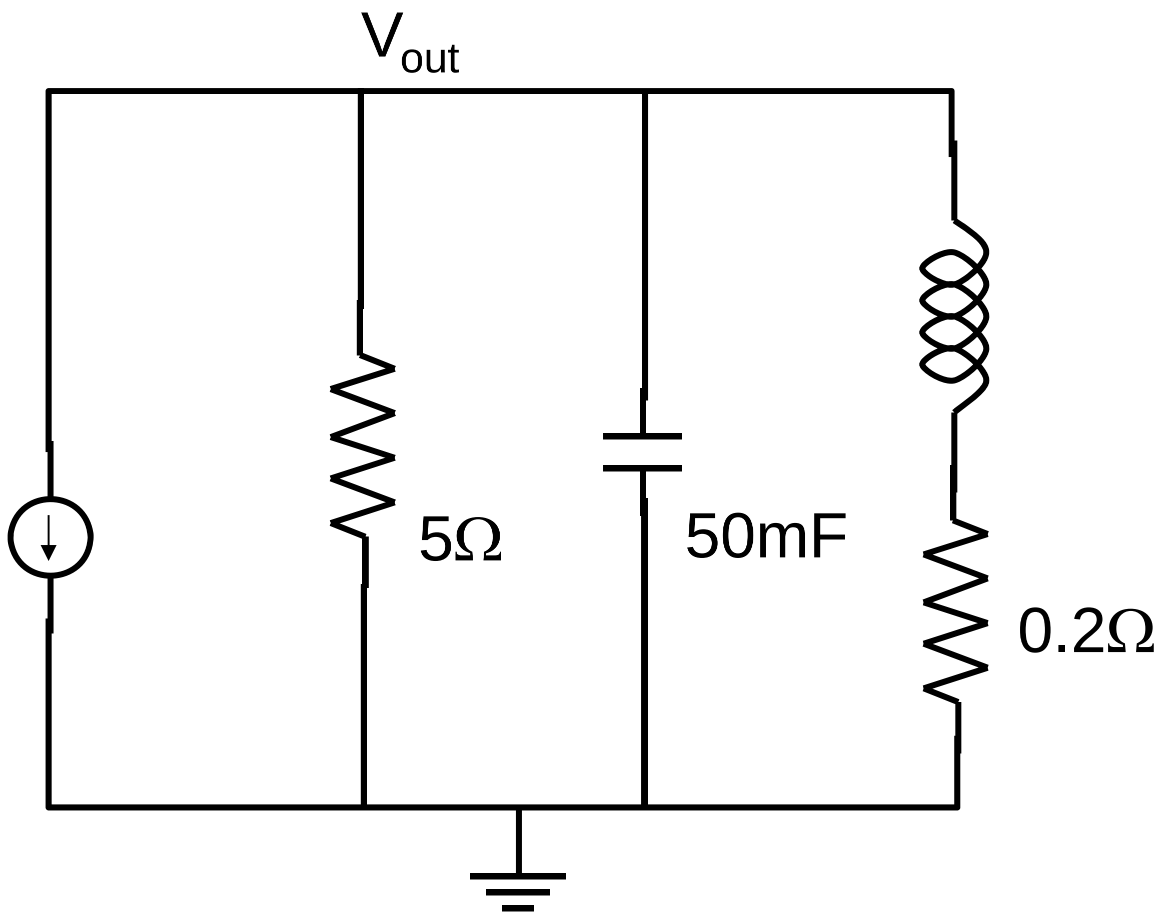
Transient Analysis

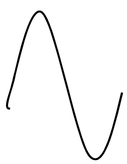


Transient Analysis (step input)



