Section 2: Ordinary Differential Equations: Introduction

A ***differential equation*** is an equation containing the derivative of one or more dependent variable, with respect to one or more independent variables.

There are two kinds of differential equations: Ordinary Differential Equations and Partial Differential Equations. What is called Ordinary Differential Equations (ODE) deals only with function of one and only one variable. On the other hand, Partial Differential Equations (PDE) deals of function of several variables. In the present course, only ODE will be considered.

The **order** of an ODE is the order of the highest *derivative*. Knowing the order help one to know what technique to use to solve efficiently a given equation. Most engineering applications deal with first and second order DE. So, many example used in this document will deal with second order ODE.

An ODE is said to be **linear** if:

- All of the y variable and all of the nth derivative of these variables are at most of order 1.
- All of the coefficient multiplying those y variables(and derivative) are, at most, function of the independent variable x.

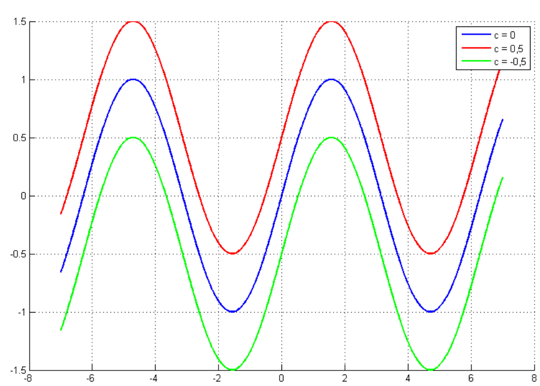
$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_0(x)y = g(x)$$

If an ODE cannot be written in that form, then the equation is non-linear.

A **solution** of a given differential equation of order n on some open interval D is a function ϕ that as at least n derivative (continuous) on that interval D.

While solving ODE, one will encounter many solutions. One can find what is called a **general solution,** which is a family of curves. In fact, the general solution is a solution having unknown constants. A **particular solution** is based upon the initial or the boundary values. It is a solution where the unknown constants are known! Also, there might exist what is called **singular solution** which are solution that can not be found like the particular solution. The most widely found one is the **Trivial solution**, y=0.

Example: for the solution of the differential equation y'=cos(x), the general solution is y = sin(x) + c



Important theorems on Initial Value Problem:

Theorem (Existence):

if f(x,y) is continuous at all points (x,y) in some rectangle R ($|x - x_0| < a\ and |y - y_0| < b$), and if f(x,y) is bounded in R ($\exists\ k :\ |f(x,y)| \leq k,\ (x,y) \in R$), then the ***initial value problem*** has **at least** one solution in $|x - x_0| < \alpha,\ \alpha$ being the smallest of a or b/k.

Theorem (Uniqueness):

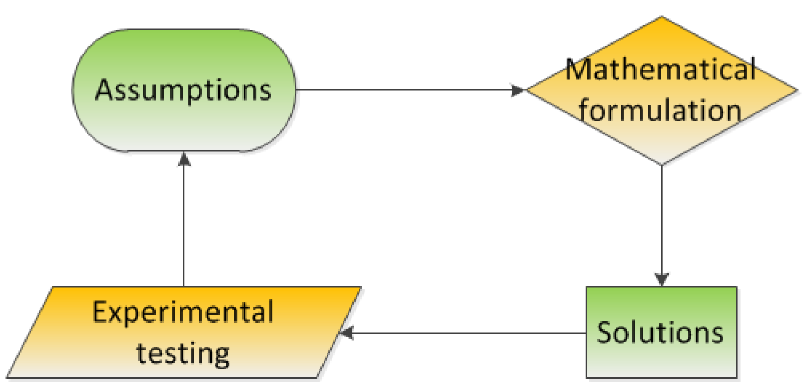
if f(x,y) and $\frac{\partial y}{\partial x}$ are continuous for all (x,y) in R and if they are bounded for all (x,y) in R, then the initial value problem has **at most** one solution y(x).

Proof of those theorems can be found in mathematics book aimed for mathematician[1]. In the textbook used for this course, no mention of such proof is present. If one wants to try to set a proof, one should try to set an integral function and show that, with successive iteration (Picard Iteration), the function leads to the solution.

Final General Remarks:

A small reminder that the reason ODE are of importance is because they are used to represent in a mathematical manner some physical phenomenon (mainly dynamics procedure). Hence, not only one has to know how to solve ODE, but also one has to be able to understand the physical representation of the phenomenon studied, to write it in mathematical language.

In a general approach, mathematical modeling can be done, for an unknown process, as described by the graph below, starting at the assumptions (hypothesis).



