Mathematical Background:

Eigenvalues/Eigenvectors/Poles/Zeros

Eigenvalues/Eigenvectors (Summary)

$$Ax = \lambda x$$

$$\left| A - \lambda I \right| = 0$$

$$A = P \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & - & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix} P^{-1}$$

$$f\left(A\right) = Pf \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & - & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix} P^{-1}$$

Eigenvalues/Eigenvectors (Summary)

$$A \sim B \quad if \quad A = QBQ^{-1}$$

$$if \quad A \sim B \quad then \quad eigenvalues(A) \equiv eigenvalues(B) \quad (prove)$$

$$if \quad A \sim B;\ what\ are\ the\ eigenvectors\ of\ B\,?$$

$$\lim_{n \to \infty} A^n b \equiv$$

The eigenvector of A corresponding to the largest eigenvalue (in magnitude)

MNA Equations

$$Cx' + Gx = bu(t); \quad y = L^t x$$

$$-G^{-1}Cx' = x - G^{-1}bu(t) \Rightarrow Ax' = x - Ru(t)$$

$$A = -G^{-1}C; \qquad R = G^{-1}b$$

$$\Rightarrow X(s) = \left(I - sA\right)^{-1} RU(s)$$

$$\boxed{Moments: \quad R, AR, A^2R, A^3R......}$$

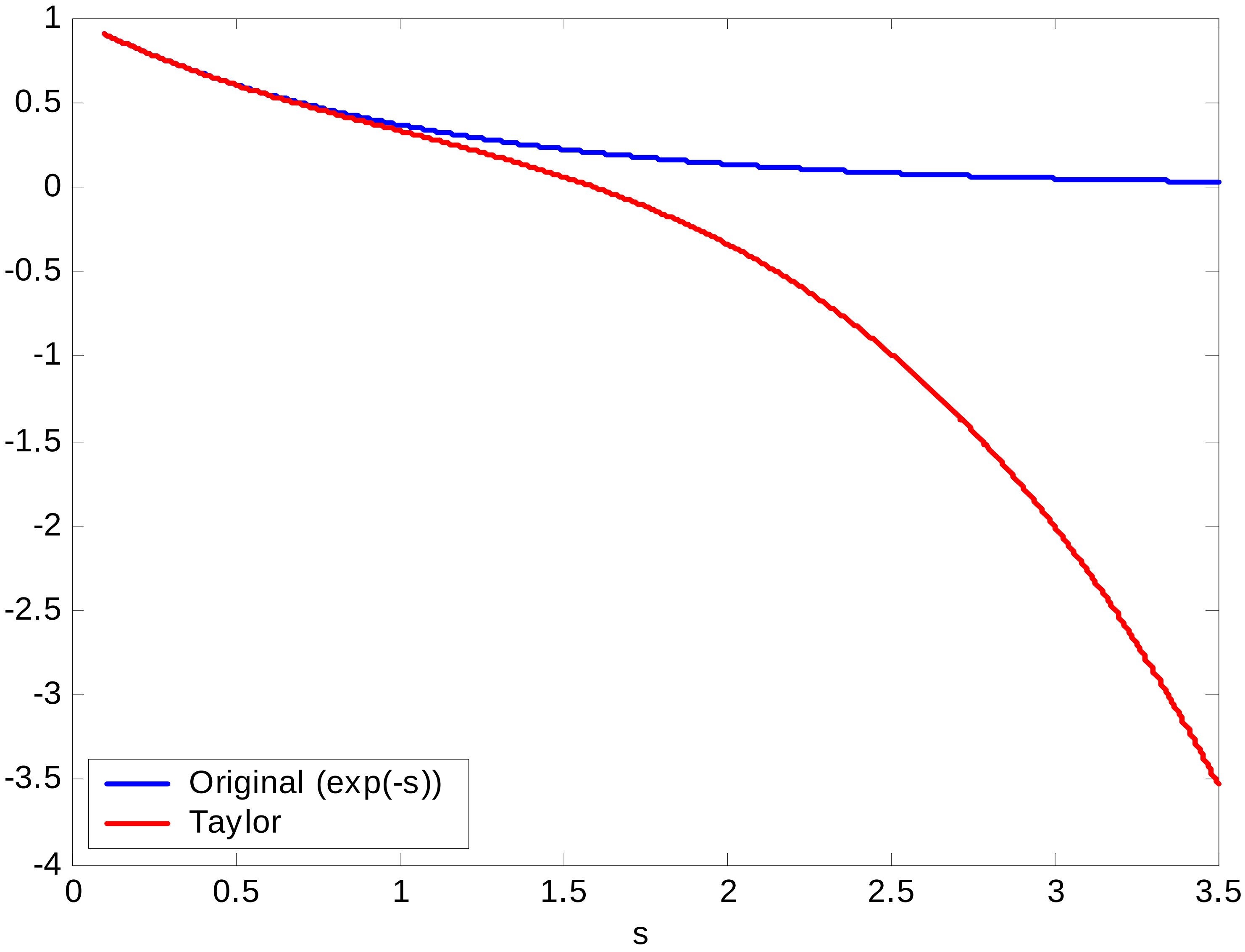
Homework:
Find the relationship between the eigenvalues of A and the poles of the circuit

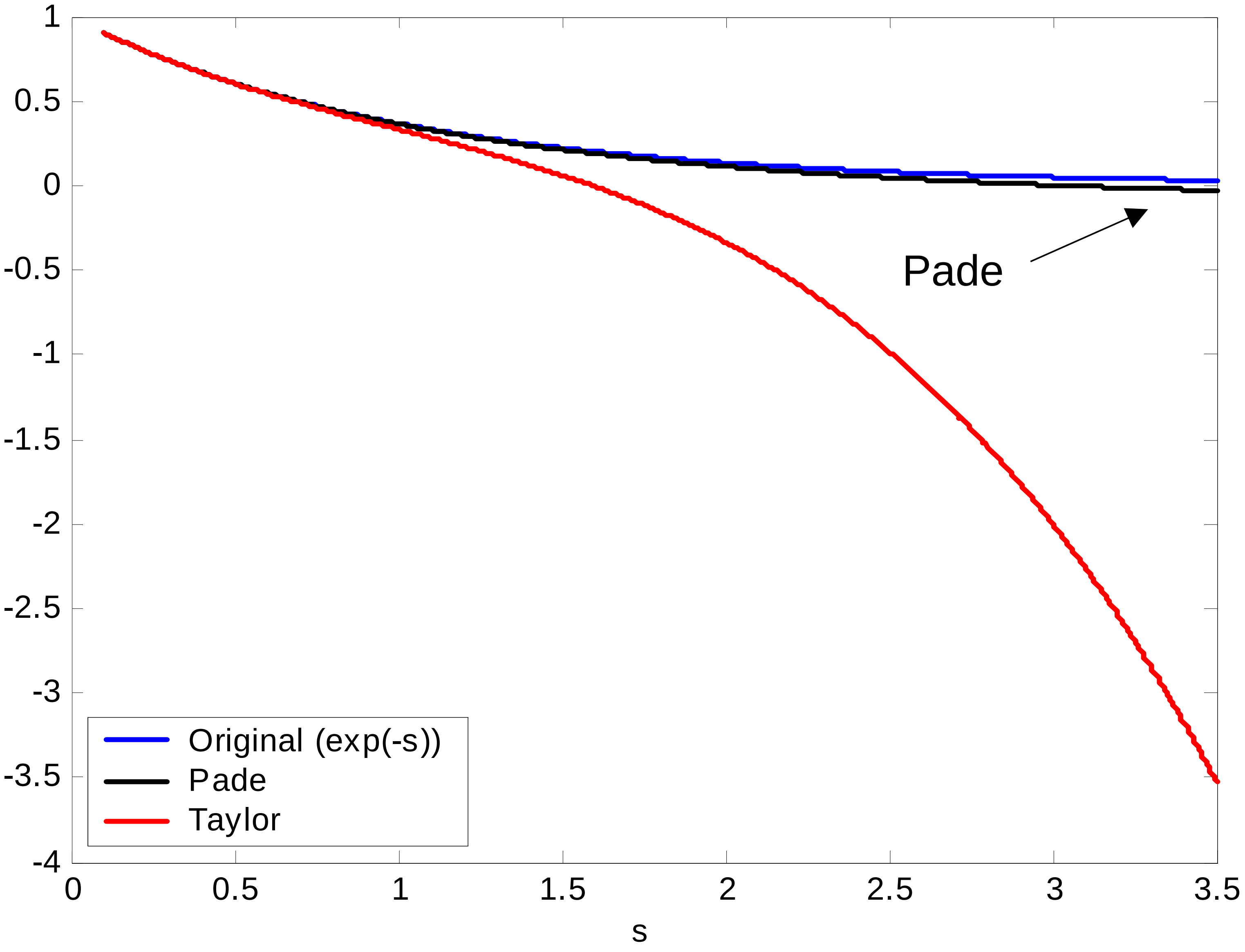
Mathematical Background:

Pade' Approximation

Henri Eugène Padé

Born: 17 Dec 1863 in Abbeville, Picardy, France,
Died: 9 July 1953 in Aix-en-Provence, France


1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
-4
0
0.5
1
1.5
2
2.5
3
3.5
s
Original (exp(-s))
Taylor


1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
-4
0
0.5
1
1.5
2
2.5
3
3.5
s
Pade
Original (exp(-s))
Pade
Taylor

Padé Approximation

$$H(s) = m_o + m_1 s + m_2 s^2 + \ldots + m_{L+M} s^{L+M} = \frac{a_o + a_1 s + a_2 s^2 + \ldots + a_L s^L}{1 + b_1 s + b_2 s^2 + \ldots + b_M s^M}$$

- The Padé approximation matches the first L+M+1 moments
- The Padé approximation is more accurate than the original Taylor expansion!
- The “poles” are the roots of the denominator.

Padé Approximation

$$h(s) = \frac{a_o + a_1 s + a_2 s^2 + \ldots}{1 + b_1 s + b_2 s^2 + \ldots}$$

$$m_o + m_1 s + m_2 s^2 + \ldots + m_n s^n = \frac{a_o + a_1 s + a_2 s^2 + \ldots + a_L s^L}{1 + b_1 s + b_2 s^2 + \ldots + b_M s^M}$$

$$\left(m_o + m_1 s + m_2 s^2 + \ldots + m_n s^n\right)\left(1 + b_1 s + b_2 s^2 + \ldots + b_M s^M\right) =$$

$$= a_o + a_1 s + a_2 s^2 + \ldots + a_L s^L$$

Padé Approximation

$$\left(m_o + m_1 s + m_2 s^2 + \ldots + m_n s^n\right)\left(1 + b_1 s + b_2 s^2 + \ldots + b_M s^M\right) =$$

$$= a_o + a_1 s + a_2 s^2 + \ldots + a_L s^L$$

Equate powers of s for powers > L

$$m_{L+1} + m_L b_1 + m_{L-1} b_2 + \ldots + m_{L-M+1} b_M = 0$$

$$m_{L+2} + m_{L+1} b_1 + m_L b_2 + \ldots + m_{L-M+2} b_M = 0$$

$$\vdots$$

$$m_{L+M} + m_{L+M-1} b_1 + m_{L+M-2} b_2 + \ldots + m_L b_M = 0$$

Padé Approximation

Special Case: L=M-1

$$\begin{bmatrix} m_0 & m_1 & \cdots & m_L \\ m_1 & m_2 & \cdots & m_{L+1} \\ \vdots & \vdots & \ddots & \vdots \\ m_L & m_{L+1} & \cdots & m_{2L} \end{bmatrix} \begin{bmatrix} b_M \\ b_{M-1} \\ \vdots \\ b_1 \end{bmatrix} = - \begin{bmatrix} m_{L+1} \\ m_{L+2} \\ \vdots \\ m_{2L+1} \end{bmatrix}$$

L=4, M=5

$$\begin{bmatrix} m_o & m_1 & m_2 & m_3 & m_4 \\ m_1 & m_2 & m_3 & m_4 & m_5 \\ m_2 & m_3 & m_4 & m_5 & m_6 \\ m_3 & m_4 & m_5 & m_6 & m_7 \\ m_4 & m_5 & m_6 & m_7 & m_8 \end{bmatrix} \begin{bmatrix} b_5 \\ b_4 \\ b_3 \\ b_2 \\ b_1 \end{bmatrix} = - \begin{bmatrix} m_5 \\ m_6 \\ m_7 \\ m_8 \\ m_9 \end{bmatrix}$$

Padé Approximation

$$m_{L+1} + m_L b_1 + m_{L-1} b_2 + \ldots + m_{L-M+1} b_M = 0$$

$$m_{L+2} + m_{L+1} b_1 + m_L b_2 + \ldots + m_{L-M+2} b_M = 0$$

Eq(98)

$$\vdots$$

$$m_{L+M} + m_{L+M-1} b_1 + m_{L+M-2} b_2 + \ldots + m_L b_M = 0$$

$$\begin{bmatrix} m_{L-M+1} & m_{L-M+2} & \cdots & m_L \\ m_{L-M+2} & m_{L-M+3} & \cdots & m_{L+1} \\ \vdots & \vdots & \ddots & \vdots \\ m_L & m_{L+1} & \cdots & m_{L+M-1} \end{bmatrix} \begin{bmatrix} b_M \\ b_{M-1} \\ \vdots \\ b_1 \end{bmatrix} = - \begin{bmatrix} m_{L+1} \\ m_{L+2} \\ \vdots \\ m_{L+M} \end{bmatrix}$$

Padé Approximation

$$(m_o + m_1 s + m_2 s^2 + \ldots + m_n s^n)(1 + b_1 s + b_2 s^2 + \ldots + b_M s^M) =$$

$$= a_o + a_1 s + a_2 s^2 + \ldots + a_L s^L$$

Equate powers of s for powers <= L

$$\left.\begin{aligned} a_0 &= m_o \\ a_1 &= m_o b_1 + m_1 \\ a_2 &= m_o b_2 + m_1 b_1 + m_2 \\ &\vdots \\ a_L &= m_L + \sum_{n=1}^{\min(L,M)} b_n m_{L-n} \end{aligned}\right\} \boldsymbol{a_k = m_k + \sum_{n=1}^{k} b_n m_{k-n}}$$

Eq.(99)