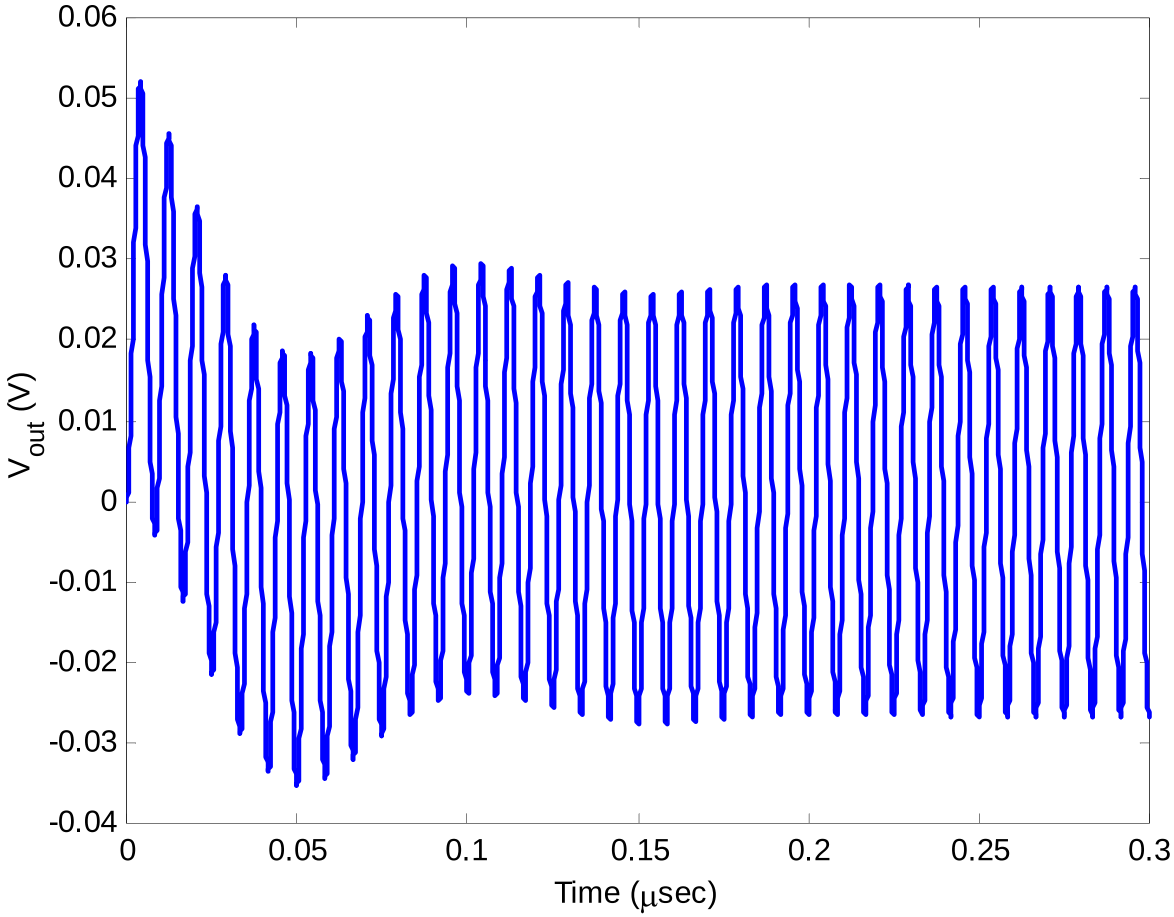
Transient Analysis



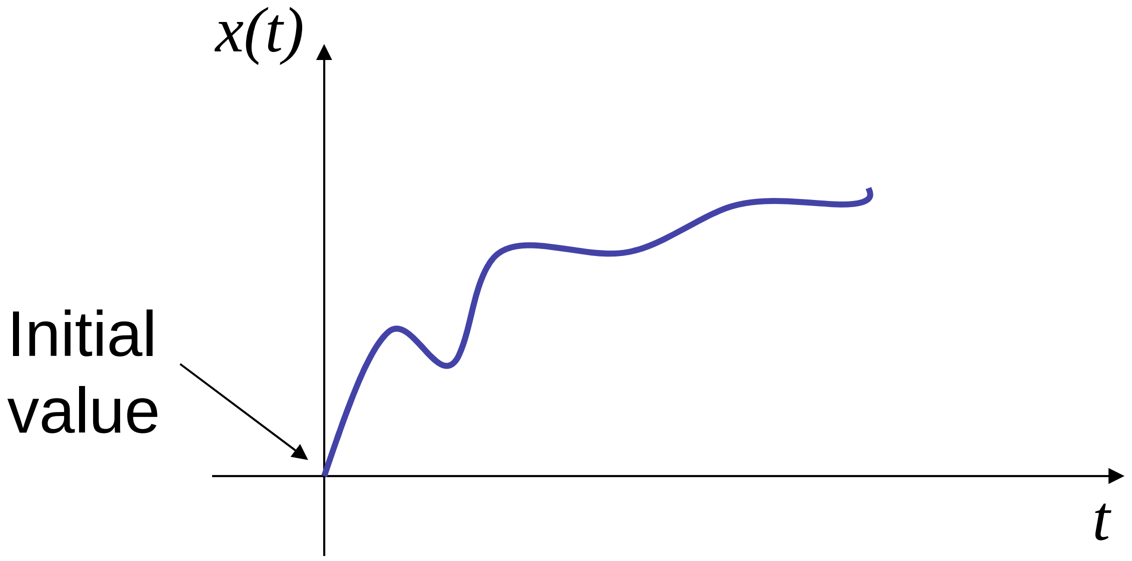


“sinusoidal” input at 120MHz

Transient Analysis (sine input)

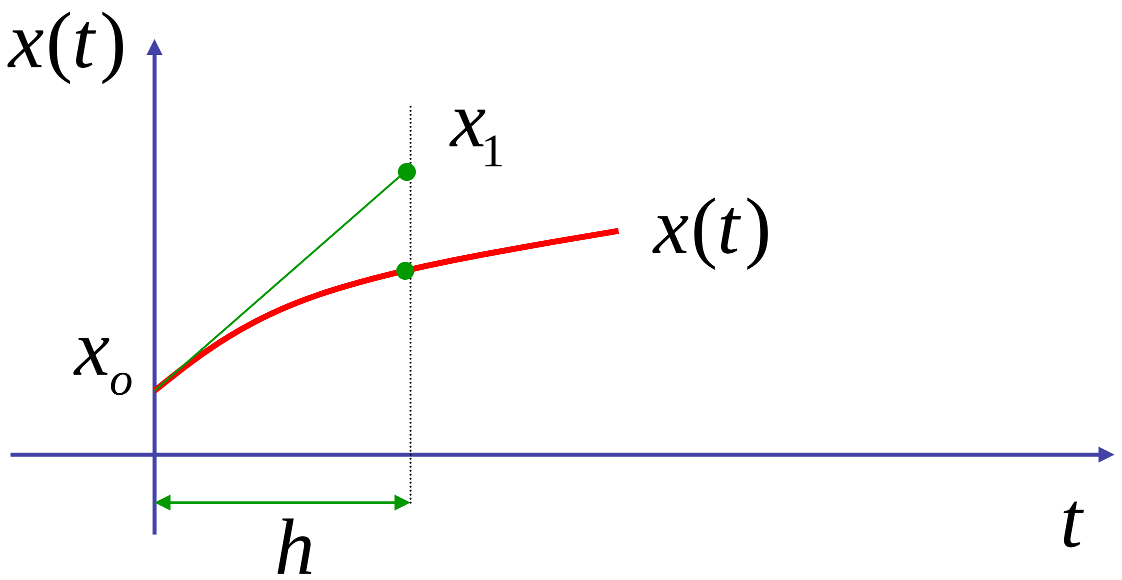


Transient Analysis



→ Initial value problem.

