

5) 1. Let  $T : V \rightarrow W$  be linear,  $X$  a subspace of  $V$  with  $X \subset \ker T$ , and (as usual), let  $q : V \rightarrow V/X$  denote the quotient mapping defined by

$$\forall v \in V, \quad q(v) := [v] = v + X$$

- a) Prove that there is a unique linear map  $S : V/X \rightarrow W$  such that  $T = S \circ q$ .
- b) Prove that the map  $S$  of part (a) is injective iff  $X = \ker T$ .

Now suppose  $V = W = \mathbf{R}^3$ ,  $X = \{(x, y, z) \in \mathbf{R}^3 \mid x - y + z = 0\}$  and  $T$  is multiplication by the matrix

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & -3 \\ 1 & 3 & 1 \end{bmatrix}.$$

c) Show that the formula

$$S(v + X) = T(v) \quad (*) \quad \textcircled{1}$$

does not define a function  $S$  from  $\mathbf{R}^3/X$  to  $\mathbf{R}^3$ .

a) Define  $S : V/X \rightarrow W$  by  $S([v]) = T(v)$ ,  $\forall [v] \in V/X$ . Since  $[v] = [v'] \Leftrightarrow v - v' \in X \Rightarrow v - v' \in \ker T \Rightarrow T(v - v') = 0 \Leftrightarrow T(v) = T(v')$ ,  $S$  is a function on  $V/X$  (i.e.  $S$  is "well-defined" by the formula  $S([v]) = T(v)$ ,  $\forall [v] \in V/X$ ).

If  $[v_1], [v_2] \in V/X$ ,  $c_1, c_2 \in F$ , then  $S(c_1[v_1] + c_2[v_2]) = S([c_1v_1 + c_2v_2]) = T(c_1v_1 + c_2v_2) = c_1T(v_1) + c_2T(v_2)$  ( $T$  is linear), but this is  $c_1S([v_1]) + c_2S([v_2])$ , hence  $S$  is linear. Clearly,  $S \circ q = T$  (by the def'n of  $S$  since  $q(v) = [v]$ ).

Moreover,  $S' \circ q = T = S \circ q \Leftrightarrow S'([v]) = S([v])$ ,  $\forall [v] \in V/X$ . Hence  $S = S'$ .

b) Note that  $S([v]) = 0 \Leftrightarrow T(v) = 0 \Leftrightarrow v \in \ker T$ . If  $X = \ker T$ , then  $S([v]) = 0 \Leftrightarrow v \in \ker T \Leftrightarrow [v] = 0$ . If  $S$  is injective, then  $S([v]) = 0 \Rightarrow [v] = 0$ , i.e.  $v \in \ker T \Rightarrow [v] = 0$ , and so  $\ker T \subseteq X$ . Since  $X \subseteq \ker T$ , this shows injective  $\Rightarrow X = \ker T$ .

c) Note that  $[0] = [(1, -1, 0)]$  in  $V/X$ , but  $S([0]) = T(0) = (0, 0, 0)$ , while  $S([(1, -1, 0)]) = T(1, -1, 0) = (-1, -2, -2) \neq (0, 0, 0)$ . Hence formula (\*) does not define a function.

2. a) Suppose that  $V$  is a vector space,  $U$  is a subspace of  $V$  and  $\{u_1, \dots, u_k, v_1, \dots, v_l\}$  is a basis of  $V$  such that  $\{u_1, \dots, u_k\}$  is a basis of  $U$ . Prove that

$$B = \{[v_1], \dots, [v_l]\} \quad \textcircled{1} + \textcircled{1}$$

is a basis of  $V/U$ .

basis of  $W$  (b) extension (c) (d) basis of  $P_3(\mathbb{R})/W$

b) Let  $W = \{p \in P_3(\mathbb{R}) \mid p(0) = p(1) = 0\}$ . Then, we know that  $W$  is a subspace of  $P_3(\mathbb{R})$ . Find a basis of the quotient space  $P_3(\mathbb{R})/W$ , and verify that  $\dim(P_3(\mathbb{R})/W) = \dim P_3(\mathbb{R}) - \dim W$ .

