Matrix Exponential

An overview

For homogeneous systems:

- Solution:

$$X = Ce^{At}$$

- With

$$e^{At} = I + At + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \cdots + A^k\frac{t^k}{k!} + \cdots = \sum_{k=0}^{\infty} A^k\frac{t^k}{k!}$$

- The goal is to compute this exponential: the trick is to look at the eigenvalues

Compute e^{At} if $A = \begin{pmatrix} -2 & 4 \\ -1 & 3 \end{pmatrix}$

① eigenvalues: $\begin{vmatrix} -2-\lambda & 4 \\ -1 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda+1)(\lambda-2) = 0$

$\lambda_1 = -1 \quad \lambda_2 = 2$

② A is $2\times 2 \Rightarrow e^{At} = b_0 I + b_1 A \quad \left(e^{At} = \sum_{j=0}^{n-1} A^j b_j(t)\right)$

using $e^{\lambda t} = \sum_{j=0}^{n-1} \lambda^j b_j(t) \Rightarrow e^{\lambda t} = b_0 + b_1 \lambda$

2 eq. $\Rightarrow e^{-t} = b_0 - b_1 \quad \& \quad e^{2t} = b_0 + 2b_1$

solving for b_0 / b_1: $b_0 = b_1 + e^{-t}$

$\hookrightarrow e^{2t} = 3b_1 + e^{-t}$

$\Rightarrow \boxed{b_1 = \dfrac{e^{2t} - e^{-t}}{3}} \quad \& \quad \boxed{b_0 = \dfrac{e^{2t} + 2e^{-t}}{3}}$

$$\Rightarrow e^{At} = \begin{pmatrix} \frac{1}{3}(e^{2t} + 2e^{-t}) & 0 \\ 0 & \frac{1}{3}(e^{2t} + 2e^{-t}) \end{pmatrix} + \begin{pmatrix} -\frac{2}{3}(e^{2t} - e^{-t}) & \frac{4}{3}(e^{2t} - e^{-t}) \\ -\frac{1}{3}(e^{2t} - e^{-t}) & (e^{2t} - e^{-t}) \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{3}e^{2t} + \frac{4}{3}e^{-t} & \frac{4}{3}e^{2t} - \frac{4}{3}e^{-t} \\ -\frac{1}{3}e^{2t} + \frac{1}{3}e^{-t} & \frac{4}{3}e^{2t} - \frac{1}{3}e^{-t} \end{pmatrix} \Leftarrow \text{answer}$$