→ know $x(0)$, calculate $x(t)$, $t>0$

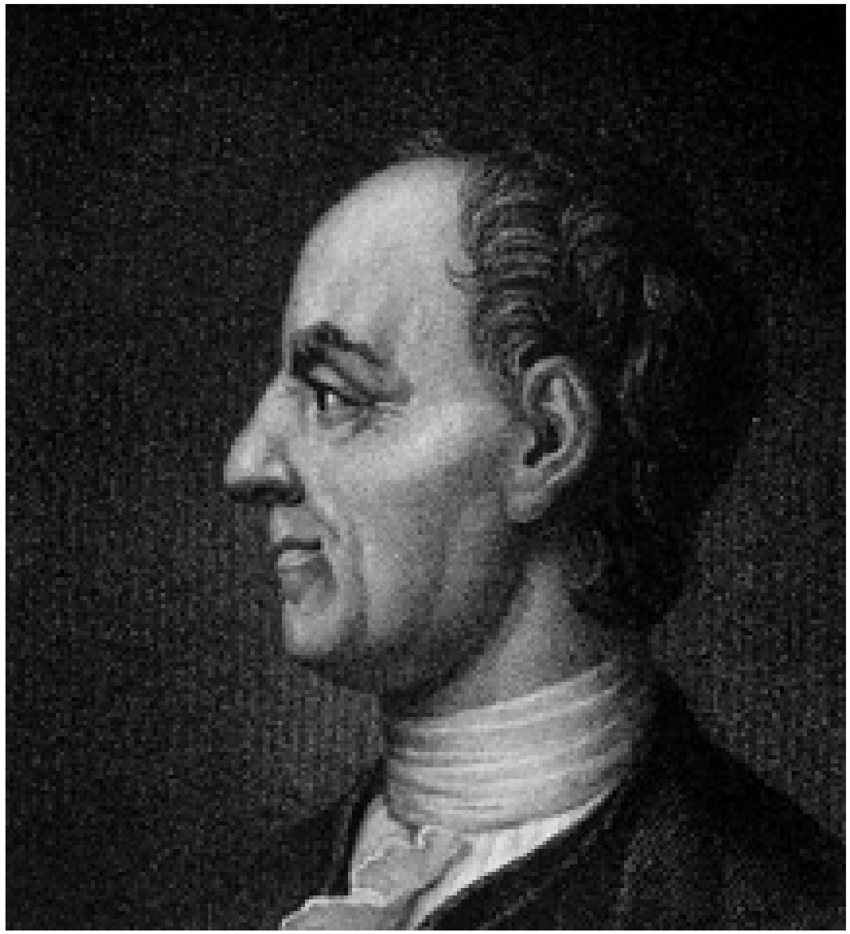


$$x_1 \cong x_o + h x_o'$$

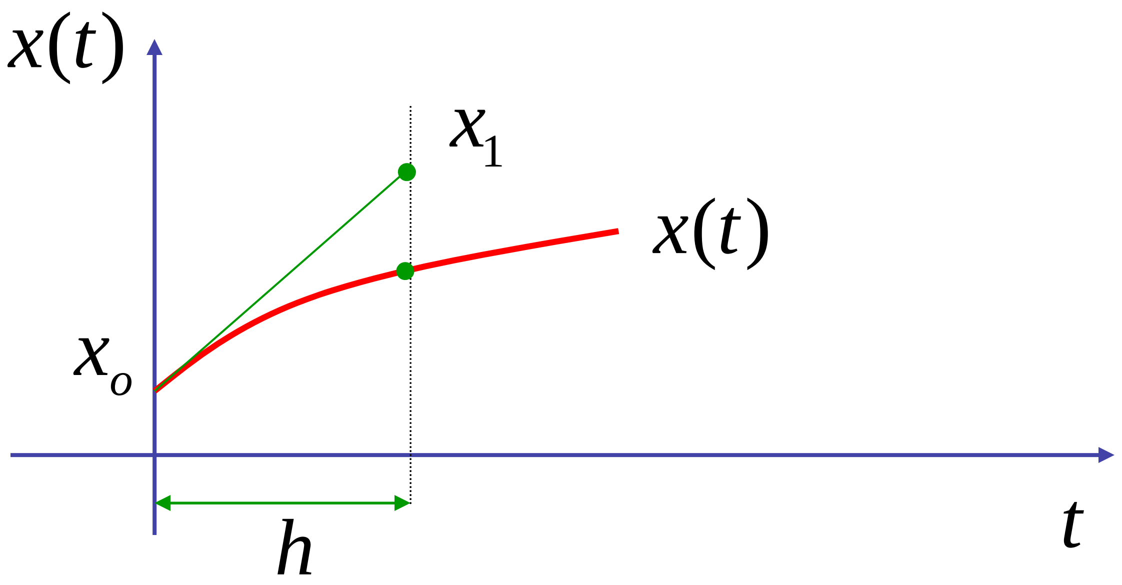
$$x_{n+1} \cong x_n + h x_n'$$

$$x_n' = \frac{x_{n+1} - x_n}{h}$$

Euler, Leonhard (1707-1783)



ForwardEuler



$$x_1 \cong x_o + h x_o'$$

$$x_{n+1} \cong x_n + h x_n'$$

$$x_n' = \frac{x_{n+1} - x_n}{h}$$

Application to Set of Differential Equations (FE)

$$\frac{dx}{dt} = Ax(t) + B(t)$$

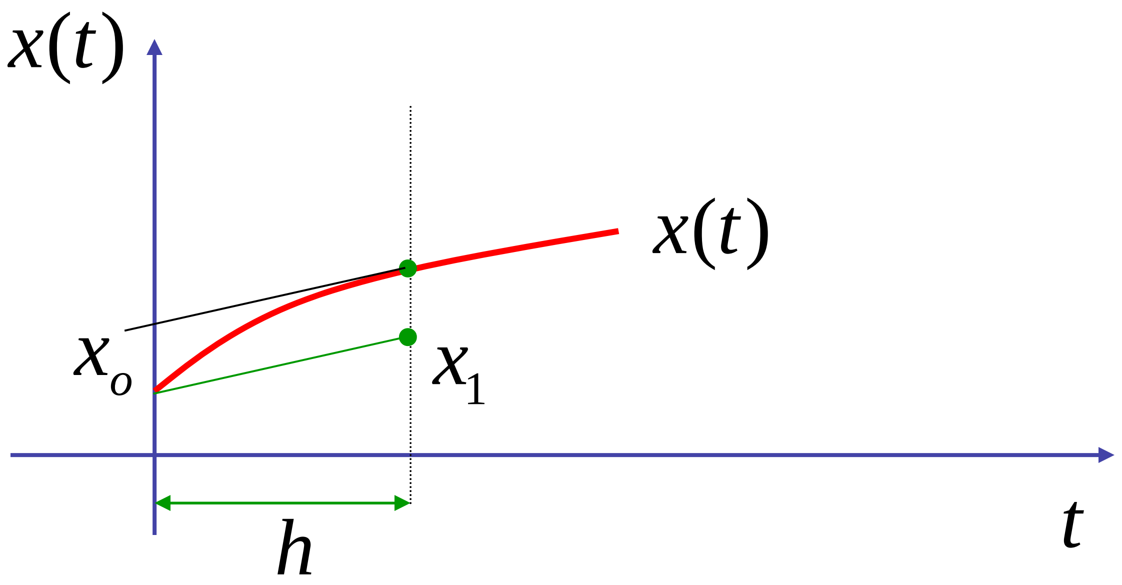
$$FE:$$

$$x_{n+1} = x_n + h\frac{dx_n}{dt}$$

$$= x_n + h\left(Ax_n + B_n\right)$$

$$x_{n+1} = \left(I + hA\right)x_n + hB_n$$

Bacward Euler



$$x_1 \cong x_o + h x_1'$$

$$x_{n+1} \cong x_n + h x_{n+1}'$$

$$x_{n+1}' = \frac{x_{n+1} - x_n}{h}$$

Application to Set of Differential Equations (BE)

$$\frac{dx}{dt} = Ax(t) + B(t)$$

$$BE:$$

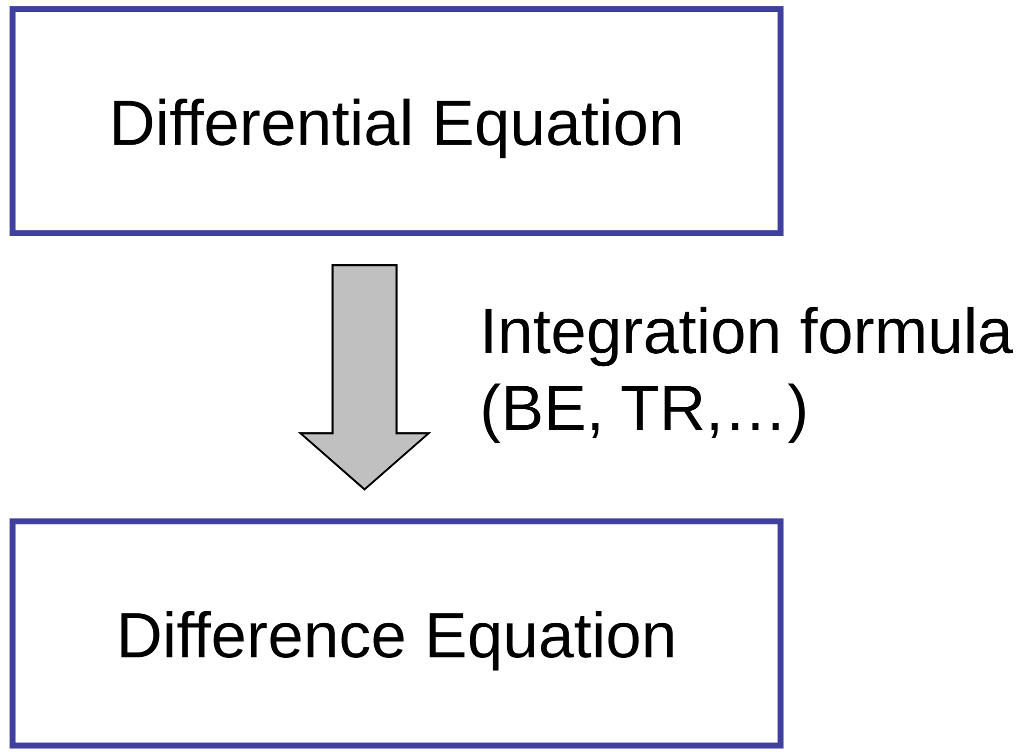
$$x_{n+1} = x_n + h\frac{dx_{n+1}}{dt}$$

$$= x_n + h\left(Ax_{n+1} + B_{n+1}\right)$$

$$\left(I - hA\right)x_{n+1} = x_n + hB_{n+1}$$

Numerical Integration

The solution of the differential equation is transformed into a solution of a system of linear equation at each time point.