[1] For example, Differebtial Equations, Hochstadt, Holt Rinehart Winston, 1965

A very common example of mathematical modeling is the Newton's law of cooling:

At t=0, you turn off the heat in your house. The temperature is 20ºC. At t = 2h, the temperature as dropped to 19ºC. What is the expected temperature in the morning, at t = 10h, if the outside temperature is 10ºC and you use Newton's law of cooling?

$$\frac{dT}{dt} = k(T - T_A) \text{ , t in hour}$$

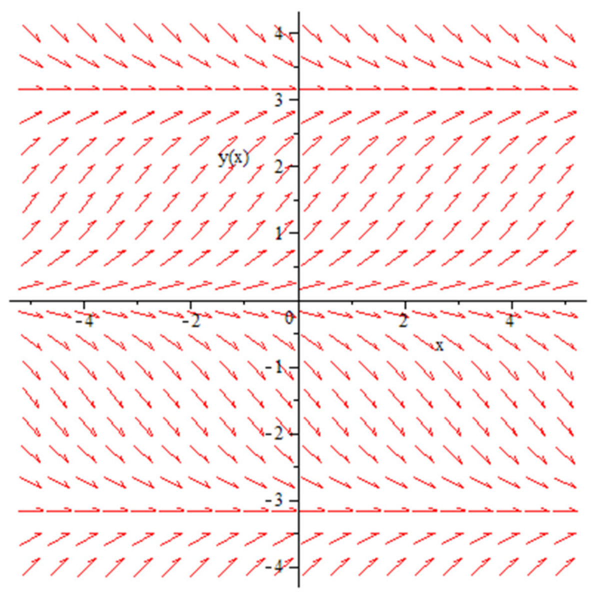
2.1 First Order ODE

2.1.1 Geometrical interpretation of a solution:

A very easy and visual method to have an idea of the solution of an ODE is by interpreting the meaning of the equation by the use of geometry.

One knows that a derivative of a function is in fact a measure of the slope of that function as a function of the variable upon which the derivation is done. Knowing this, it is possible to draw the look of an ODE simply by rewriting the equation in the implicit form (F(x,y)=0) and by changing the value of the slope obtained. This will allow one to find Direction Field.

Direction field of $\frac{dy}{dx} = \sin y$ with Maple 15



2.1.2: Separation of variable:

One of the first method that can be used to solve ODE of the first order is called the separation of variables. As the name says, it consist at bringing all variable of the same nature to the same side of the equation.

For example, all dependant y variable to the left side and all independent x variable to the right. This allow one to simply integrate both side.

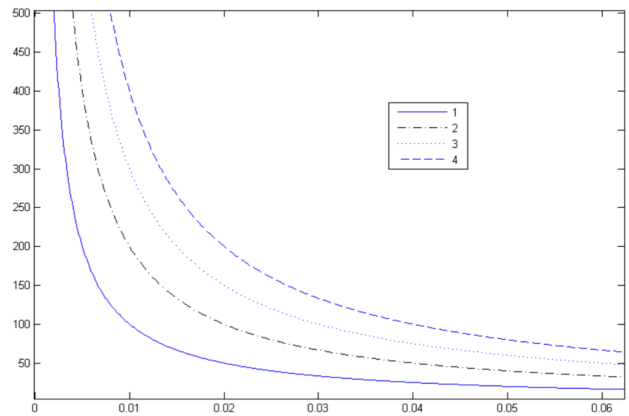
Example: $Solve\ \frac{dy}{dx} = -\frac{y}{x}$

Solution:

Using the separation of variable: $\frac{dy}{y} = -\frac{dx}{x}$

By integration: $\int \frac{dy}{y} = -\int \frac{dx}{x}$ $\ which\ gives:\ \ \ln y = -\ln x + c$ or $y = \frac{c}{x}$

From there, one can draw the general solution curves:



2.1.2 Linear Equation

A first order differential equation is said to be linear if it can be written in the form:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

The standard form is found by dividing this equation by $a_1(x)$, provided that it is not 0.

$$\frac{dy}{dx} + P(x)y = f(x)$$

The solution is found by separation if f(x) is zero. Otherwise, the trick is to multiply the whole equation by a factor and to solve.

Factor: $e^{\int P(x)dx}$

Solution: $y(x) = \frac{\left\{\int e^{\int P(x)dx} f(x)dx\right\}+c}{e^{\int P(x)dx}}$

2.1.3 Exact Equation

To use the theory of exact equation, one needs to know what is an exact differential. The following theorem tells us how to define such equation.

Theorem:

Let us call M(x,y)dx + N(x,y)dy = 0 an ***exact equation*** if M(x,y)dx + N(x,y)dy is an *exact differential*. A necessary and sufficient condition that M(x,y)dx + N(x,y)dy be an ***exact differential***, provided that M(x,y) and N(x,y) be continuous and have continuous first partial derivative on R, is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

That being known, a quick method to solve an exact equation is to solve, partially, for one variable and then to find the value for the other variable, using the above-mentionned theorem.

Without proof, the solution is found, starting with the M(x,y):

$$f(x,y) = \int M(x,y)dx + g(y)$$

$$g'(y) = N(x,y) - \frac{\partial}{\partial x}\left[\int M(x,y)dx\right]$$

Then, one has a function f(x,y). To find the complete solution, one need to say that this f(x,y) is equal to a general constant and then solve for y, if one can.

Another method using exact equation is to make an equation exact. To do so, one need to find integrating factor, either in x or y, and to multiply the equation by this factor. After that, the equation can be solved using the method described above.

2.1.4 Substitution:

The general idea is to use a simpler equation to simplify the differential equation to solve. It is sometime called the reduction to separable form, but it is not always true.

A very useful substitution is to use either: $y = ux \;\; or \;\; x = vy$

Other substitution:

1- Bernoulli equation:

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

Use: $u = y^{1-n},\; n \neq 0 \; or \; n \neq 1$

2- $\frac{dy}{dx} = f(Ax + By + C) \qquad use \qquad u = Ax + By + C, B \neq 0$