a) We know from class that  $\dim V/U = \dim V - \dim U = k+l - k = l$ , so it suffices to prove that  $B$  above is l.i. So suppose  $c_1, \dots, c_l \in F$  and  $c_1[v_1] + \dots + c_l[v_l] = [0]$ . Then  $[\sum_{i=1}^l c_i v_i] = [0]$ , which implies  $\sum_{i=1}^l c_i v_i \in U$ . Hence,  $\exists b_1, \dots, b_k \in F$  s.t.  $\sum_{i=1}^l c_i v_i = \sum_{j=1}^k b_j u_j$ . But then  $\sum_{j=1}^k b_j u_j + \sum_{i=1}^l (-c_i) v_i = 0$ ; Since  $\{u_1, \dots, u_k, v_1, \dots, v_l\}$  is l.i., this implies  $(b_1 = b_2 = \dots = b_k = 0 = c_1 = c_2 = \dots = c_l)$ . Hence  $B$  is l.i. and so is a basis of  $V/U$ .

b) We use part (a) by extending a basis of  $W$  to a basis of  $P_3(\mathbb{R})$ . By the remainder theorem,  $p \in W \Leftrightarrow p(t) = t(t-1)q(t)$  for some  $q(t)$  of degree at most one. Hence  $W = \{t(t-1)(a+tb) \mid a, b \in F\}$   
 $= \{a(t^2-t) + b(t^3-t^2) \mid a, b \in F\}$   
 $= \text{span}\{t^2-t, t^3-t^2\}$ .

Moreover,  $a(t^2-t) + b(t^3-t^2) = 0 \Rightarrow \begin{cases} (t=-1) & 2a - 2b = 0 \\ (t=2) & 2a + 4b = 0 \end{cases} \Rightarrow a = b = 0$ .

Hence  $\{t^2-t, t^3-t^2\}$  is a basis for  $W$ . Now  $t \notin \text{span}\{t^2-t, t^3-t^2\} = W$  since  $p(t) = t$  does not satisfy  $p(1) = 0$ . Hence  $\{t^2-t, t^3-t^2, t\}$  is l.i. Moreover  $1 \notin \text{span}\{t^2-t, t^3-t^2, t\}$  since each of  $t^2-t, t^3-t^2, t$  is zero for  $t=0$ , while  $1$  is not. Hence  $\{t^2-t, t^3-t^2, t, 1\}$  is a basis of  $P_3(\mathbb{R})$ . By (a)  $\{[t], [1]\}$  is a basis of  $P_3(\mathbb{R})/W$ . Clearly,  $2 = \dim P_3(\mathbb{R})/W$ .

4

$\dim V < \infty$

3. Let  $V$  be a vector space over a field  $F$ . As before we denote the dual space of  $V$  by  $V^*$ .

(1) 1st iso thm or if  
(1)  $\text{img} = F$

a) If  $g \in V^*$  and  $g \neq 0$ , prove that  $V/(\ker g) \cong F$ .

b) If  $v \in V$  is any non-zero vector, prove that there is a linear form  $a \in V^*$  such that  $a(v) \neq 0$ .

(1) - realizing  $\{v\}$  can be extended to a basis  
(1) - using dual basis or another method

c) If  $\{v, w\}$  is linearly independent in  $V$ , prove that there are two linear forms  $f, g \in V^*$  such that  $f(w) = 0 = g(v)$  and  $f(v) = 1 = g(w)$ .

(1)

d) If  $V = \mathbf{R}^3$ ,  $v = (1, 0, -1)$  and  $w = (1, 1, 0)$ , give explicit formulae for two linear forms  $f, g \in V^*$  such that  $f(w) = 0 = g(v)$  and  $f(v) = 1 = g(w)$ .

$\begin{matrix} J \\ D \cdot g \end{matrix}$  (1) - just

a) Since  $g \neq 0$ ,  $\exists v_0 \in V$  s.t.  $g(v_0) = c_0 \neq 0$ ,  $c_0 \in F$ . If  $c \in F$  is arbitrary,  $g(\frac{c v_0}{c_0}) = c$ , and so  $g$  is surjective as a map from  $V$  to  $F$ . (Note: we can divide by  $c_0$  because  $c_0 \neq 0$ .) Now, by the 1st isomorphism theorem,

$$V/\ker g \cong \text{img } g \quad \text{ie} \quad V/\ker g \cong F.$$

b) Since  $v \neq 0$ ,  $\{v\}$  is l.i. Extend  $\{v\}$  to a basis  $\{v\} \cup R$  of  $V$ , and let  $X = \text{span } R$ . Then  $V = \text{span}\{v\} + X$ , since  $\{v\} \cup R$  spans  $V$ , and  $\text{span}\{v\} \cap X = \{0\}$  because  $\{v\} \cup R$  is l.i. Thus,  $V = \text{span}\{v\} \oplus X$ . Now define  $a : V \rightarrow F$  by  $a(cv + x) = c$ , for  $c \in F, x \in X$ . Then  $a$  is clearly linear, so  $a \in V^*$ . Moreover  $a(v) = 1 \neq 0$ .