Circuit Equations: Backward Euler

$$\mathbf{G}\mathbf{x}(t)+\mathbf{C}\dot{\mathbf{x}}(t)=\mathbf{b}(t)$$ → MNA equations

$$\mathbf{x}_{n+1}\cong\mathbf{x}_n+h\dot{\mathbf{x}}_{n+1}$$ → BE formula

$$\dot{\mathbf{x}}_{n+1}=\frac{1}{h}\left(\mathbf{x}_{n+1}-\mathbf{x}_n\right)$$

$$\mathbf{G}\mathbf{x}(t_{n+1})+\mathbf{C}\dot{\mathbf{x}}(t_{n+1})=\mathbf{b}(t_{n+1})$$

$$\mathbf{G}\mathbf{x}_{n+1}+\mathbf{C}\dot{\mathbf{x}}_{n+1}=\mathbf{b}_{n+1}$$

Circuit Equations: Backward Euler

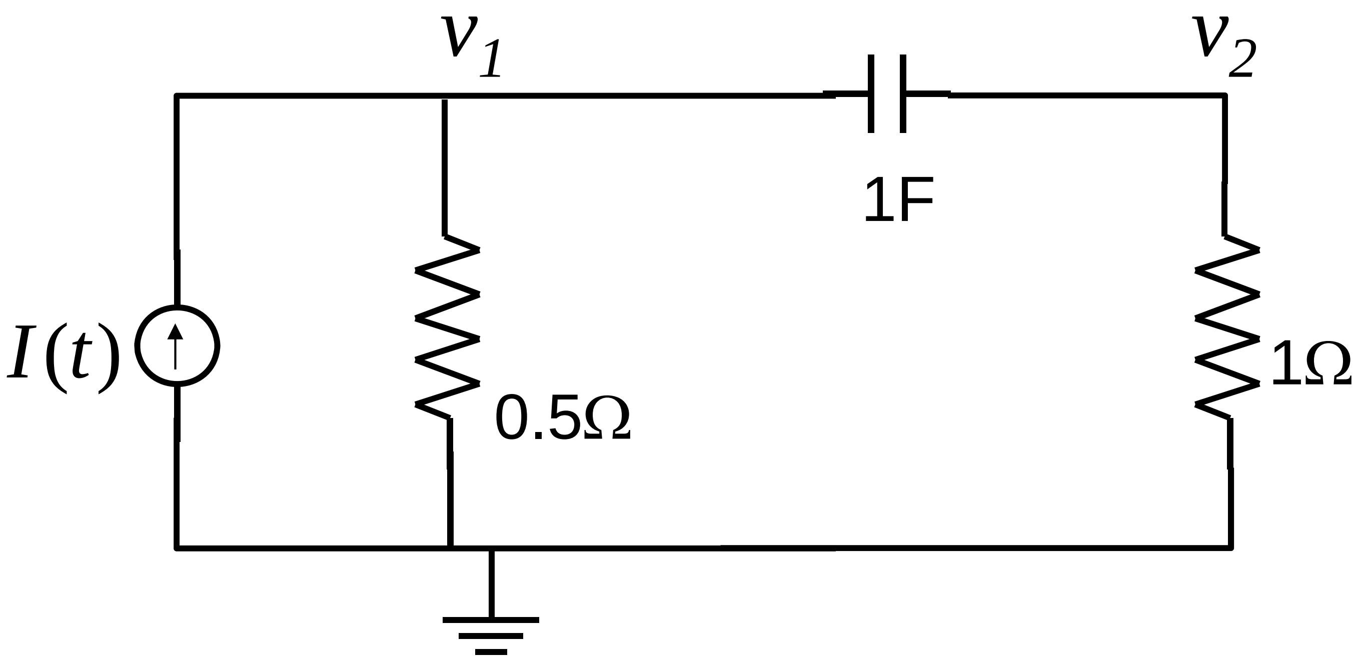
$$\begin{cases} \dot{\mathbf{x}}_{n+1} = \dfrac{1}{h}\left(\mathbf{x}_{n+1} - \mathbf{x}_n\right) \\ \mathbf{G}\mathbf{x}_{n+1} + \mathbf{C}\dot{\mathbf{x}}_{n+1} = \mathbf{b}_{n+1} \end{cases}$$

$$\Rightarrow \quad \mathbf{G}\mathbf{x}_{n+1} + \frac{\mathbf{C}}{h}\left(\mathbf{x}_{n+1} - \mathbf{x}_n\right) = \mathbf{b}_{n+1}$$

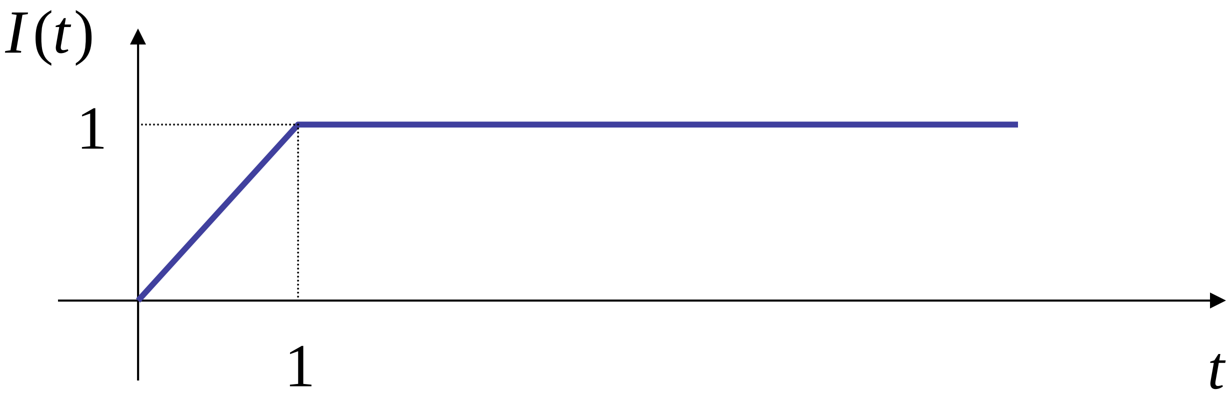
$$\Rightarrow \quad \boxed{\left(\mathbf{G} + \frac{\mathbf{C}}{h}\right)\mathbf{x}_{n+1} = \mathbf{b}_{n+1} + \frac{\mathbf{C}}{h}\mathbf{x}_n}$$

BE Difference Equation

Example



Zero initial condition



Example

$$\mathbf{G}\mathbf{x}(t) + \mathbf{C}\dot{\mathbf{x}}(t) = \mathbf{b}(t)$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} I(t) \\ 0 \end{bmatrix}$$

- Use Backward Euler
- Time step h=0.5

Example

Backward Euler

$$\left(\mathbf{G}+\frac{\mathbf{C}}{h}\right)\mathbf{x}_{n+1}=\frac{\mathbf{C}}{h}\mathbf{x}_n+\mathbf{b}_{n+1}$$

$$\mathbf{x}_0=\begin{bmatrix}0\\0\end{bmatrix}$$

$$\left(\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}+\frac{1}{0.5}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\right)\mathbf{x}_1=\frac{1}{0.5}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix}+\begin{bmatrix}0.5\\0\end{bmatrix}$$

$$\begin{bmatrix}4 & -2\\-2 & 3\end{bmatrix}\mathbf{x}_1=\begin{bmatrix}0.5\\0\end{bmatrix}\quad\Rightarrow\quad\mathbf{x}_1=\begin{bmatrix}0.1875\\0.125\end{bmatrix}$$

