Higher Order DE

In this section, we will mainly focus on second order linear ordinary differential equation. However, the theory will be developed in the goal to use it correctly even with higher order ODE. A nth order linear ordinary differential equation can be written as:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

If g(x) is equal to zero, then one deals with a homogeneous equation. If not, one has a non-homogeneous equation.

A nice theorem that was presented in class is the existence and uniqueness of the solution of the ODE in the case of an Initial value problem (IVP). This theorem states that a solution exist if the coefficient a are continuous on a given interval an if a_n is not zero. This solution around x_0 is unique when x_0 is in that interval.

The most important aspect of higher order ODE to know is the fact that there will be more than one solution for each ODE. The general solution is presented as the sum of all the solution in a particular interval, if those solutions are linearly independent.

Superposition Principle:

A complete solution of an ODE is the sum of the set of independent solutions:

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x)$$

For the case of a non-homogeneous equation, the solution will be the sum of the homogeneous solution and the non-homogeneous particular solution:

$$y = y_c + y_p$$

To know if the set of solution found when solving an ODE are linearly independent, a nice method to know is to use the Wronskian:

$$W(f_1, f_2, \dots, f_n) = \begin{vmatrix} f_1 & \cdots & f_n \\ \vdots & \ddots & \vdots \\ f_1{}^{(n-1)} & \cdots & f_n{}^{(n-1)} \end{vmatrix}$$

Let us now look at some methods to solve second order ODE

2.2.1: Reduction of order

The reduction of order is not a technique that will be used in this course. However, the idea might be used to present part of solution. The idea is to start with one solution and find a new one which will be independent of the first one:

$$y_2 = u(x)y_1 \ , \ \ with \quad u(x) = \int \frac{e^{-\int P(x)dx}}{y_1^2} dx$$

The P(x) comes from the standard form of the ODE (the linear ODE divided by a_2, P(x) being the first term).

2.2.2 : Homogeneous Linear ODE with constant coefficient

When one has to solve an ODE with all the coefficients are constant, the trick to solve is to suppose that the solution will be of the form:

$$y \equiv e^{mx}$$

With this, one can write the characteristic equation, which is a polynomial function in m. Solving thus reduce to finding the roots of a polynomial function of the same order has the ODE. For second order ODE, it is solving a quadratic equation.

Three particular cases may appear. Either the roots are real (separate or double) or are complex (complex and complex conjugate). Different solutions may be found using the different type of roots:

$$3\ cases: \begin{cases} m & y = c1e^{mx} + c2xe^{mx} \\ m1, m2\ real & y = c1e^{m1x} + c2e^{m2x} \\ m1, m2\ comp & y = e^{\alpha x}(c1\cos(\beta x) + c2\sin(\beta x)) \end{cases}$$

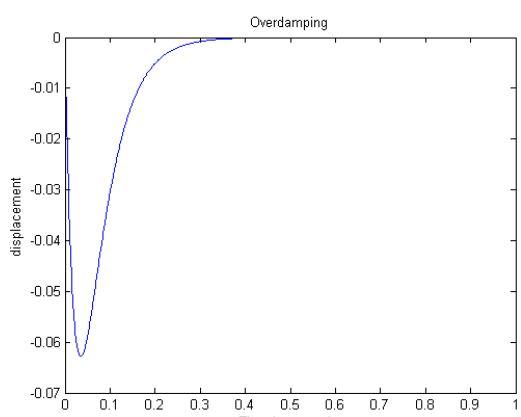
Common application:

Spring and mass system on an horizontal plane.

This is one of the most famous applications in engineering. Using Hooke's law, one has the equation (in the case of damped motion):

$$\ddot{x} + 2\lambda\dot{x} + \omega^2 x = 0$$

The value of λ is related to the damping factor and the ω is related to the frequency of the oscillation. Here, the three possible cases leads to overdamping (real distinct), critical damping (real double) and underdamping (complex). The figure below presents a case of overdamping.



2.2.3: Undetermined Coefficients

This method is sometime referred to as the “educated guess” technique. It actually is a nice method to solve non-homogeneous ODE if one has a table of typical solution. This method can be summarized in 4 steps.

