UNIVERSITÉ D'OTTAWA • UNIVERSITY OF OTTAWA
Sciences • Science
Mathématiques et statistique
Mathematics and Statistics

MAT 2141 Assignment 5

Due 4:20 pm, Dec. 1, 2008.   Instructor: Barry Jessup

Family Name:______________________

First Name:________________________

Student number:____________________

(For the marker's use only →)

1	
2	
3	
(Bonus) 4	
Total	

PLEASE READ THESE INSTRUCTIONS CAREFULLY.

1. Read each question carefully, and answer all questions in the space provided after each question. You may use the backs of pages if necessary, but be sure to indicate to the marker that you have done this.
2. Part marks can be earned. The correct answers require justification written legibly and logically: you must convince the marker that you know why your solution is correct.
3. *Bonus* marks are much more difficult to earn than those in previous questions.
4. You may submit this assignment during the preceding DGD, or by 4:20pm on the due date, at the secretariat of the department of Mathematics (KED103A), which closes at 4:30pm each weekday.

1. Let $T : \mathcal{P}_2(\mathbf{R}) \to \mathbf{R}^3$ be the function defined by

$$T(p) = \begin{bmatrix} p(1) \\ p(0) + p'(1) \\ p(1) + \frac{p''(0)}{2} \end{bmatrix},$$

where p' and p'' respectively denote the first and second derivatives of p.

a) Prove that T is a linear map.

b) Find the matrix of T with respect to the standard bases of $\mathcal{P}_2(\mathbf{R})$ and $\mathbf{R}^2$.

c) Is T an isomorphism? If so, give the matrix of T^{-1} with respect to the standard bases of $\mathbf{R}^2$ and $\mathcal{P}_2(\mathbf{R})$. If not, find a basis for $\ker(T)$.

2. Let $B = \{\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$ and $D = \{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$ be ordered bases of $\mathbf{R}^2$, and define a linear map $T : \mathbf{R}^2 \to \mathbf{R}^2$ by

$$T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} x - y \\ 2x - y \end{bmatrix}.$$

a) Find the matrix $X = M_B(T)$ of T with respect to the basis B.

b) Find the matrix $Y = M_D(T)$ of T with respect to the basis D.

c) Find an invertible matrix P such that $P^{-1}XP = Y$.

3. Let

$$A = \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}$$

a) Show that the eigenvalues of A are 0 and 9.

b) Find an orthogonal basis for each eigenspace of A.

c) Find an orthogonal matrix P such that $P^{-1}AP$ is diagonal, and give this diagonal matrix.

d) Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be defined by $T(v) = A.v$. Find a basis B of $\mathbf{R}^3$ such that the matrix $M_B(T)$ is diagonal, and give this diagonal matrix.

4. (Bonus) Regard $\mathbf{C}^n$ as a vector space over $\mathbf{C}$ and define a function $\langle\,,\rangle : \mathbf{C}^n \times \mathbf{C}^n \to \mathbf{C}$ by

$$\left\langle \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}, \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \right\rangle = \sum_{i=1}^{n} z_i \bar{w}_i,$$

where $\bar{w}$ denotes the complex conjugate of $w \in \mathbf{C}$. (Such a function is called an complex inner product, and $\mathbf{C}^n$, with this complex inner product is a finite-dimensional *Hilbert space.*)

a) Suppose a (complex-) linear map $T : \mathbf{C}^n \to \mathbf{C}^n$ satisfies

$$\langle Tv, w\rangle = \langle v, Tw\rangle, \quad \forall v, w \in \mathbf{C}^n.$$

(That is, T is self-adjoint.) Prove that the eigenvalues of T are real.

b) Suppose A is an $n \times n$ symmetric matrix with real entries. Prove that the eigenvalues of A are real.

c) Now suppose A and B are both real $n \times n$ symmetric matrices which commute, i.e. $AB = BA$. Prove that A and B are *simultaneously diagonalizable*, i.e. there is an invertible matrix P such that both $P^{-1}AP$ and $P^{-1}BP$ are diagonal.

(The generalization of (c) is that any (finite) set of commuting self-adjoint operators can be simultaneously diagonalized.)