Example

L=4, M=5

$$\begin{bmatrix} m_o & m_1 & m_2 & m_3 & m_4 \\ m_1 & m_2 & m_3 & m_4 & m_5 \\ m_2 & m_3 & m_4 & m_5 & m_6 \\ m_3 & m_4 & m_5 & m_6 & m_7 \\ m_4 & m_5 & m_6 & m_7 & m_8 \end{bmatrix} \begin{bmatrix} b_5 \\ b_4 \\ b_3 \\ b_2 \\ b_1 \end{bmatrix} = - \begin{bmatrix} m_5 \\ m_6 \\ m_7 \\ m_8 \\ m_9 \end{bmatrix}$$

$$a_0 = m_o$$

$$a_1 = m_o b_1 + m_1$$

$$a_2 = m_o b_2 + m_1 b_1 + m_2$$

$$a_3 = m_o b_3 + m_1 b_2 + m_2 b_1 + m_3$$

$$a_4 = m_4 + \sum_{n=1}^{4} b_n m_{4-n}$$

Animated examples

http://math.fullerton.edu/mathews/numerical.html

Poles/Residues

$$\begin{bmatrix} \frac{1}{p_1} & \frac{1}{p_2} & \cdots & \frac{1}{p_q} \\ \frac{1}{(p_1)^2} & \frac{1}{(p_2)^2} & \cdots & \frac{1}{(p_q)^2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{(p_1)^q} & \frac{1}{(p_2)^q} & \cdots & \frac{1}{(p_q)^q} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ \cdots \\ k_q \end{bmatrix} = - \begin{bmatrix} m_0 \\ m_1 \\ \cdots \\ m_{q-1} \end{bmatrix}$$

Eq(102)

Pole-Residue Model

$$\mathbf{X(s)} = \sum_{\mathbf{i}} \frac{\mathbf{k_i}}{\mathbf{s - p_i}}$$

Frequency Response

$$\mathbf{G}\mathbf{X}(s) + s\mathbf{C}\mathbf{X}(s) = \mathbf{b}$$

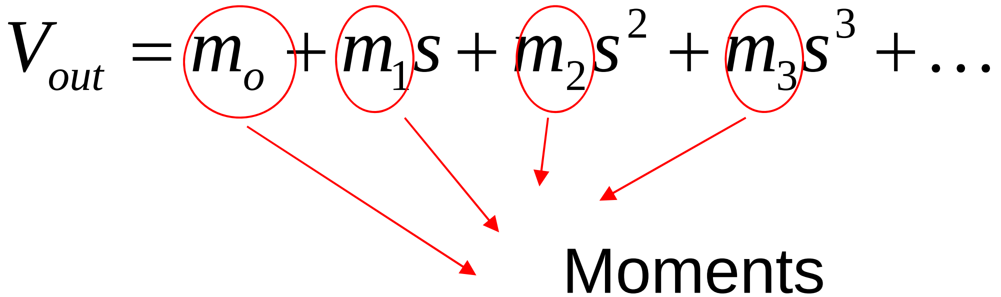
$$\mathbf{X}(s) = \begin{bmatrix} \vdots \\ V_{out} \\ \vdots \end{bmatrix}$$

At each frequency point:

$$\mathbf{X}(j\omega) = \left(\mathbf{G} + j\omega\mathbf{C}\right)^{-1}\mathbf{b}$$

100 frequency points → 100 LU decompositions

Asymptotic Waveform Evaluation



$$V_{out} = m_o + m_1 s + m_2 s^2 + m_3 s^3 + \ldots$$

Moments

$$\left(\mathbf{G} + s\mathbf{C}\right)\mathbf{X}(s) = \mathbf{b}$$

$$\mathbf{X}(s) = M_o + M_1 s + M_2 s^2 + M_3 s^3 + \ldots$$

$$\left(\mathbf{G} + s\mathbf{C}\right)\left(M_o + M_1 s + M_2 s^2 + M_3 s^3 + \ldots\right) = \mathbf{b}$$

Moment Calculation

Impulse response →

$$\mathbf{b} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

$$\left(\mathbf{G} + s\mathbf{C}\right)\left(M_o + M_1 s + M_2 s^2 + M_3 s^3 + \ldots\right) = \mathbf{b}$$

$$\mathbf{G}M_o = \mathbf{b}$$

$$\mathbf{G}M_1 = -\mathbf{C}M_o \quad \Longrightarrow \quad M_1 = -\mathbf{G}^{-1}\mathbf{C}M_o$$

Moment Calculation

$$(\mathbf{G}+s\mathbf{C})\left(M_o+M_1s+M_2s^2+M_3s^3+\ldots\right)=\mathbf{b}$$

$$\mathbf{G}M_o=\mathbf{b}$$

$$\begin{aligned}
\mathbf{G}M_1&=-\mathbf{C}M_o &&\Rightarrow && M_1=-\mathbf{G}^{-1}\mathbf{C}M_o\\
\mathbf{G}M_2&=-\mathbf{C}M_1 &&\Rightarrow && M_2=-\mathbf{G}^{-1}\mathbf{C}M_1\\
& &&\vdots && \\
\mathbf{G}M_{n+1}&=-\mathbf{C}M_n &&\Rightarrow && M_{n+1}=-\mathbf{G}^{-1}\mathbf{C}M_n
\end{aligned}$$

Cost of moments → 1 LU decomposition

Moment Calculation

$$\mathbf{X}(s) = \begin{bmatrix} \vdots \\ V_{out} \\ \vdots \end{bmatrix} \qquad V_{out} = m_o + m_1 s + m_2 s^2 + m_3 s^3 + \ldots$$

$$M_o = \begin{bmatrix} \vdots \\ m_o \\ \vdots \end{bmatrix} \qquad M_1 = \begin{bmatrix} \vdots \\ m_1 \\ \vdots \end{bmatrix} \qquad M_2 = \begin{bmatrix} \vdots \\ m_2 \\ \vdots \end{bmatrix}$$

Padé Approximation

$$V_{out}(s) = h(s) = m_o + m_1 s + m_2 s^2 + m_3 s^3 + \ldots$$

$$h(j\omega) = m_o + m_1 (j\omega) + m_2 (j\omega)^2 + m_3 (j\omega)^3 + \ldots$$

↳ Taylor series is ill conditioned

Padé Approximation:

$$h(s) = \frac{a_o + a_1 s + a_2 s^2 + \ldots}{1 + b_1 s + b_2 s^2 + \ldots}$$

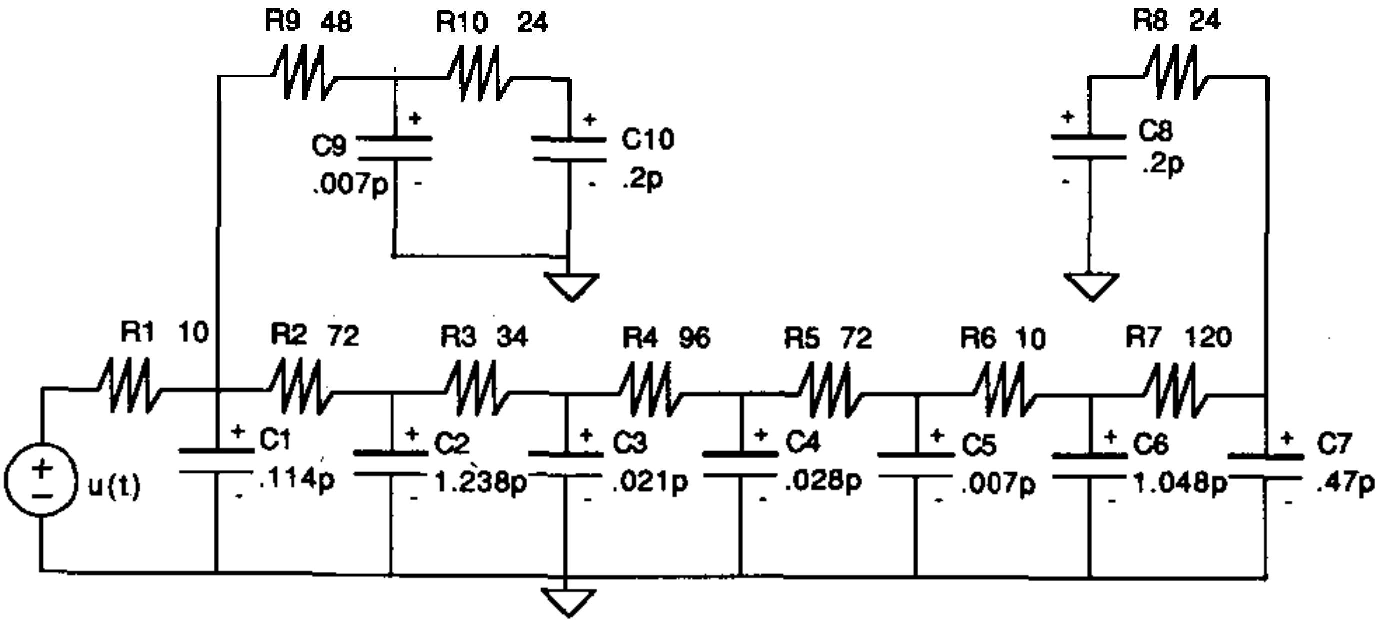


Fig. 16. *RC* tree with widely varying time constants.

