Solutions to Math 105 Midterm 2 (Version I)

1. $f'(x) = (2x+1)' \int_3^{2x+1} t^{25} \ln\left(\frac{et}{3}\right) dt + (2x+1) \frac{d}{dx}\left(\int_3^{2x+1} t^{25} \ln\left(\frac{et}{3}\right) dt\right)$

$$= 2\int_3^{2x+1} t^{25} \ln\left(\frac{et}{3}\right) dt + (2x+1)\cdot(2x+1)^{25} \ln\left(\frac{e(2x+1)}{3}\right)\cdot 2$$

Hence, $f'(1) = 2\int_3^3 t^{25} \ln\left(\frac{et}{3}\right) dt + 3\cdot 3^{25}(\ln e)\cdot 2 = 2(3^{26})$. //

2. (a) Substitute $y = \sin x$, $dy = \cos x\, dx$.

$$\int \sin^{84} x \cos^3 x\, dx = \int \sin^{84} x(1-\sin^2 x)\cos x\, dx = \int y^{84}(1-y^2)\, dy$$

$$= \frac{y^{85}}{85} - \frac{y^{87}}{87} + C = \boxed{\frac{\sin^{85} x}{85} - \frac{\sin^{87} x}{87} + C}$$

(b). Substitute $y = \tan x$, $dy = \sec^2 x\, dx$.

$$\int \tan^{42} x \sec^4 x\, dx\, dx = \int \tan^{42} x(1+\tan^2 x)\sec^2 x\, dx = \int y^{42}(1+y^2)\, dy$$

$$= \frac{y^{43}}{43} + \frac{y^{45}}{45} + C = \boxed{\frac{\tan^{43} x}{43} + \frac{\tan^{45} x}{45} + C}$$

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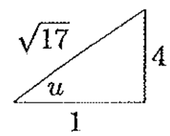
3. Substitute $x = \frac{1}{2}\tan u$, $dx = \frac{1}{2}\sec^2 u\, du$.

$$\int_0^2 \frac{1}{[1+4x^2]^{3/2}}\, dx = \int_0^{\arctan 4} \frac{1}{[1+\tan^2 u]^{3/2}} \frac{1}{2}\sec^2 u\, du$$
$$= \frac{1}{2}\int_0^{\arctan 4} \frac{\sec^2 u}{\sec^3 u}\, du$$
$$= \frac{1}{2}\int_0^{\arctan 4} \cos u\, du$$
$$= \frac{1}{2}\sin(\arctan 4) = \frac{2}{\sqrt{17}}$$



(b) $$\frac{5x^2-9x+3}{x^3-2x^2+x} = \frac{5x^2-9x+3}{(x-1)^2 x} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x}$$

$$5x^2-9x+3 = A(x-1)x + Bx + C(x-1)^2. \qquad (1)$$

Letting
$$= Ax^2 - Ax + Bx + Cx^2 - 2Cx + C$$
$$= (A+C)x^2 + (-A+B-2C)x + C$$

$\Rightarrow$
$$A + C = 5 \qquad (1)$$
$$-A + B - 2C = -9 \qquad (2)$$
$$C = 3 \qquad (3)$$

By (3) and (1), $A = 5 - C = 5 - 3 = 2 \qquad (4)$.

Using (3) and (4), (2) becomes $-2 + B - 2(3) = -9$
$\Rightarrow B = -1$.

Thus, $$\frac{5x^2-9x+3}{x^3-2x^2+3} = \frac{2}{x-1} - \frac{1}{(x-1)^2} + \frac{3}{x}.$$

4. (a) We can factorize $x^2 - 3x + 2 = (x-1)(x-2)$. Now we want

$$\frac{2}{x^2-3x+2} = \frac{A}{x-1} + \frac{B}{x-2} = \frac{A(x-2)+B(x-1)}{(x-1)(x-2)} = \frac{x(A+B)-2A-B}{(x-1)(x-2)}.$$

This means that

$$A + B = 0$$
$$-2A - B = 2$$

Solving the system we get $A = -2$ $B = 2$.Hence

$$\int \frac{2}{x^2-3x+2}dx = -2\ln|x-1| + 2\ln|x-2| + C. \quad (1)$$

Since the function $\frac{2}{x^2-3x+2}$ is continuous on the interval $[0, 0.5]$, we get

$$\int_0^{0.5} \frac{2}{x^2-3x+2}dx = \left(-2\ln|x-1| + 2\ln|x-2|\right)\Big|_0^{0.5}$$
$$= -2\ln 0.5 + 2\ln 1.5 - 2\ln 2.$$

(b) Since $\frac{2}{x^2-3x+2} = \frac{2}{(x-1)(x-2)}$ is not defined at $x=1$, $\int_0^1 \frac{2}{x^2-3x+2}dx$ is an improper integral. By (1) in (a), we have

$$\int_0^1 \frac{2}{x^2-3x+2}dx = \lim_{b\to 1^-} \int_0^b \frac{2}{x^2-3x+2}dx$$
$$= \lim_{b\to 1^-} \left(-2\ln|x-1| + 2\ln|x-2|\right)\Big|_0^b = \lim_{b\to 1^-}\left(-2\ln|b-1| + 2\ln|b-2| - \ln 2\right)$$
$$= \infty, \text{ i.e. } \int_0^1 \frac{2}{x^2-3x+2}dx \text{ diverges.}$$

5. $\dfrac{dy}{dx} = -xy^3$, $\quad \dfrac{dy}{y^3} = -x\,dx$, $\quad \displaystyle\int \frac{dy}{y^3} = -\int x\,dx$,

$$-\frac{1}{2y^2} = -\frac{x^2}{2} + c \Rightarrow y^2 = \frac{1}{x^2 - 2c}.$$

Since $y(0) = f(0) = -\frac{1}{2}$, we get

$$\left(-\frac{1}{2}\right)^2 = \frac{1}{0^2 - 2c} \Rightarrow c = -2.$$

Thus, $y^2 = \dfrac{1}{x^2+4}$. Hence, $f(x) = -\sqrt{\dfrac{1}{x^2+4}} = -\dfrac{1}{\sqrt{x^2+4}}$.

6. (a) $\displaystyle\int_0^{1.6} g(x) \approx \frac{0.8}{3}\left(g(0) + 4g(0.8) + g(1.6) \right)$

$$= \frac{0.8}{3}\left(12.1 + 4(11.7) + 13.2 \right).$$

(b) Since $|g^{(4)}(x)| \leq 5$ for $0 \leq x \leq 1.6$, we get that the error $E_{S(2)} \leq \dfrac{5 \cdot (1.6 - 0)}{180} \cdot (0.8)^4$.

7. Integrating by parts twice, first with $u = \sin x$, $v = f'(x)$ and then with $u = \cos x$, $v = f(x)$,

$$\begin{aligned}
\int_0^{\pi} f''(x) \sin x\,dx &= \sin x\, f'(x)\big|_0^{\pi} - \int_0^{\pi} f'(x) \cos x\,dx \\
&= -\int_0^{\pi} f'(x) \cos x\,dx \\
&= -\cos(x) f(x)\big|_0^{\pi} - \int_0^{\pi} f(x) \sin x\,dx \\
&= 9 - \int_0^{\pi} f(x) \sin x\,dx
\end{aligned}$$

Adding $\int_0^{\pi} f(x) \sin x\,dx$ to both sides gives

$$\int_0^{\pi} [f(x) + f''(x)] \sin x\,dx = \boxed{9}$$