Section 1: Complex Numbers

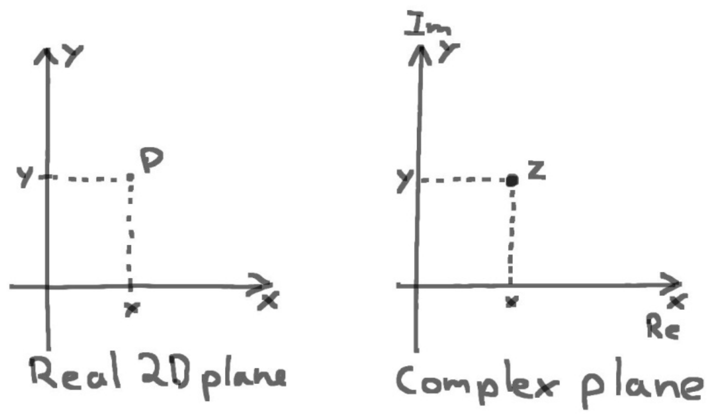
In a part, let us introduce the concept of complex numbers. The idea of complex numbers was introduce for solving polynomial equation by an Italian mathematician Girolamo Cardono in the 16 century, but it was generalized by Gauss and Euler.

The basic idea is to solve the following equation: $x^2 + 1 = 0$

The answer is of course that $x = \sqrt{-1}$. But what is the meaning of this? In standard real numbers, this solution does not exist. However, one can imagine that it is a viable solution. Therefore, it was proposed that this value should be named i ($i = \sqrt{-1}$)[1]. Using this new symbol, it is possible to introduce complex number. It is of usage to write those number as z, in a complex plane:

$$z = x + iy$$

Where x is said to be the Real part of z (x=Re z) and y is the imaginary part of z (y = Im z).



Two complex numbers are said to be equal if and only if there real and imaginary parts are equal. It is similar to a point in a Cartesian rectangular plane. Two points are the same if and only if they have the same coordinates.

Another definition very useful for the complex number is the complex conjugate. The complex conjugate of a complex number is noted $\bar{z}$ and is simply the imaginary part that change sign.

$$z = x + iy \quad so \quad \bar{z} = x - iy$$

[1] In electrical engineering, the letter j is used to avoid confusion with the current.

To play with complex numbers, one needs to know the rules of addition, difference, multiplication and division.

Addition:

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

The difference is the same as the addition.

Multiplication:

$$z_1 * z_2 = (x_1 + iy_1) * (x_2 + iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + y_1x_2)$$

It is similar to the multiplication of two polynomial functions.

Division:

For the division, it is a bit trickier. It is not possible to divide by an expression with more then one term. So, one has to get rid of the imaginary part in the denominator.

$$\frac{z_1}{z_2} = \frac{z_1\overline{z_2}}{z_2\overline{z_2}} = \frac{x_1x_2+y_1y_2}{x_2^2+y_2^2} + i\,\frac{x_2y_1-x_1y_2}{x_2^2+y_2^2}$$

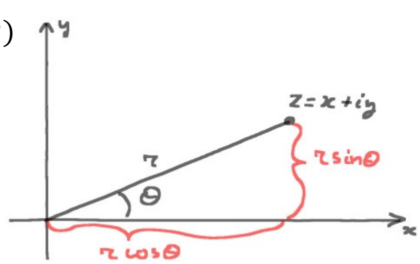
Also, it is important to mention that the associative, commutative and distributive law still holds for the complex numbers. A last thing to mention is how to find the modulus of the complex number. Following the analogy with the rectangular plane, $|z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$. Also, there is a useful inequality called the triangle inequality:

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$$

Now, if one want to use complex number in there full extend, it is important to introduce the polar form of the complex numbers. With the polar form, the introduction of powers and roots for the complex number will be easy to do.

Using the same representation as the representation of the polar coordinates in a plane, the complex number can be rewritten as:

$$z = r(\cos\theta + i\sin\theta)$$



It is interesting to note that the value of r is the modulus of the complex number, and the angle is normally referred to as the argument of the complex number.

Let us now rewrite the complex number, using the Euler formula, as mentioned above:

$$z = |z|e^{i\theta}$$

Using this form of the complex number is very interesting for many reasons:

1- The power of a complex number:

The n power of a complex number is:

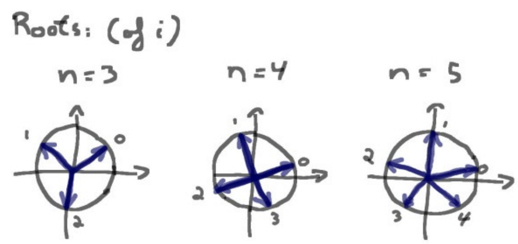
$$z^n = |z|^n e^{ni\theta} = |z|^n(\cos n\,\theta + i \sin n\theta)$$

Hence, the power of a complex number is still a complex number, and it is a "spiral" value of the original number.

2- Roots:

The n roots of a complex number is quite interesting. By using the Euler formula again, and by analogy with the power described above, if z is the n root of z^n, then:

$$z = \sqrt[n]{z^n} = |z|e^{i\theta} = |z|(\cos\theta + i\sin\theta)$$



Example: above is the representation of the n = 3,4,5, roots of the complex number i.