1. Solve the homogeneous part
2. Choose in table 3.4.1 of the textbook a solution corresponding to g(x)
3. IF the solution of the particular solution is the same as for the homogeneous solution, multiply the particular solution by x—The goal being to have independent solutions
4. If g(x) is a combination (sum or difference) of function from 3.4.1, add/substract solutions

Application:

Spring-mass with a driving motion: this is a very common application.

Electrical circuitry: also a very common example, if the circuit has an applied external voltage.

2.2.4: Variation of Parameters (Lagrange Methods)

This is an easy and powerful algorithm to know. However, even if the algorithm is easy, it might be hard to compute.

The trick: find particular solution using the homogeneous solution:

$$y_p = u(x)y_1 + v(x)y_2$$

To compute the values u(x) and v(x), one can use the following formulas, knowing that the W is for the Wronskian of the two homogeneous solution.

$$u = -\int \frac{y_2 f(x)}{W} dx \ , \qquad v = \int \frac{y_1 f(x)}{W} dx$$

2.2.5: Cauchy-Euler Equation:

A common type of equation encounter in engineering is the Cauchy-Euler equation. For the order 2 case, this equation is simply:

$$x^2 y'' + axy' + by = 0$$

The idea is to suppose a solution in the form:

$$y \equiv x^m$$

Using this solution in the starting equation leads to solving the following equation:

$$m^2 + (a-1)m + b = 0$$

Now, like for the case of the second order ODE using exponential form to solve, we have three different cases for the different roots of this equation:

$$3\ cases: \begin{cases} m \quad y = (c1 + c2\ln(x))x^m \\ m1, m2\ real \quad y = c1x^{m1} + c2x^{m2} \\ m1, m2\ \in \mathbb{C} \quad y = x^{\mu}(c1\cos(\nu\ln(x)) + c2\sin(\nu\ln(x))) \end{cases}$$

Hence, a solution is found simply by choosing the right value for the m found.

2.2.6: Boundary Value Problem

A Boundary Value Problem (BVP) is a problem using ODE and having condition given at more then one point for the particular solution. This type of problem is often encounter when physical object are studied (beam, string, heat transfer,…).

There is no particular informatino to know for this type of problem, only that one may come across a set of solution, called the eigenfunctions. This fact was presented in an example during the lecture. One has to verify what are the possible solution.

A very, very common ODE that is useful for the PDE solutions is the so-called Sturm-Liouville Problem. The description of such problem can be found in the textbook. In many applications, it can be considered that the Sturm-Liouville problem is simply the following equation:

$$y'' + \lambda y = 0$$

The solution of that problem depends on the value of the constant λ.

If λ is negative: $y = \mathrm{a} * \cosh\sqrt{\lambda}x + b * \mathrm{sinh}\sqrt{\lambda}\mathrm{x}$, or $\mathrm{y} = \mathrm{ae}^{\sqrt{\lambda}x} + \mathrm{be}^{-\sqrt{\lambda}x}$

If λ is positive: $y = a * cos\sqrt{\lambda}x + b * sin\sqrt{\lambda}x$

One can recall that there are many possible solutions depending on the initial value:

- y(0)=y(L)=0

  eigenvalue: $\lambda_n = \left(\frac{n\pi}{L}\right)^2$

  eigenfunction: $\sin(\frac{n\pi x}{L})$

- y'(0)=y'(L)=0 $\lambda > 0$

  eigenvalue: $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ $\lambda = 0 \rightarrow y = cste$

  eigenfunction: $\cos(\frac{n\pi x}{L})$

- y(0)=y'(L)=0 $\lambda > 0$

eigenvalue: $\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2$

eigenfunction: $\sin(\frac{(2n-1)\pi x}{2L})$

- y'(0)=y(L)=0 $\lambda > 0$

eigenvalue: $\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2$

eigenfunction: $\cos(\frac{(2n-1)\pi x}{2L})$

Those solutions will become very useful in the chapter on PDE. For now, it is just a good thing to remember them.

Section 3: On the use of Power Series

Another technique was presented in order to be able to solve ODE. This technique was the use of power series. Of course this technique involve the knowledge of series theory. If one is interested in general series theory, one can consult a book on mathematical analysis. Here, no proof will given on the few theorem presented.