L. Pillage, R. Rohrer, "Asymptotic Waveform Evaluation for Timing Analysis", IEEE Transactions on CAD, April 1990

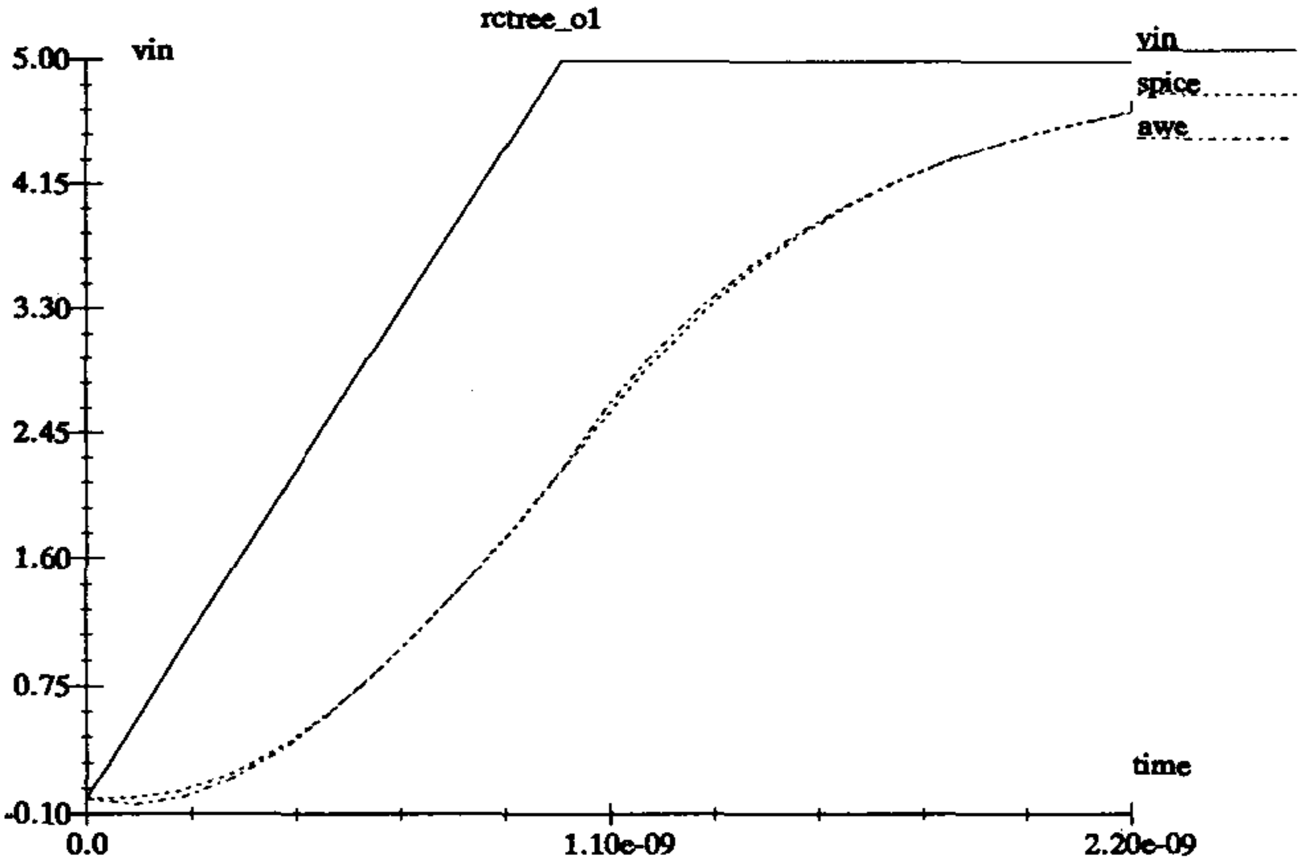


Fig. 17. First-order approximation for the voltage at capacitor C_7 in the circuit of Fig. 16.

L. Pillage, R. Rohrer, "Asymptotic Waveform Evaluation for Timing Analysis", IEEE Transactions on CAD, April 1990

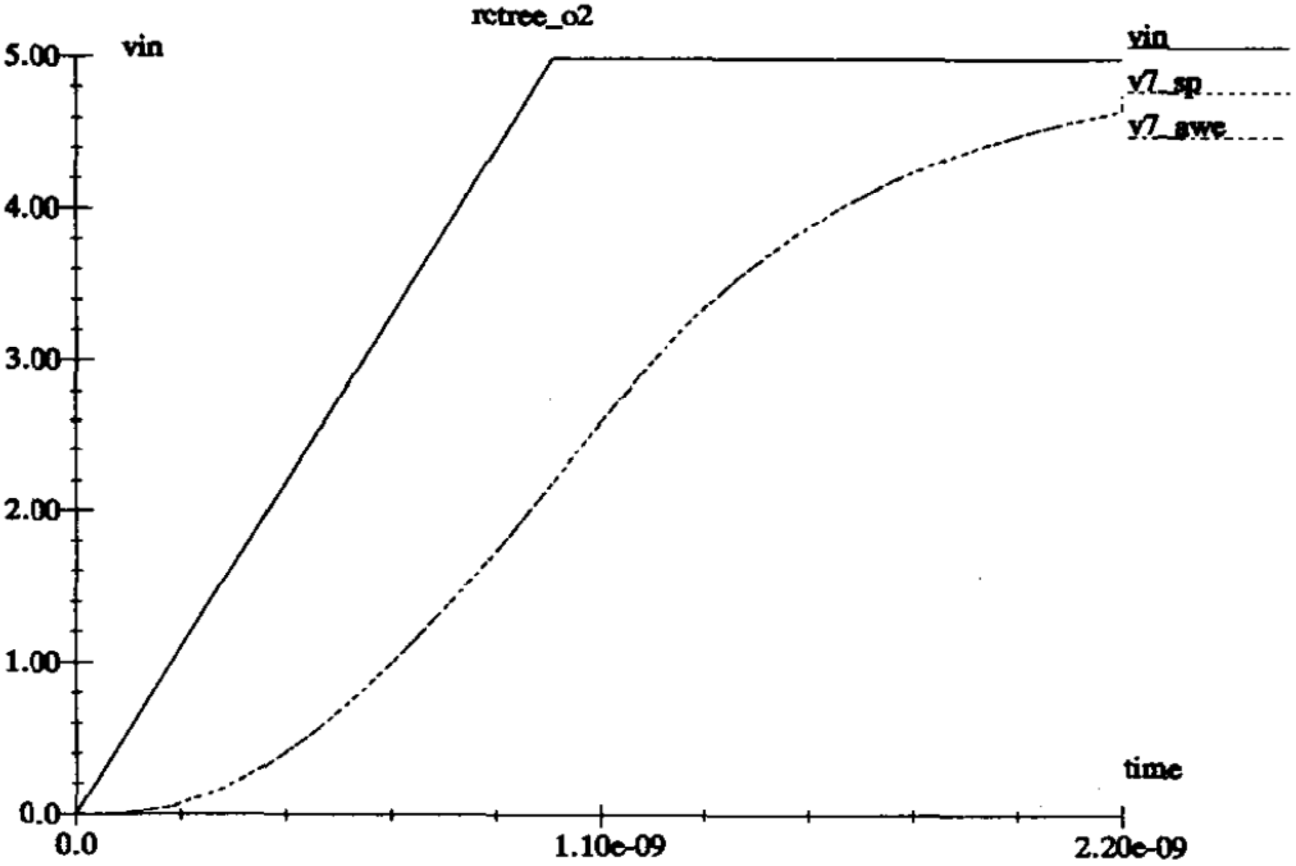


Fig. 18. Second-order approximation for the voltage at capacitor C_7 in the circuit of Fig. 16.