Example

Backward Euler

$$\left(\mathbf{G}+\frac{\mathbf{C}}{h}\right)\mathbf{x}_{n+1}=\frac{\mathbf{C}}{h}\mathbf{x}_n+\mathbf{b}_{n+1}$$

$$\mathbf{x}_1=\begin{bmatrix}0.1875\\0.125\end{bmatrix}$$

$$\left(\begin{bmatrix}2&0\\0&1\end{bmatrix}+\frac{1}{0.5}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}\right)\mathbf{x}_2=\frac{1}{0.5}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}\begin{bmatrix}0.1875\\0.125\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}$$

$$\begin{bmatrix}4&-2\\-2&3\end{bmatrix}\mathbf{x}_2=\begin{bmatrix}1.125\\-0.125\end{bmatrix} \Rightarrow \mathbf{x}_2=\begin{bmatrix}0.3906\\0.2188\end{bmatrix}$$

Example

Backward Euler

$$\left(\mathbf{G}+\frac{\mathbf{C}}{h}\right)\mathbf{x}_{n+1}=\frac{\mathbf{C}}{h}\mathbf{x}_n+\mathbf{b}_{n+1}$$

$$\mathbf{x}_2=\begin{bmatrix}0.3906\\0.2188\end{bmatrix}$$

$$\left(\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}+\frac{1}{0.5}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\right)\mathbf{x}_3=\frac{1}{0.5}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}0.3906\\0.2188\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}$$

Circuit Equations: Trapezoidal Rule

$$\mathbf{G}\mathbf{x}(t)+\mathbf{C}\dot{\mathbf{x}}(t)=\mathbf{b}(t)$$ → MNA equations

$$\mathbf{x}_{n+1}\cong\mathbf{x}_n+\frac{h}{2}\left(\dot{\mathbf{x}}_n+\dot{\mathbf{x}}_{n+1}\right)$$ → TR formula

$$\dot{\mathbf{x}}_{n+1}+\dot{\mathbf{x}}_n=\frac{2}{h}\left(\mathbf{x}_{n+1}-\mathbf{x}_n\right)$$

Circuit Equations: Trapezoidal Rule

$$\mathbf{Gx}(t) + \mathbf{C}\dot{\mathbf{x}}(t) = \mathbf{b}(t)$$ → MNA equations

$$\begin{cases} \mathbf{Gx}(t_n) + \mathbf{C}\dot{\mathbf{x}}(t_n) = \mathbf{b}(t_n) \\ \mathbf{Gx}_n + \mathbf{C}\dot{\mathbf{x}}_n = \mathbf{b}_n \end{cases}$$

$$\begin{cases} \mathbf{Gx}(t_{n+1}) + \mathbf{C}\dot{\mathbf{x}}(t_{n+1}) = \mathbf{b}(t_{n+1}) \\ \mathbf{Gx}_{n+1} + \mathbf{C}\dot{\mathbf{x}}_{n+1} = \mathbf{b}_{n+1} \end{cases}$$

Circuit Equations: Trapezoidal Rule

$$\begin{cases} \mathbf{G}\mathbf{x}_n + \mathbf{C}\dot{\mathbf{x}}_n = \mathbf{b}_n \\ \mathbf{G}\mathbf{x}_{n+1} + \mathbf{C}\dot{\mathbf{x}}_{n+1} = \mathbf{b}_{n+1} \end{cases}$$

$$\mathbf{G}\left(\mathbf{x}_n + \mathbf{x}_{n+1}\right) + \mathbf{C}\left(\dot{\mathbf{x}}_n + \dot{\mathbf{x}}_{n+1}\right) = \mathbf{b}_n + \mathbf{b}_{n+1}$$

$$\dot{\mathbf{x}}_{n+1} + \dot{\mathbf{x}}_n = \frac{2}{h}\left(\mathbf{x}_{n+1} - \mathbf{x}_n\right)$$

Circuit Equations: Trapezoidal Rule

$$\begin{cases} \mathbf{G}\left(\mathbf{x}_n + \mathbf{x}_{n+1}\right) + \mathbf{C}\left(\dot{\mathbf{x}}_n + \dot{\mathbf{x}}_{n+1}\right) = \mathbf{b}_n + \mathbf{b}_{n+1} \\ \dot{\mathbf{x}}_{n+1} + \dot{\mathbf{x}}_n = \dfrac{2}{h}\left(\mathbf{x}_{n+1} - \mathbf{x}_n\right) \end{cases}$$

$$\mathbf{G}\left(\mathbf{x}_n + \mathbf{x}_{n+1}\right) + \frac{2\mathbf{C}}{h}\left(\mathbf{x}_{n+1} - \mathbf{x}_n\right) = \mathbf{b}_n + \mathbf{b}_{n+1}$$

$$\boxed{\left(\mathbf{G} + \frac{2\mathbf{C}}{h}\right)\mathbf{x}_{n+1} = \left(\frac{2\mathbf{C}}{h} - \mathbf{G}\right)\mathbf{x}_n + \mathbf{b}_n + \mathbf{b}_{n+1}}$$

Difference Equations

Backward Euler

$$\left(\mathbf{G}+\frac{\mathbf{C}}{h}\right)\mathbf{x}_{n+1}=\frac{\mathbf{C}}{h}\mathbf{x}_n+\mathbf{b}_{n+1}$$

Trapezoidal Rule

$$\left(\mathbf{G}+\frac{2\mathbf{C}}{h}\right)\mathbf{x}_{n+1}=\left(\frac{2\mathbf{C}}{h}-\mathbf{G}\right)\mathbf{x}_n+\mathbf{b}_n+\mathbf{b}_{n+1}$$