Theorem 1: Wierestrass Criteria

> Let us look at a series of function $\sum_{n=1}^{\infty} f_n(x)$. If there exist a convergent serie such as all the term of that convergent serie are greater or equal to all the function of the initial serie, for all n and all x in the domain, then $\sum_{n=1}^{\infty} f_n(x)$ is absolutely and uniformly convergent

A power series can be written as:

$$y \equiv \sum_{n=0}^{\infty} a_n x^n$$

To know on what domain that series is a convergent series (for what x part of the real realms), a quick criteria to use is a similar criteria to the ratio test (d'Alembert criteria):

Theorem 2:

> Let us study the power series $y \equiv \sum_{n=0}^{\infty} a_n x^n$. The radius of convergence is given by:
>
> $$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$$
>
> Provided that the limit exist.

In order to use power series in ODE, one need to be able to do the derivative of the power series:

Theorem 3:

> Let the power series $f(x) \equiv \sum_{n=0}^{\infty} a_n x^n$ have a radius of convergence R not zero. Then, f is differentiable on (-R,R) and, for x in the interval,
>
> $$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Then, using the first and the second, the third, … derivative of the power series, one can rewrite the ODE using power series and thus, solving the equation become simply finding the correct values for the coefficient of the series.

Section 4: Linear System of First-Order ODE

This section is focusing on the general theory of linear system of ODE. Since this course is an introductory course, the main focus will be put on homogeneous system of first order ODE.

A system of ordinary diffrential equation is generally presented as:

$$\frac{dy_1}{dt} = a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n$$

$$\frac{dy_2}{dt} = a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n$$

$$\frac{dy_n}{dt} = a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n$$

A very useful manner to deal with this system is to present it using matrix:

$$Y' = AY + f(t)$$

Where A is a matrix, Y is a column vector and f(t) is also a column vector representing the non-homogeneous component.

Let us start by looking at ***the homogeneous systems***:

Fundamental Theorem of Linear Systems:

Let A be an nxn matrix. Then, for a given $x_0 \in \mathbb{R}^n$, the initial problem: $\dot{X} = AX\ ,\ X(0) = x_0$, has a unique solution given by:

$$X(t) = e^{At}x_0$$

This theorem is very powerful and useful if one can work with matrix exponential. If not, a very easy solution is to use a similar expression has the one found for the higher order ODE, mainly:

$$X(t) = \begin{pmatrix} k1 \\ \vdots \\ kn \end{pmatrix} e^{\lambda t}$$

Where the λ are the eigenvalues of the matrix A and the K vector are the eigenvectors. One can find the eigenvalue by solving:

$$\det(A - \lambda I) = 0$$

And the eigenvectors are found by solving $(A - I\lambda)K = 0$ for each eigenvalues.

3 cases:

1- Distinct real eigenvalues: The complete solution is :
$$X(t) = c1K1e^{\lambda 1t} + c2K2e^{\lambda 2t} + \cdots + cnKne^{\lambda nt}$$
2- Repeated real eigenvalues: Either find different eigenvector with that repeated eigenvalue OR solution of the form:
$$X2 = Kte^{\lambda 1t} + Pe^{\lambda 1t}, with\ (A - I\lambda)P = K$$
$$X3 = K\frac{t^2}{2}e^{\lambda 1t} + Pte^{\lambda 1t} + Qe^{\lambda 1t}, with\ (A - I\lambda)Q = P$$
3- Complex eigenvalues: Having $\lambda = \alpha + i\beta$, finding K and B1=Re(K), B2=Im(K), one has:
$$X1 = e^{\alpha t}[B1\cos(\beta t) - B2\sin(\beta t)]$$
$$X2 = e^{\alpha t}[B2\cos(\beta t) + B1\sin(\beta t)]$$

For Non-Homogeneous systems, two methods were presented during this course:

a) Undetermined coefficients: This method consist of finding the homogeneous solution and then one can suppose, by educated guess, the form of the non-homogeneous solution. Then, by using this solution, one can calculate the undetermined coefficients.
b) Variation of parameters: After finding the solution of the homogeneous equation related to the non-homogeneous system, one must find the fundamental matrix ϕ(t), which is non-singular. Then, the solution of the non-homogeneous part is found by:
$$X_p = \Phi \int \Phi^{-1}F(t)dt$$