c) We proceed as in (b). Since  $\{v, w\}$  is l.i., extend it to a basis  $\{v, w\} \cup C$  of  $V$ , and let  $Y = \text{span } C$ . Then, as in (b),  $V = \text{span}\{v, w\} \oplus Y$ , and if we define  $f, g : V \rightarrow F$  by  $f(av + bw + y) = a$  and  $g(av + bw + y) = b$  for  $a, b \in F, y \in Y$ , then  $f, g \in V^*$  and

no other conditions required.

3d)  $B = \{e_1, w, v\}$  is a basis of  $\mathbb{R}^3$  because  $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$  clearly has

rank 3. If  $\{f_1, f_2, f_3\}$  is the corresponding dual basis,

then  $f_i(v) = a_i \cdot v$  for  $a_i \in M_{1,3}(\mathbb{R})$ ,  $i=1,2,3$ . The fact

that  $\{f_1, f_2, f_3\}$  is the dual basis to  $B$  is equivalent to

to fact that  $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \cdot \begin{bmatrix} e_1 & w & v \end{bmatrix} = I_3$ .

So we invert  $A$  to find  $a_2 = :g$  and  $a_3 = :f$ .

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \end{array} \right] \sim \left[ \begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array} \right] \sim \left[ \begin{array}{ccc|ccc} I_3 & & & 1 & -1 & 1 \\ & & & 0 & 1 & 0 \\ & & & 0 & 0 & -1 \end{array} \right]$$

$$\text{Thus } f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = [0 \ 0 \ -1] \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -z$$

$$\text{and } g\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = [0 \ 1 \ 0] \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = y$$

will suffice.

① -  $\rho$

① -  $\delta$

① -  $z$

4

# Alternative solutions for (b) and (c); when $\dim V < \infty$

3. Let  $V$  be a vector space over a field  $\mathbf{F}$ . As before we denote the dual space of  $V$  by  $V^*$ .

- a) If  $g \in V^*$  and  $g \neq 0$ , prove that  $V/(\ker g) \cong \mathbf{F}$ .
- b) If  $v \in V$  is any non-zero vector, prove that there is a linear form  $a \in V^*$  such that  $a(v) \neq 0$ .
- c) If  $\{v, w\}$  is linearly independent in  $V$ , prove that there are two linear forms  $f, g \in V^*$  such that  $f(w) = 0 = g(v)$  and  $f(v) = 1 = g(w)$ .
- d) If  $V = \mathbf{R}^3$ ,  $v = (1, 0, -1)$  and  $w = (1, 1, 0)$ , give explicit formulae for two linear forms  $f, g \in V^*$  such that  $f(w) = 0 = g(v)$  and  $f(v) = 1 = g(w)$ .

b) (Alternative sol<sup>n</sup> when  $\dim V < \infty$ ) Let  $\{v, v_1, \dots, v_{n-1}\}$  be a basis of  $V$  containing  $v$ . (possible, since  $\{v\} \rightarrow$  l.i.). Let  $\{f, f_1, \dots, f_{n-1}\}$  be the dual basis of  $V^*$ . Then, by def<sup>n</sup>,  $f(v) = 1 \neq 0$ , and  $f =: a \in V^*$

c) (Alternative sol<sup>n</sup> when  $\dim V < \infty$ ) Let  $\{v, w, w_2, \dots, w_{n-1}\}$  be a basis of  $V$  extending  $\{v, w\}$ , and let  $\{f_1, g, g_2, \dots, g_{n-1}\}$  be the dual basis of  $V^*$ . Then, by def<sup>n</sup>,  $f(v) = 1$ ,  $f(w) = 0$   
 $g(w) = 1$ ,  $g(v) = 0$ ;

and  $f, g \in V^*$ .

4. a) Given a vector  $v \in \mathbf{R}^3$ , define an element  $f_v \in (\mathbf{R}^3)^*$  by

$$f_v(w) = v \cdot w \quad \forall w \in \mathbf{R}^3,$$

where  $v \cdot w$  denotes the dot product.

