Math 222 Midterm Exam Version 1, May 2014.

1. The series $\displaystyle\sum_{n=100}^{\infty} \frac{3^n}{4^{n+2}}$

   (a) converges to 0

   (b) converges to $\dfrac{1}{1-\frac{3}{4}}$

   (c) converges to $\dfrac{\frac{3^{100}}{4^{102}}}{1-\frac{3}{4}}$

   (d) converges to something other than the previous choices

   (e) diverges

2. The sequence $\left(1+\dfrac{2}{n}\right)^n$

   (a) converges to 1

   (b) converges to e^2

   (c) converges to 2

   (d) diverges

   (e) none of the above

3. The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}$

   (a) converges to 1

   (b) converges to $\frac{1}{2}$

   (c) converges to $\frac{1}{\sqrt{2}}$

   (d) converges to 0

   (e) diverges

4. The p-series test claims that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for

   (a) $p \leq 1$
   (b) $p \geq 1$
   (c) $p < 1$
   (d) $p > 1$
   (e) none of the above

5. The series $\sum_{n=1}^{\infty} \frac{n^{100} 100^n}{n!}$

   (a) is conditionally convergent
   (b) is absolutely convergent
   (c) diverges
   (d) none of the above

6. What is the least amount of terms of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}$ we must add to approximate the sum to within $\frac{1}{1000}$?

   (a) 2
   (b) 3
   (c) 4
   (d) 5
   (e) 6

7. What is the coefficient of x^2 in the MacLaurin series expansion of $\frac{x}{\sqrt{4+x^2}}$?

   (a) 0
   (b) 1
   (c) 2
   (d) 3
   (e) none of the above

8. If $u = (3, 6, 2)$ and $v = (1, 2, 3)$, $\text{proj}_u v =$

(a) 0
(b) $\frac{9}{14}(1, 2, 3)$
(c) $\frac{9}{49}(3, 6, -2)$
(d) $\frac{9}{14}(3, 6, -2)$
(e) none of the above

9. The equation of the line through $(1, -1, 1)$ and parallel to $x + 2 = \frac{1}{2}y = z - 3$ is given by

(a) $x = 1 + t, y = -1 + 2t, z = 1 + t$
(b) $x = 1 - t, y = -1 + 2t, z = 1 - t$
(c) $x = 1 + t, y = -1 + 2t, z = 1 - t$
(d) $x = 1 - t, y = -1 - 2t, z = 1 + t$
(e) none of the above

10. The plane through the points $(3, -1, 2), (8, 2, 4)$ and $(-1, -2, -3)$ has equation

(a) $-8x + 4y + 14z = 0$
(b) $x + y - z = 0$
(c) $20y - 14z = -48$
(d) $-13x + 17y + 7z = -42$
(e) none of the above

1. $\sum_{n=100}^{\infty} \frac{3^n}{4^{n+2}} = \sum_{n=100}^{\infty} \frac{1}{4^2} \cdot \left(\frac{3}{4}\right)^n$ $\quad \left|\frac{3}{4}\right| < 1$

so series converges

$$= \frac{1}{4^2} \cdot \frac{\left(\frac{3}{4}\right)^{100}}{1-\frac{3}{4}}$$

2. $\lim_{n\to\infty} \left(1+\frac{2}{n}\right)^n \quad (1^\infty)$

$$\lim_{n\to\infty} \ln\left(1+\frac{2}{n}\right)^n$$

$$= \lim_{n\to\infty} n \ln\left(1+\frac{2}{n}\right) \quad (\infty \cdot 0)$$

$$= \lim_{n\to\infty} \frac{\ln\left(1+\frac{2}{n}\right)}{\frac{1}{n}} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{n\to\infty} \frac{\frac{1}{1+\frac{2}{n}} \cdot \cancel{-}\frac{2}{\cancel{n^2}}}{\cancel{-}\frac{1}{\cancel{n^2}}} = 2$$

thus $\lim_{n\to\infty} \left(1+\frac{2}{n}\right)^n = e^2$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}$

$S_1 = 1 - \frac{1}{\sqrt{2}}$

$S_2 = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} = 1 - \frac{1}{\sqrt{3}}$

$S_4 = 1 - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} = 1 - \frac{1}{\sqrt{4}}$

$\vdots$

$S_n = 1 - \frac{1}{\sqrt{n+1}} \rightarrow 1$

so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} = 1$

4. the p-series test says $\sum \frac{1}{n^p}$ converges if $p > 1$

5. $$\sum_{n=1}^{\infty} \frac{n^{100}\, 100^n}{n!}$$

$$\lim_{n\to\infty} \frac{(n+1)^{100}\, 100^{\cancel{n+1}}}{(n+1)\cancel{!}} \cdot \frac{\cancel{n!}}{n^{100}\, \cancel{100^n}}$$

$$= \lim_{n\to\infty} \frac{100\,(n+1)^{99}}{n^{100}} = 0 < 1$$

so series converges absolutely

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}$ is a convergent alternating series. $\displaystyle\left| \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6} - \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n^6} \right| \le \frac{1}{(N+1)^6}$

so if $\dfrac{1}{(N+1)^6} < \dfrac{1}{1000}$ we are fine!

$N=2$: $\dfrac{1}{3^6} = \dfrac{1}{81 \cdot 9} > \dfrac{1}{1000}$

$N=3$: $\dfrac{1}{4^6} = \dfrac{1}{16 \cdot 16 \cdot 16} < \dfrac{1}{1000}$ $\quad$ $N=3$ is correct answer.

7. $(4+x^2)^{-1/2} = \frac{1}{2}\left(1+\frac{x^2}{4}\right)^{-1/2}$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\binom{-1/2}{n}\left(\frac{x^2}{4}\right)^n$$

$$= \sum_{n=0}^{\infty}\binom{-1/2}{n}\frac{x^{2n}}{2^{2n+1}}$$

so $x(4+x^2)^{-1/2} = \sum_{n=0}^{\infty}\binom{-1/2}{n}\frac{x^{2n+1}}{2^{2n+1}}$

ie, no ~~odd~~ even terms so coefficient of all the even powers of x are 0, including that of x^2!

8. $\text{proj}_u v = \frac{v \cdot u}{\|u\|^2} u = \frac{21}{49}(3,6,2)$

9. $x+2 = \frac{1}{2}y = z-3$ is the symmetric equation of the line

$x+2=t$ $\qquad$ $x = -2+t$

$\frac{1}{2}y = t$ $\qquad$ ie, $\qquad$ $y = 2t$

$z-3=t$ $\qquad$ $z = 3+t$

so direction vector $d = (1, 2, 1)$

and line is
$$x = 1+t$$
$$y = -1+2t$$
$$z = 1+t$$

10. pts $(3,-1,2)$, $(8,2,4)$, $(-1,-2,-3)$

give rise to vectors $(5,3,2)$ and
$(-4,-1,-5)$

cross product $(-13, 17, 7)$

so plane: $-13x + 17y + 7z = D$ plug in point

$-13(-1) + 17(-2) + 7(-3) = -42$

ie, $-13x + 17y + 7z = -42$