L. Pillage, R. Rohrer, “Asymptotic Waveform Evaluation for Timing Analysis”, IEEE Transactions on CAD, April 1990

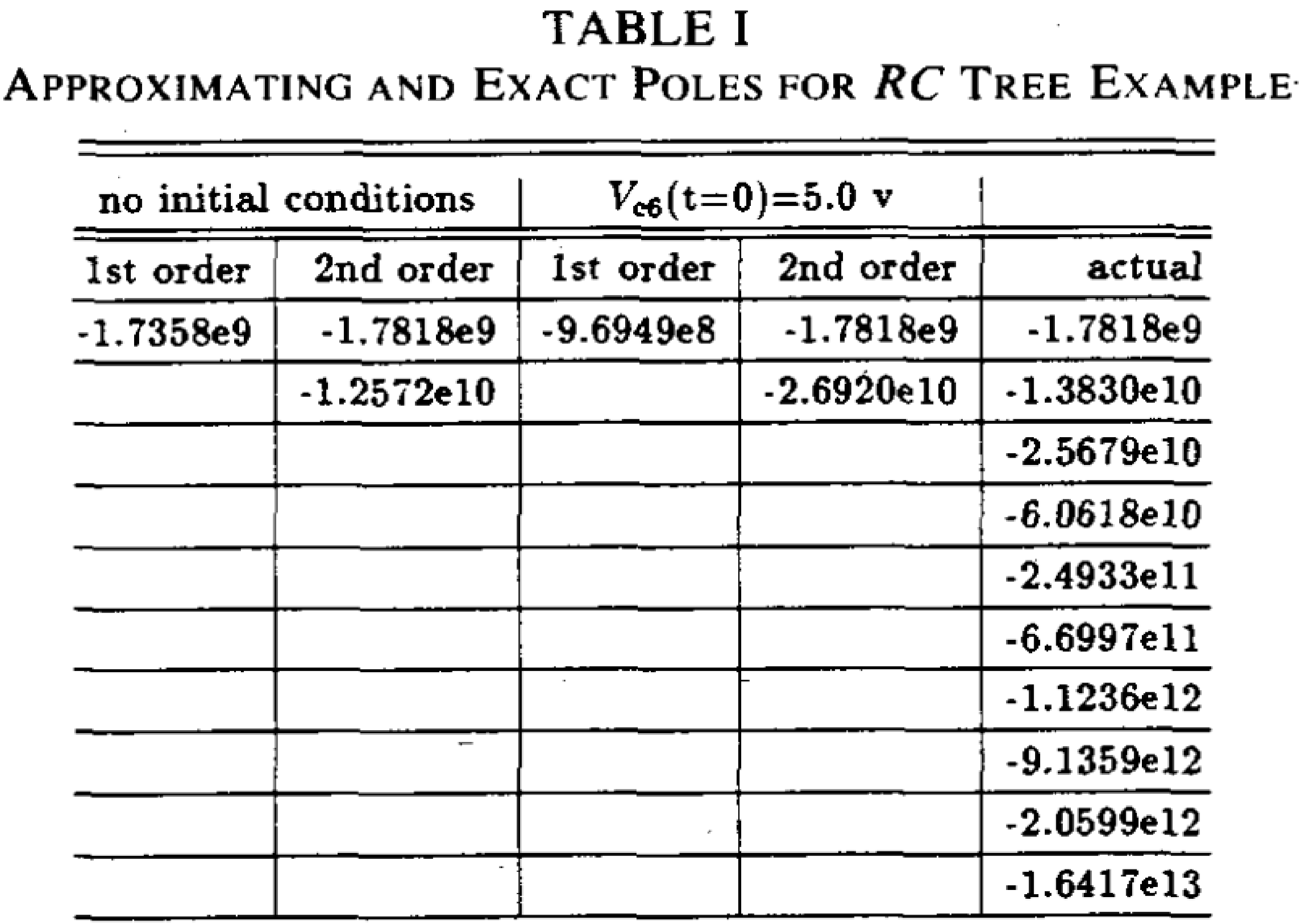


TABLE I

APPROXIMATING AND EXACT POLES FOR *RC* TREE EXAMPLE

no initial conditions		$V_{c6}(t=0)=5.0$ v		
1st order	2nd order	1st order	2nd order	actual
-1.7358e9	-1.7818e9	-9.6949e8	-1.7818e9	-1.7818e9
	-1.2572e10		-2.6920e10	-1.3830e10
				-2.5679e10
				-6.0618e10
				-2.4933e11
				-6.6997e11
				-1.1236e12
				-9.1359e12
				-2.0599e12
				-1.6417e13

L. Pillage, R. Rohrer, “Asymptotic Waveform Evaluation for Timing Analysis”, IEEE Transactions on CAD, April 1990

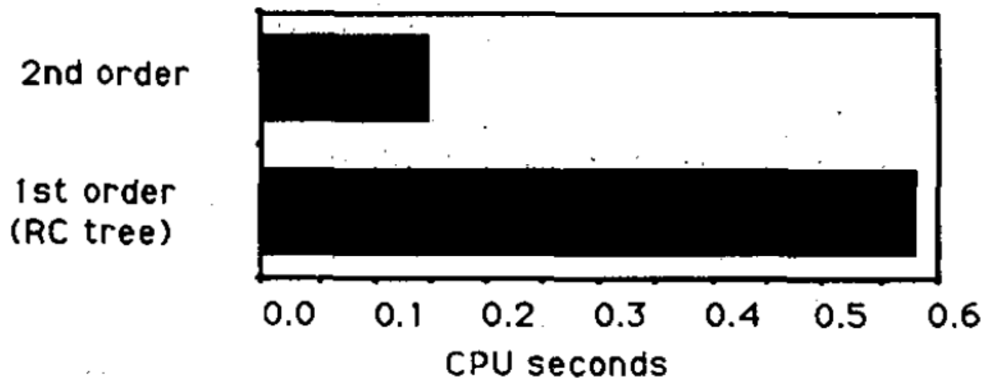


Fig. 19. CPU time comparison between first- and second-order approximation for the circuit of Fig. 16.