Prove that the map  $S : \mathbf{R}^3 \rightarrow (\mathbf{R}^3)^*$  defined by

$$S(v) = f_v, \quad \forall v \in \mathbf{R}^3$$

is an isomorphism.

b) The isomorphism  $S$  of part (a) has an inverse  $S^{-1} : (\mathbf{R}^3)^* \rightarrow \mathbf{R}^3$ . Prove that if  $f \in (\mathbf{R}^3)^*$ , then  $S^{-1}(f) \in \mathbf{R}^3$  is the vector

$$S^{-1}(f) = \begin{bmatrix} f(e_1) \\ f(e_2) \\ f(e_3) \end{bmatrix}, \quad \text{---} \star$$

where  $\{e_1, e_2, e_3\}$  is the standard ordered basis of  $\mathbf{R}^3$ .

a) If  $a, b \in \mathbf{R}$  and  $u, v \in \mathbf{R}^3$ ,  $S(au + bv) = f_{(au+bv)}$ . But,  $\forall w \in \mathbf{R}^3$ ,

$$f_{(au+bv)}(w) = (au + bv) \cdot w = a(u \cdot w) + b(v \cdot w) = a f_u(w) + b f_v(w)$$

$$\therefore f_{au+bv} = a f_u + b f_v = a S(u) + b S(v).$$

Hence,  $S$  is linear. To see that  $S$  is a bijection, we could use the formulae of (b), or, since  $\dim \mathbf{R}^3 = \dim (\mathbf{R}^3)^*$ , it suffices to show  $S$  is injective. But  $S(v) = 0 \Rightarrow f_v(w) = 0, \forall w \in \mathbf{R}^3 \Rightarrow f_v(e_i) = 0 \Rightarrow v \cdot e_i = 0 \Rightarrow v = 0$ . Hence  $S$  is injective, and is thus an iso.

$$b) \text{ Let } v \in \mathbf{R}^3. \text{ Then } (S^{-1} \circ S)(v) = S^{-1}(f_v) = \begin{bmatrix} f_v(e_1) \\ f_v(e_2) \\ f_v(e_3) \end{bmatrix} = \begin{bmatrix} v \cdot e_1 \\ v \cdot e_2 \\ v \cdot e_3 \end{bmatrix} = v.$$

If  $f \in (\mathbf{R}^3)^*$ , then  $S(S^{-1}(f))(w) = S^{-1}(f) \cdot w, \forall w \in \mathbf{R}^3$ . If  $w = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ ,

$$S^{-1}(f) \cdot w = f(e_1)y_1 + f(e_2)y_2 + f(e_3)y_3 = f(y_1 e_1 + y_2 e_2 + y_3 e_3) = f(w). \text{ Hence,}$$

$$S(S^{-1}(f))(w) = f(w), \quad \forall w \in \mathbf{R}^3, \text{ and so } S(S^{-1}(f)) = f, \quad \forall f \in (\mathbf{R}^3)^*.$$

5. (Bonus) Suppose that  $T : \mathbf{R}^3 \rightarrow \mathbf{R}^3$  is multiplication by the matrix

$$B = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & -3 \\ 1 & 3 & 1 \end{bmatrix},$$

and let  $T^* : (\mathbf{R}^3)^* \rightarrow (\mathbf{R}^3)^*$  be the linear map defined by

$$T^*(f)(v) = f(T(v)), \quad \forall v \in \mathbf{R}^3, \forall f \in (\mathbf{R}^3)^*.$$

Then, if  $S$  is the isomorphism of Q4, we know that  $S^{-1} \circ T^* \circ S : \mathbf{R}^3 \rightarrow \mathbf{R}^3$  is a linear map from  $\mathbf{R}^3$  to  $\mathbf{R}^3$ , and hence is multiplication by some matrix  $A$ . Find the matrix  $A$ , and carefully check that  $S^{-1}(T^*(S(v))) = Av, \forall v \in \mathbf{R}^3$ .

Let  $v = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ . Then

$$S^{-1}(T^*(S(v))) = \begin{bmatrix} T^*(S(v))(e_1) \\ T^*(S(v))(e_2) \\ T^*(S(v))(e_3) \end{bmatrix}$$

(3)

$$= \begin{bmatrix} T^*(f_v)(e_1) \\ T^*(f_v)(e_2) \\ T^*(f_v)(e_3) \end{bmatrix}$$

$$= \begin{bmatrix} f_v(Te_1) \\ f_v(Te_2) \\ f_v(Te_3) \end{bmatrix}$$

$$= \begin{bmatrix} v \circ Te_1 \\ v \circ Te_2 \\ v \circ Te_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + x_2 + x_3 \\ 2x_1 + 3x_2 - 3x_3 \\ -x_1 - 3x_2 + x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & -3 \\ -1 & -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= P^t \quad \quad \quad A = P^t$$