Concordia University
Department of Mathematics & Statistics

Course	**Number**	**Section**
MATH	365	A
Examination	**Date**	**Pages**
Final	April 2008	3
Instructor		**Course Examiner**
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Special Instructions

▷ None.

MARKS

[20] **1.** a) State the definition of Riemann integrability of a bounded function f on an interval $[a, b]$ in terms of upper and lower integrals, and then give a criterion for integrability in terms of upper and lower sums.

b) Prove that if

$$f(x) = \begin{cases} 3 - x & \text{if} \quad x = 0 \text{ or } 1 \\ 1 & \quad\text{otherwise} \end{cases}$$

then $f(x) \in \mathcal{R}[0, 1]$.

c) Now prove that if

$$f(x) = \begin{cases} 3 - x & \text{if} \quad x \in \mathbb{Q} \\ 1 & \quad\text{otherwise} \end{cases}$$

then $f(x) \notin \mathcal{R}[0, 1]$.

d) For the function $f(x)$ in (b), find $F(x) = \int_0^x f$. Is it true that $F'(x) = f(x)$ for each $x \in [0, 1]$? Explain.

[12] **2.** a) If $f(x)$ is continuous on $[a, b]$ and $\int_a^x f = \int_x^b f \ \forall x \in [a, b]$ prove that $f(x) = 0$ on $[a, b]$.

b) If $f(x) \in \mathcal{R}[a, b]$ and $\int_a^x f = \int_x^b f \ \forall x \in [a, b]$, is it still true that $f(x) = 0$ on $[a, b]$ (and therefore continuous)? Prove or give a counterexample.

c) Evaluate without integrating:

$$\frac{d}{dx}\int_{x^3}^{x^4}\sqrt{1+u^5}du$$

[8] **3.** a) State the definition of a Riemann sum of a bounded function $f(x)$ over an interval $[a,b]$.

b) Show that $L(P,f) \le S(P,f) \le U(P,f)$ is always true.

c) Express

$$\sum_{i=0}^{n-1}\frac{1}{n+i}$$

as a Riemann sum. Then write down the Riemann integral (but don't evaluate it) by taking the appropriate limit.

[10] **4.** a) Show that $\int_{-\infty}^{0} e^x dx$ converges and explain why this implies that $I = \int_0^1 |\ln x| dx$ also converges. Evaluate the integral I.

b) Suppose $f(x) \ge 0$ is continuous on $[1,\infty)$, and for some $p > 1$, we have that $\lim\limits_{x\to\infty} x^p f(x) = l$ where $0 < l < \infty$. State the Limit Comparison test for type 1 improper intergrals and use it to show that $\int_1^\infty f(x)dx$ converges.

[12] **5.** Determine if each series converges absolutely, converges conditionally or diverges

a) $\sum\limits_{n=1}^{\infty}\dfrac{(n!)^3}{(3n)!}$

b) $\sum\limits_{n=1}^{\infty}\sin(1/n^2)$

c) $\sum\limits_{n=3}^{\infty}(-1)^{n+1}\dfrac{\ln(\ln n)}{\sqrt{\ln n}}$

[12] **6.** a) Show that if $\sum\limits_{n=1}^{\infty} a_n$ converges and each $a_n \ge 0$, then $\sum\limits_{n=1}^{\infty}\dfrac{a_n}{1+a_n}$ also converges. What about $\sum\limits_{n=1}^{\infty}\sqrt{a_n}$? Explain or give a counterexample.

b) State Dirichlet's test for the convergence of a series $\sum\limits_{n=1}^{\infty} a_n b_n$

c) Use part b) to show that if $\sum\limits_{n=1}^{\infty} a_n$ converges and $x \in [0,1]$ then $\sum\limits_{n=1}^{\infty} a_n x^n$ converges.

[18] **7.** a) Let

$$f_n(x) = \begin{cases} 2n^2x & \text{if} \quad 0 \le x < \dfrac{1}{2n} \\ 2n - 2n^2x & \text{if} \quad \dfrac{1}{2n} \le x < \dfrac{1}{n} \\ 0 & \text{if} \quad \dfrac{1}{n} \le x \le 1 \end{cases}$$

i) Find $F(x) = \lim\limits_{n\to\infty} f_n(x)$.

ii) Show that the convergence is not uniform on $[0, 1]$.

iii) Show that the convergence is uniform on every interval $[a, 1]$, $0 < a < 1$.

b) Use the Weierstrass M-test to prove uniform convergence of $\sum\limits_{n=0}^{\infty} \dfrac{1}{1+n^2x}$ on $[a, \infty)$ for any $a > 0$. Is the convergence uniform on $(0, \infty)$? Explain.

[8] **8.** a) Find the radius and interval of convergence of the power series $\sum\limits_{n=2}^{\infty} \dfrac{(x-2)^n}{n(\ln n)^{3/2}}$

b) Find the Taylor series that represents $\sin x$ centered at $\pi/2$

Bonus Question *(5 marks)*

Suppose $a_n > 0$ for each n, and $\sum\limits_{n=1}^{\infty} a_n$ diverges. Let $s_n = \sum\limits_{k=1}^{n} a_k$. Prove that $\sum\limits_{n=1}^{\infty} a_n/s_n^2$ converges [*Hint:* Prove $a_n/s_n^2 = (s_n - s_{n-1})/s_n^2) \le 1/s_{n-1} - 1/s_n$ for $n \ge 2$. Then show that $\sum\limits_{n=2}^{\infty} (1/s_{n-1} - 1/s_n)$ converges.]