L. Pillage, R. Rohrer, "Asymptotic Waveform Evaluation for Timing Analysis", IEEE Transactions on CAD, April 1990

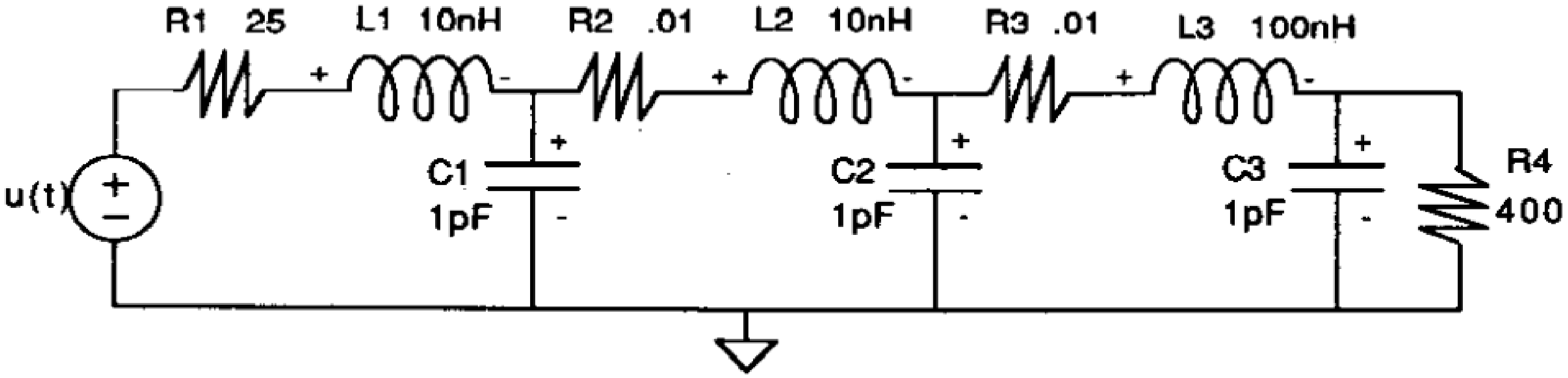


Fig. 25. *RLC* underdamped circuit with complex dominant poles.