Forward Euler Example

$$\begin{cases} v' + v = t^2 \\ v(0) = 1 \end{cases}$$

$$v_{n+1} \cong (1-h)v_n + ht_n^2 \qquad h = 0.025$$

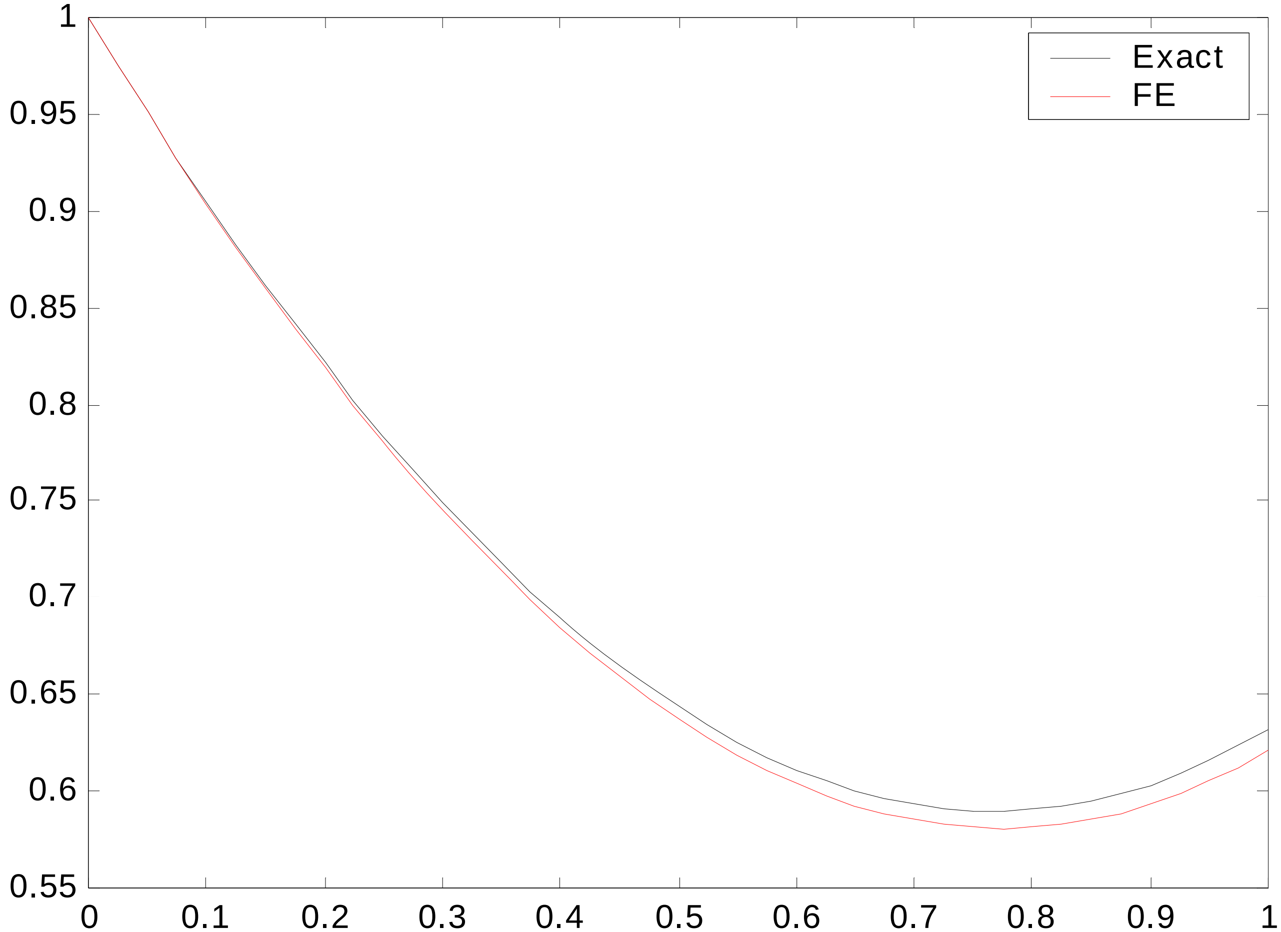
$$v_o \cong 1$$

$$v_1 \cong (1-0.025)v_o + 0.025 \times 0^2 = 0.975$$ **0.9753**

$$\mathrm{v}_2 \cong (1-0.025) \times 0.975 + 0.025 \times 0.025^2 = 0.9507$$

$\vdots$

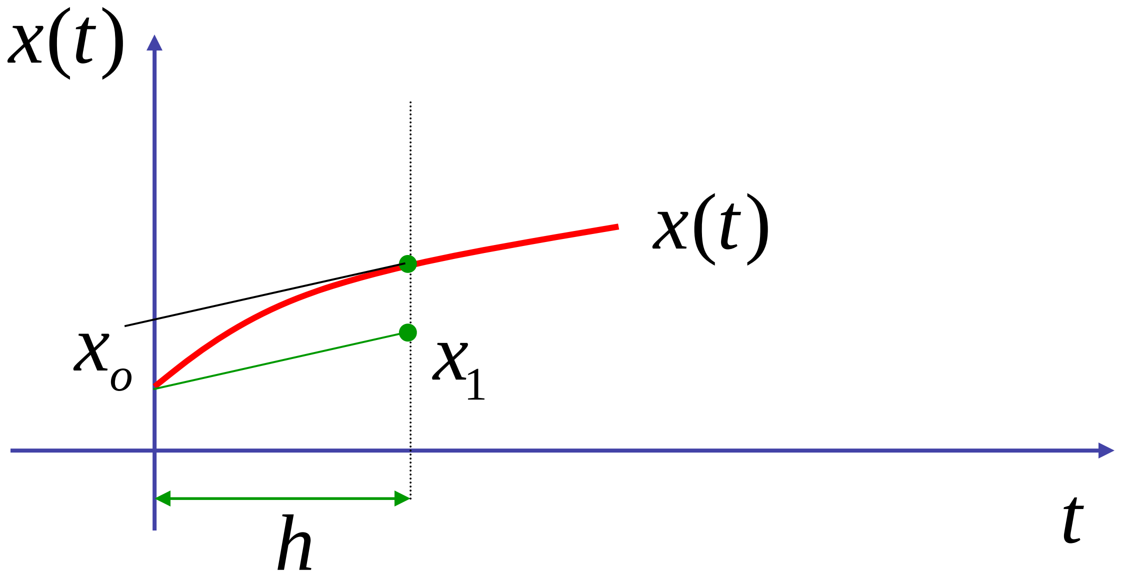
0.9513



Exact
FE
1
0.95
0.9
0.85
0.8
0.75
0.7
0.65
0.6
0.55
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Time	Vexact	VFE	|error|
0	1.0000	1.0000	0
2.5000e-002	9.7532e-001	9.7500e-001	3.1509e-004
5.0000e-002	9.5127e-001	9.5064e-001	6.2995e-004
7.5000e-002	9.2788e-001	9.2694e-001	9.4440e-004
1.0000e-001	9.0516e-001	9.0390e-001	1.2583e-003
1.2500e-001	8.8313e-001	8.8156e-001	1.5714e-003
1.5000e-001	8.6179e-001	8.5991e-001	1.8836e-003
1.7500e-001	8.4117e-001	8.3897e-001	2.1948e-003
2.0000e-001	8.2127e-001	8.1876e-001	2.5048e-003
2.2500e-001	8.0211e-001	7.9930e-001	2.8134e-003
2.5000e-001	7.8370e-001	7.8058e-001	3.1206e-003
2.7500e-001	7.6605e-001	7.6263e-001	3.4262e-003

Bacward Euler



$$x_1 \cong x_o + h x_1'$$

$$x_{n+1} \cong x_n + h x_{n+1}'$$

$$x_{n+1}' = \frac{x_{n+1} - x_n}{h}$$

Backward Euler

$$\dot{v} + v = t^2$$ → Differential Equation

$$v_{n+1} \cong v_n + h v'_{n+1}$$ → Integration formula (Backward Euler)

$$(1+h)v_{n+1} \cong \left(v_n + h t_{n+1}^2\right) \qquad v_{n+1} \cong (1+h)^{-1}\left(v_n + h t_{n+1}^2\right)$$