L. Pillage, R. Rohrer, “Asymptotic Waveform Evaluation for Timing Analysis”, IEEE Transactions on CAD, April 1990

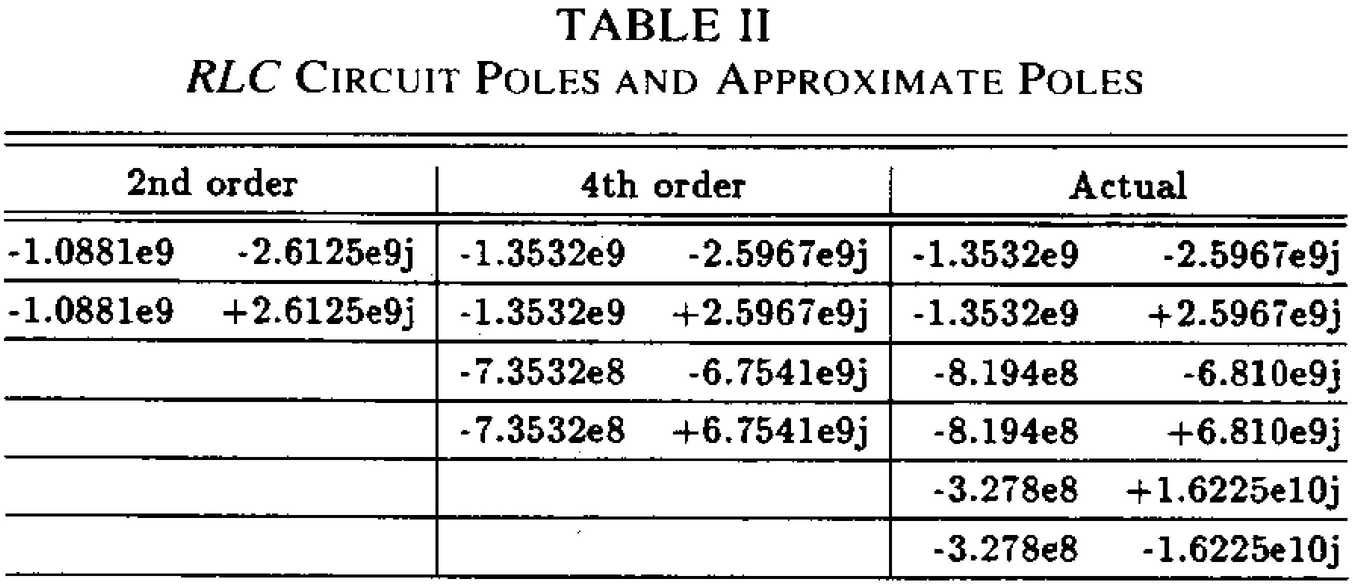
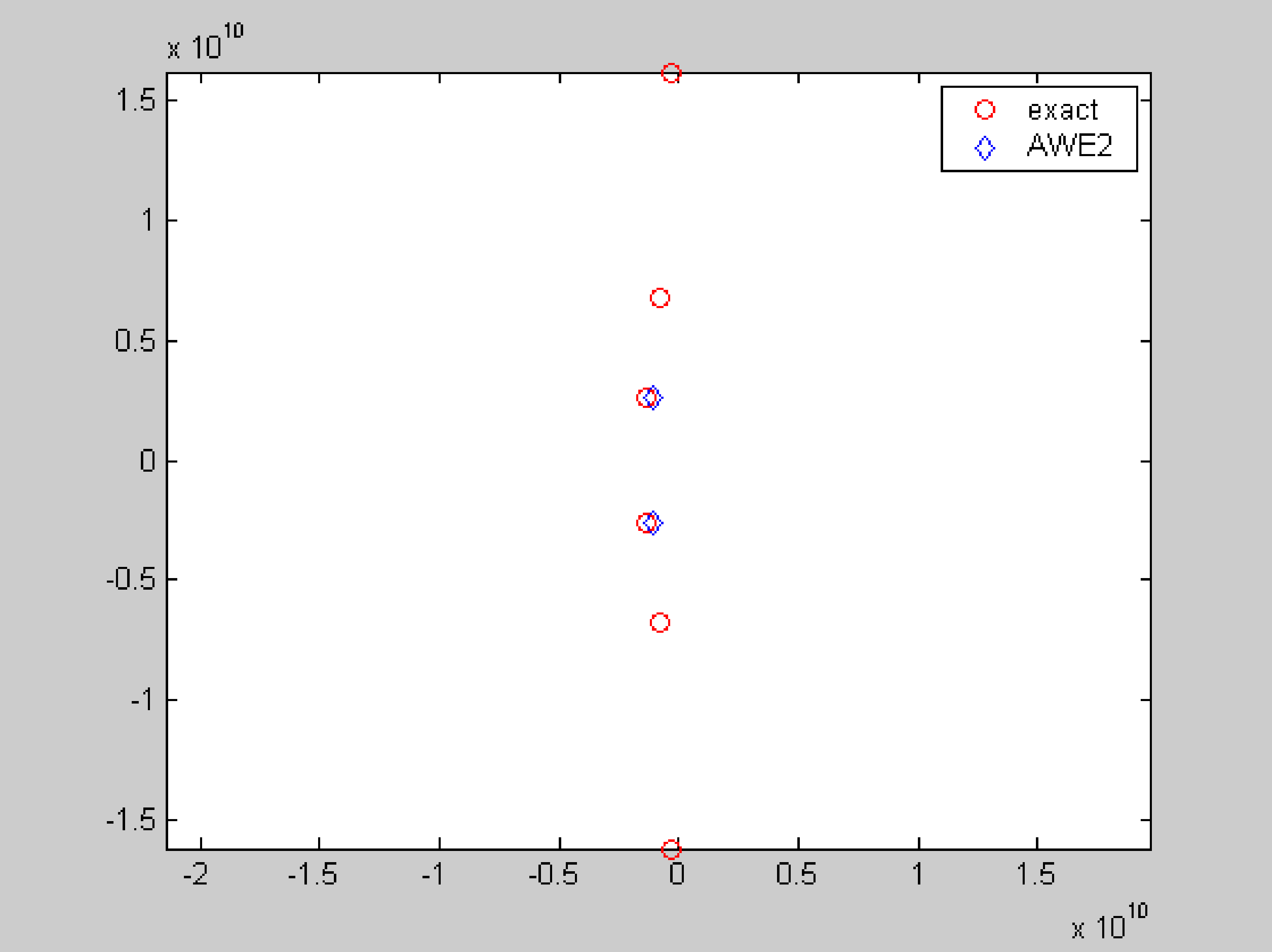
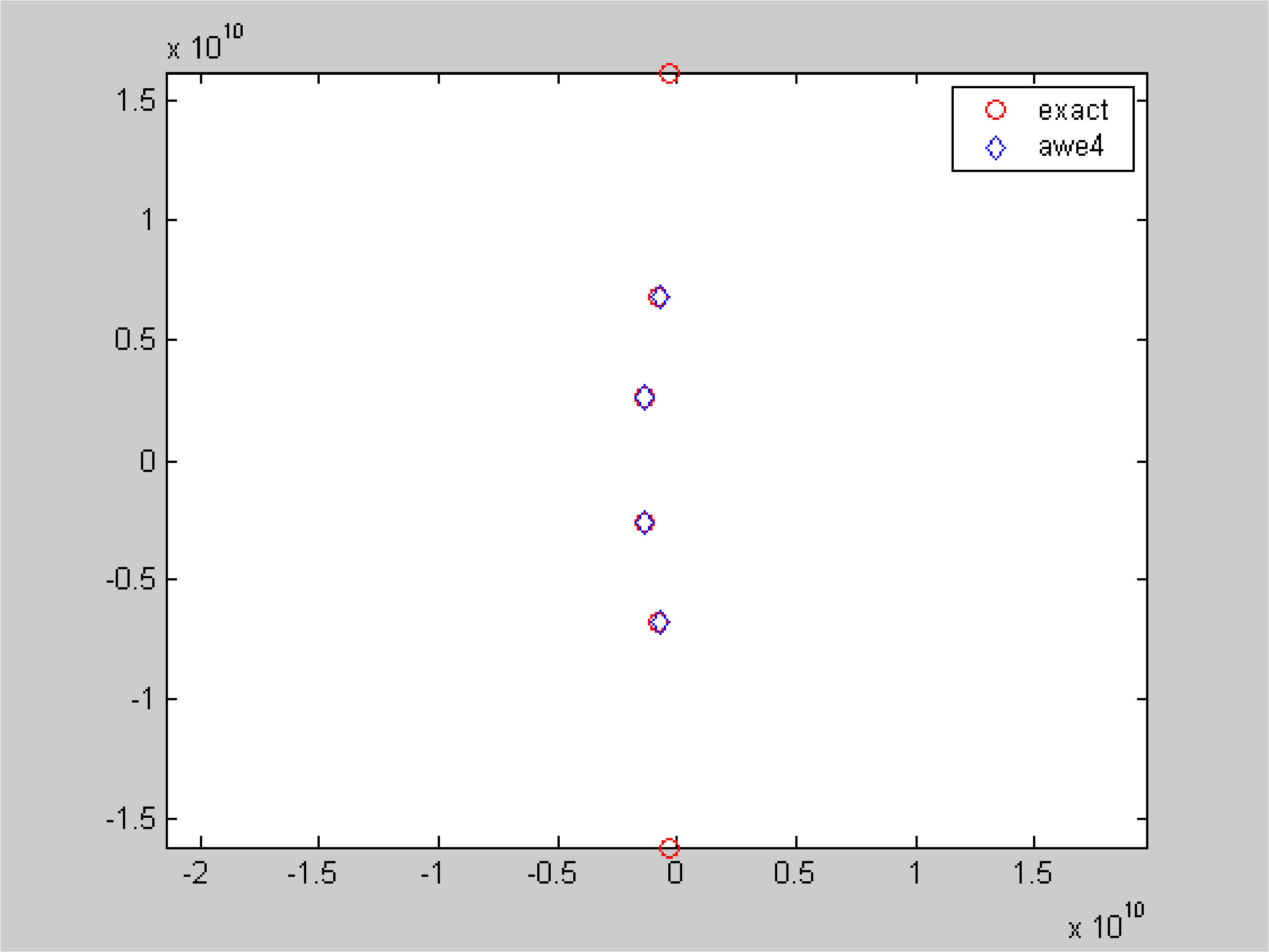