Backward Euler Example

$$\begin{cases} \dot{v} + v = t^2 \\ v(0) = 1 \end{cases}$$

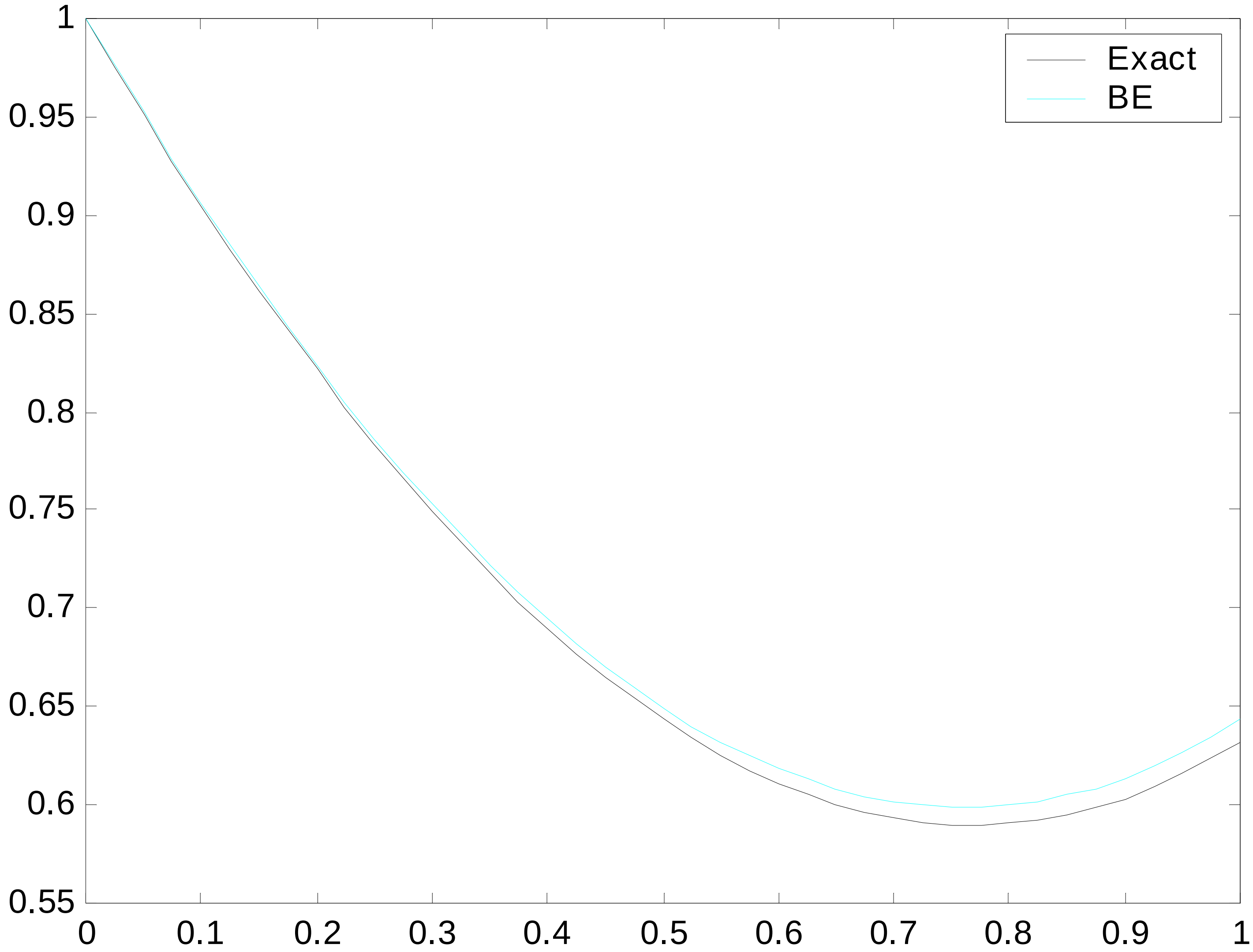
$$v_{n+1} \cong (1+h)^{-1}\left(v_n + h t_{n+1}^2\right) \qquad h = 0.025$$

$$v_o \cong 1$$

$$v_1 \cong (1+0.025)^{-1}\left(v_o + 0.025 \times 0.025^2\right) = 0.9756$$

$$v_2 \cong (1+0.025)^{-1}\left(0.9756 + 0.025 \times 0.05^2\right) = \ldots$$

$$\vdots$$


Exact
BE
1
0.95
0.9
0.85
0.8
0.75
0.7
0.65
0.6
0.55
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Time	Vexact	VBE	|error|
0	1.0000	1.0000	0
2.5000e-002	9.7532e-001	9.7563e-001	3.0991e-004
5.0000e-002	9.5127e-001	9.5189e-001	6.1967e-004
7.5000e-002	9.2788e-001	9.2881e-001	9.2909e-004
1.0000e-001	9.0516e-001	9.0640e-001	1.2380e-003
1.2500e-001	8.8313e-001	8.8467e-001	1.5463e-003
1.5000e-001	8.6179e-001	8.6365e-001	1.8537e-003
1.7500e-001	8.4117e-001	8.4333e-001	2.1602e-003
2.0000e-001	8.2127e-001	8.2373e-001	2.4655e-003
2.2500e-001	8.0211e-001	8.0488e-001	2.7696e-003
2.5000e-001	7.8370e-001	7.8677e-001	3.0724e-003
2.7500e-001	7.6605e-001	7.6943e-001	3.3737e-003

Trapezoidal Rule

$$\dot{v} + v = t^2$$

$$x_{n+1} \cong x_n + \frac{h}{2}\left(x'_n + x'_{n+1}\right)$$

$$\left(1 + \frac{h}{2}\right) v_{n+1} \cong \left(1 - \frac{h}{2}\right) v_n + \frac{h}{2}(t_n^2 + t_{n+1}^2)$$

$$v_{n+1} \cong \left(1 + \frac{h}{2}\right)^{-1} \left[\left(1 - \frac{h}{2}\right) v_n + \frac{h}{2}(t_n^2 + t_{n+1}^2)\right]$$

TR Example

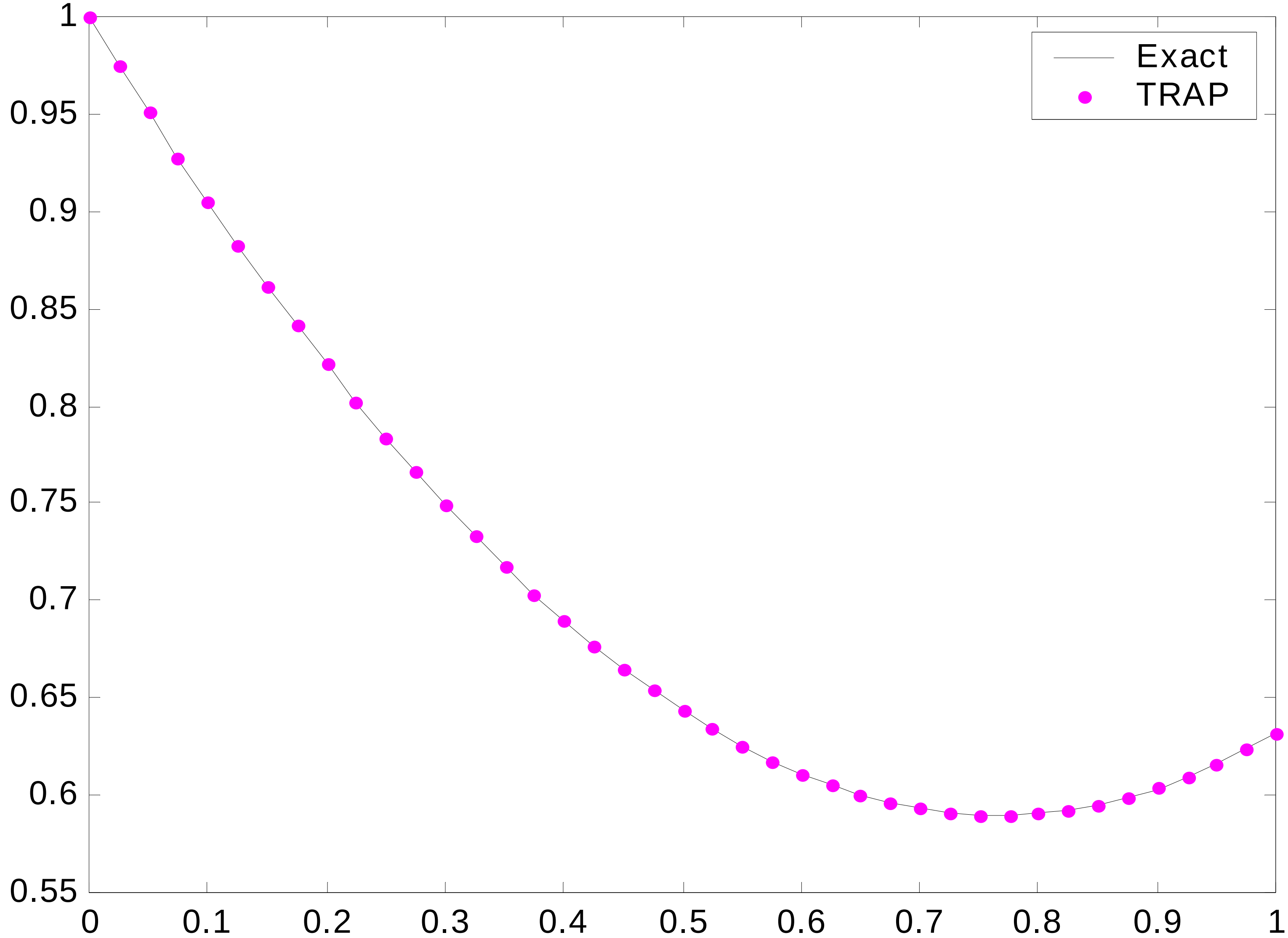
$$\begin{cases} \dot{v} + v = t^2 \\ v(0) = 1 \end{cases} \qquad h = 0.025$$