TABLE II
RLC CIRCUIT POLES AND APPROXIMATE POLES

2nd order		4th order		Actual	
-1.0881e9	-2.6125e9j	-1.3532e9	-2.5967e9j	-1.3532e9	-2.5967e9j
-1.0881e9	+2.6125e9j	-1.3532e9	+2.5967e9j	-1.3532e9	+2.5967e9j
		-7.3532e8	-6.7541e9j	-8.194e8	-6.810e9j
		-7.3532e8	+6.7541e9j	-8.194e8	+6.810e9j
				-3.278e8	+1.6225e10j
				-3.278e8	-1.6225e10j

L. Pillage, R. Rohrer, “Asymptotic Waveform Evaluation for Timing Analysis”, IEEE Transactions on CAD, April 1990


x 10^10
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
x 10^10
○ exact
◇ AWE2


x 10^10
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
x 10^10
○ exact
◇ awe4

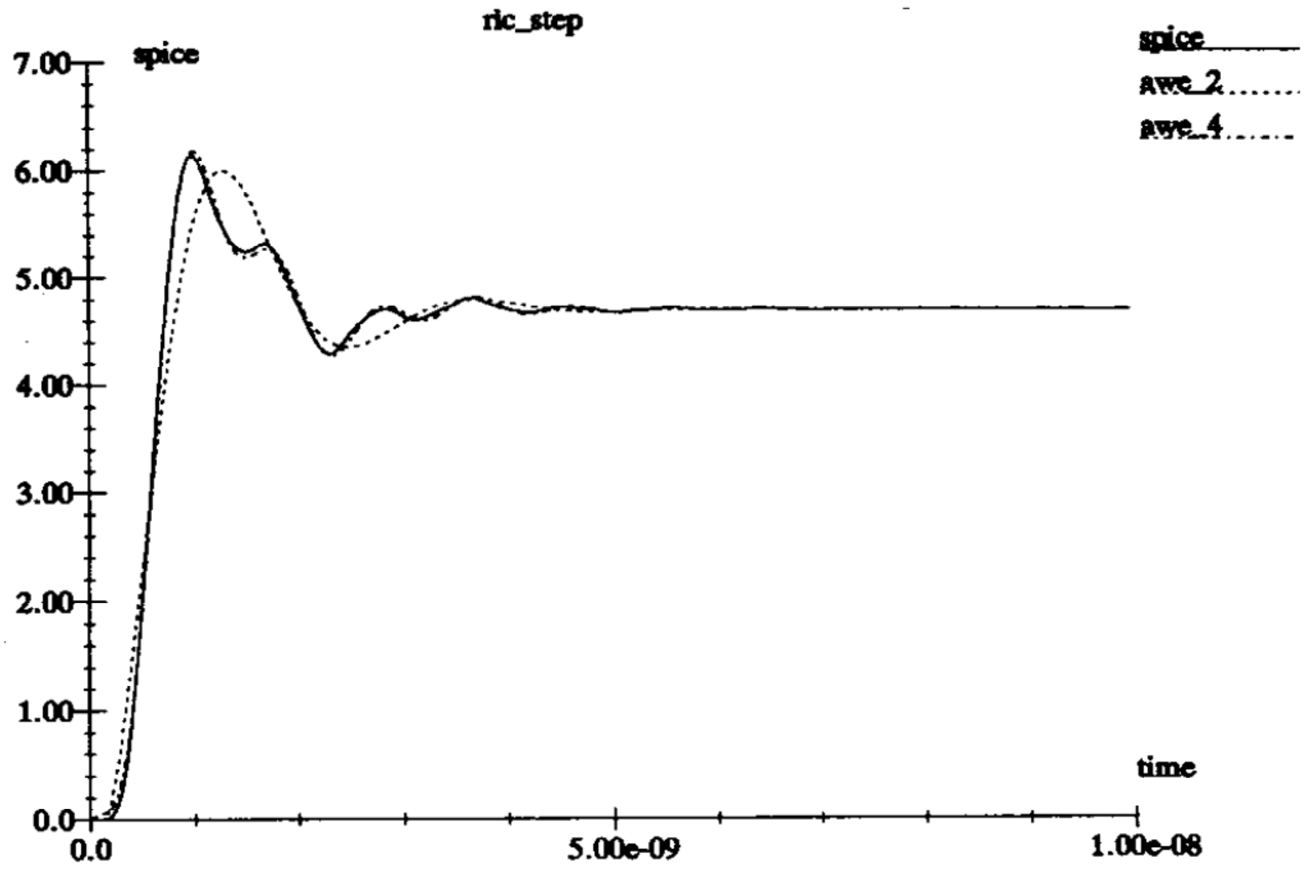
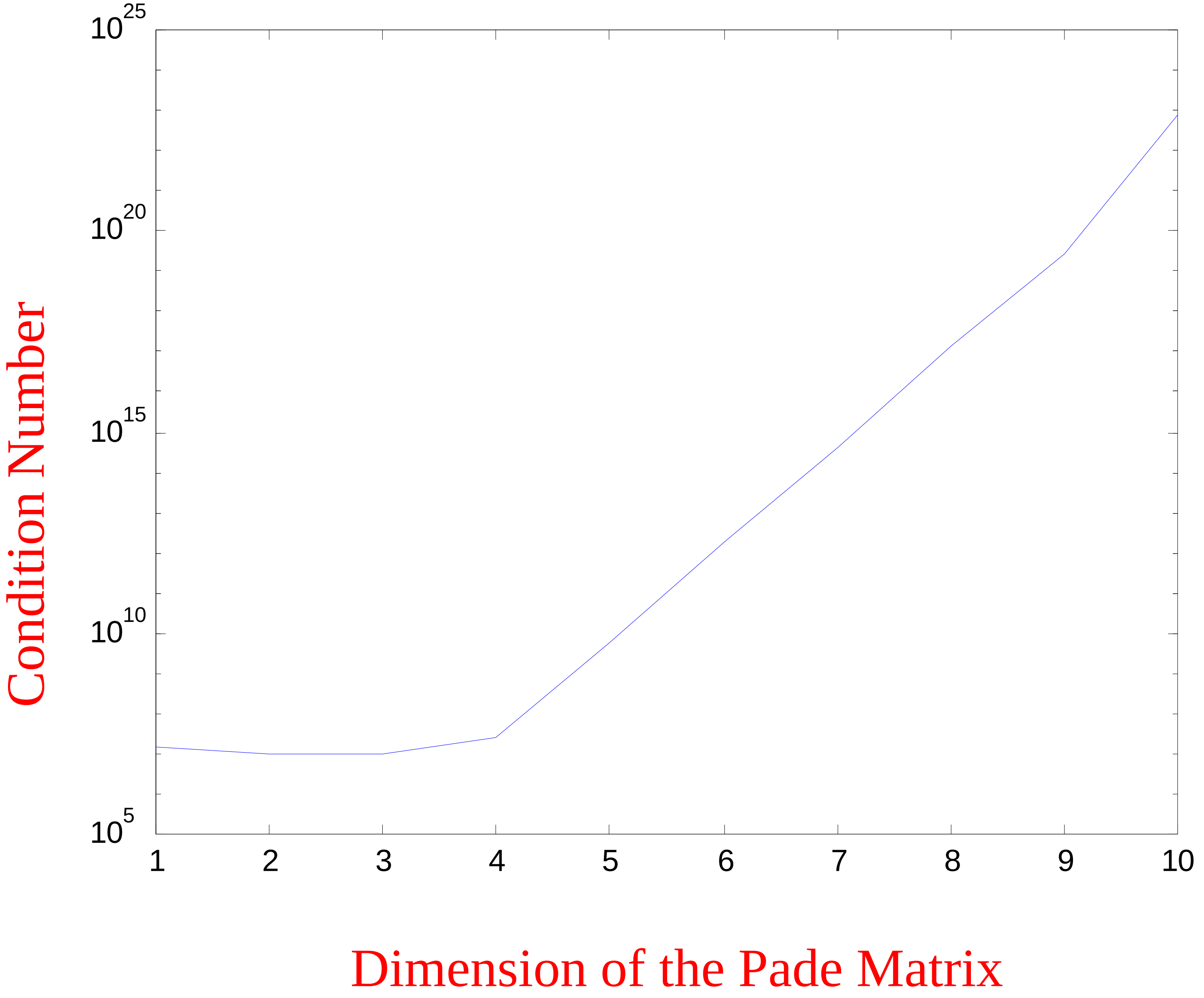


Fig. 26. AWE second- and fourth-order approximations for the step response of the *RLC* circuit in Fig. 25.

L. Pillage, R. Rohrer, “Asymptotic Waveform Evaluation for Timing Analysis”, IEEE Transactions on CAD, April 1990



10^{25}
10^{20}
10^{15}
10^{10}
10^{5}
1
2
3
4
5
6
7
8
9
10
Condition Number
Dimension of the Pade Matrix

Direct (Explicit) Moment Matching

$$C\frac{dx}{dt}+Gx=b$$

$$\left(sC+G\right)X=b$$

$$X=M_0+M_1s+M_2s^2+\ldots\ldots$$

Pade' Approximation in the Form of Rational Functions

Direct (Explicit) Moment Matching

$$C\frac{dx}{dt} + Gx = b$$

$$\left(sC + G\right)X = b$$

$$X = M_0 + M_1 s + M_2 s^2 + \ldots\ldots.$$

Pade' Approximation in the Form of Rational Functions

ill-conditioned numerical process !!

Moment Calculation

$$(\mathbf{G}+s\mathbf{C})\left(M_o + M_1 s + M_2 s^2 + M_3 s^3 + \dots\right) = \mathbf{b}$$

$$\mathbf{G}M_o = \mathbf{b}$$

$$A = -G^{-1}C$$

$$\mathbf{G}M_1 = -\mathbf{C}M_o \Longrightarrow M_1 = -\mathbf{G}^{-1}\mathbf{C}M_o = AM_0$$

$$\mathbf{G}M_2 = -\mathbf{C}M_1 \Longrightarrow M_2 = -\mathbf{G}^{-1}\mathbf{C}M_1 = A^2M_0$$

$$\vdots$$

$$\mathbf{G}M_{n+1} = -\mathbf{C}M_n \Longrightarrow M_{n+1} = -\mathbf{G}^{-1}\mathbf{C}M_n = A^{n+1}M_0$$

Direct (Explicit) Moment Matching

- The moments of x are M_0=b, M_1=Ab, M_2=A^2b . . . . M_n=A^nb
- The sequence {b, Ab, A^2b, . . . A^nb} converges to an eigenvector corresponding to the largest eigenvalue of A

(WHY ?)

no more useful information is obtained by adding new moments !!