$$v_{n+1} \cong \left(1 + \frac{h}{2}\right)^{-1} \left[\left(1 - \frac{h}{2}\right) v_n + \frac{h}{2}(t_n^2 + t_{n+1}^2) \right]$$

$$v_o \cong 1$$

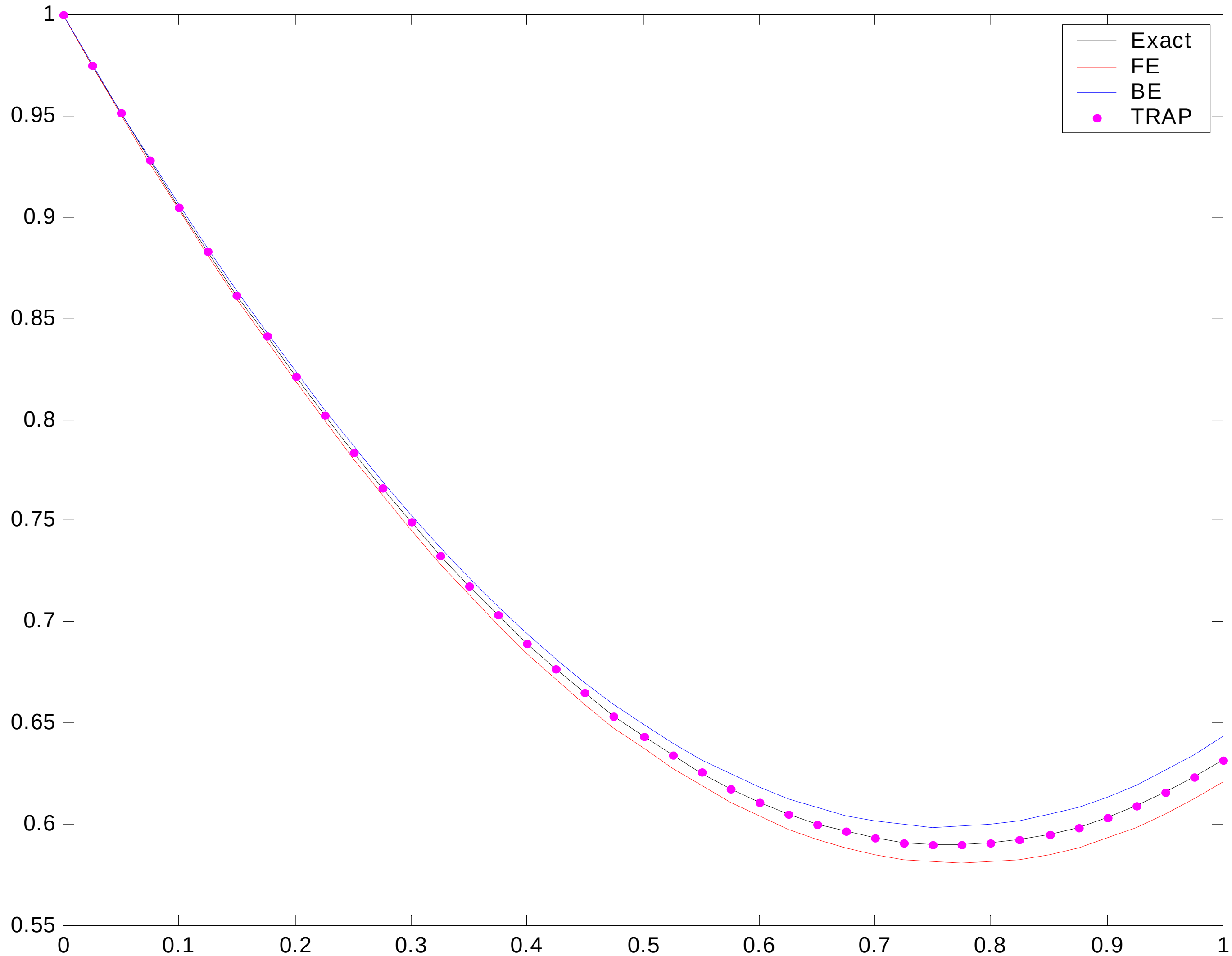
$$v_1 \cong \left(1 + \frac{0.025}{2}\right)^{-1} \left[\left(1 - \frac{0.025}{2}\right) 1 + \frac{h}{2}(0^2 + 0.025^2) \right]$$

⋮



Exact
TRAP
1
0.95
0.9
0.85
0.8
0.75
0.7
0.65
0.6
0.55
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Time	**Vexact**	**VTrap**	**|error|**
0	1.0000	1.0000	0
2.5000e-002	9.7532e-001	9.7532e-001	1.2701e-006
5.0000e-002	9.5127e-001	9.5127e-001	2.4774e-006
7.5000e-002	9.2788e-001	9.2789e-001	3.6243e-006
1.0000e-001	9.0516e-001	9.0517e-001	4.7131e-006
1.2500e-001	8.8313e-001	8.8313e-001	5.7459e-006
1.5000e-001	8.6179e-001	8.6180e-001	6.7249e-006
1.7500e-001	8.4117e-001	8.4118e-001	7.6520e-006
2.0000e-001	8.2127e-001	8.2128e-001	8.5292e-006
2.2500e-001	8.0211e-001	8.0212e-001	9.3584e-006
2.5000e-001	7.8370e-001	7.8371e-001	1.0142e-005
2.7500e-001	7.6605e-001	7.6606e-001	1.0880e-005


Exact
FE
BE
TRAP
1
0.95
0.9
0.85
0.8
0.75
0.7
0.65
0.6
0.55
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1