

MATH 365
2007 FINAL EXAM SOLUTIONS

MARKS

- 20 1. (a) Given a bounded function f on $[a, b]$, state a necessary and sufficient condition (n.a.s.c.) in terms of upper and lower sums for f to be Riemann integrable on $[a, b]$.

Solution A n.a.s.c. (Theorem 6.2) is $\forall \varepsilon > 0$ There is a partition P such that

$$U(P, f) - L(P, f) < \varepsilon$$

- (b) Use part (a) to prove that if

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

then $f(x) \notin \mathcal{R}[0, 1/2]$.

Solution For this function $U(P, f) = 1$ for every partition, since $M_i = \sup \{f(x) : x_{i-1} \leq x \leq x_i\}$ is always 1. On the other hand, since $x \leq 1/2$ we see that $m_i = \inf \{f(x) : x_{i-1} \leq x \leq x_i\} \leq 1/2$. So

$$U(P, f) = 1 \cdot \left(\frac{1}{2} - 0\right) = \frac{1}{2} \text{ and}$$

$$L(P, f) = \sum_{i=1}^n m_i \Delta x_i \leq \frac{1}{2} \sum_{i=1}^n \Delta x_i = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}, \text{ so}$$

$$U(P, f) - L(P, f) \geq \frac{1}{4} \geq \varepsilon$$

for any $\varepsilon \leq 1/4$ and any partition. So the condition is not satisfied for $\varepsilon = .25$

- (c) If $f(x) \notin \mathcal{R}[0, 1/2]$, is it possible that $f(x) \in \mathcal{R}[0, 1]$? Explain.

Solution No, it is impossible, because according to Prop. 6.5 if $f \in \mathcal{R}[0, 1]$, then (since $1/2 \in [0, 1]$) $f \in \mathcal{R}[0, 1/2]$ and $f \in \mathcal{R}[1/2, 1]$.

- (d) Express

$$\sum_{i=1}^n \frac{n}{3n^2 - (n+i)^2}$$

as a Riemann sum $S(P, f)$. Then write down the Riemann integral (but

don't evaluate it) by taking the appropriate limit.

Solution

$$\begin{aligned}\sum_{i=1}^n \frac{n}{3n^2 - (n+i)^2} &= \sum_{i=1}^n \frac{1}{3 - (1+i/n)^2} \cdot \frac{1}{n} \\ &\rightarrow \int_0^1 \frac{1}{3 - (1+x)^2} dx\end{aligned}$$

where in the Riemann sum $S(P, f)$ we have $f(x) = \frac{1}{3 - (1+x)^2}$;

P is the partition of $[0, 1]$ into equal subintervals $x_i = i/n$;

$\Delta x_i = 1/n$ and $t_i = x_i$.

(e) Prove that if $f(x) \in \mathcal{R}[a, b]$ then

$$\int_a^b f = \lim_{c \rightarrow a^+} \int_c^b f$$

[Hint - consider $\left| \int_a^b f - \int_c^b f \right|$ and remember that f is bounded].

Solution Since $|f|$ is bounded (by M , say) we have

$$\begin{aligned}\left| \int_a^b f - \int_c^b f \right| &= \left| \int_a^c f \right| \leq \int_a^c |f| \\ &\leq \int_a^c M = M(c-a) < \varepsilon \text{ if} \\ c-a &< \frac{\varepsilon}{M}\end{aligned}$$

i.e. if we take c to be within ε/M to the right of a , the difference is less than ε .

10 (a) If u and v are differentiable functions on an interval I with values in $[a, b]$ and

f is continuous on $[a, b]$, use the fact that a continuous function has an antiderivative

to show that

$$\frac{d}{dx} = f(v(x))v'(x) - f(u(x))u'(x)$$

Solution Letting $F(x) = \int_a^x f$, we know (Thm. 6.12) that F is differentiable since f is continuous on $[a, b]$, and $F'(x) = f(x)$ on $[a, b]$. We also know that

$$\begin{aligned} \int_{u(x)}^{v(x)} f &= \int_{u(x)}^a f + \int_a^{v(x)} f = \int_a^{v(x)} f - \int_a^{u(x)} f \\ &= F(v(x)) - F(u(x)) \end{aligned}$$

Now simply apply the chain rule.

(b) If $f(x)$ is continuous and strictly positive on $[a, b]$, show that $F(x) = \int_a^x f$ is strictly increasing on $[a, b]$.

Solution As in part (a), we see that $F'(x) = f(x) > 0$ on $[a, b]$ since $f(x)$ is continuous and strictly positive on $[a, b]$. If $x_1, x_2 \in [a, b]$ and $x_1 < x_2$ then

$$\begin{aligned} F(x_2) - F(x_1) &= F'(c)(x_2 - x_1) \\ &= f(c)(x_2 - x_1) \end{aligned}$$

for some $c \in (x_1, x_2)$ by the Mean Value Theorem (for derivatives). Since $f(c) > 0$ it follows that $F(x_2) - F(x_1) > 0$, i.e. $F(x_2) > F(x_1)$ and so F is strictly increasing.

10 3. (a) Show that

$$\int_1^\infty \frac{\cos(x)}{x^p} dx$$

converges absolutely if $p > 1$.

Solution This is pretty easy: for $x \geq 1$

$$\begin{aligned} \left| \frac{\cos(x)}{x^p} \right| &\leq \frac{1}{x^p} \text{ and so by the Comparison Test, since} \\ \int_1^\infty \frac{1}{x^p} dx &\text{ converges for } p > 1, \text{ so does} \\ \int_1^\infty \frac{\cos(x)}{x^p} dx \end{aligned}$$

(b) Determine whether

$$\int_0^1 \frac{x}{\sqrt{1-x}} dx$$

converges or diverges.

Solution The integrand is undefined at $x = 1$. Put $u = 1 - x$, etc. and the integral becomes

$$\int_0^1 \frac{1-u}{\sqrt{u}} du = \int_0^1 \frac{1}{u^{1/2}} du - \int_0^1 \sqrt{u} du$$

The first is a type 2 improper integral which converges because $p = 1/2 < 1$ and the second is an ordinary Riemann integral.

12 4. Determine if each series converges absolutely, converges conditionally or diverges

(a) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

Solution Use the ratio test:

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{(n+1)!^2 (2n)!}{(2n+2)! (n!)^2} \\ &= \frac{(n+1)^2}{(2n+1)(2n+2)} = \frac{(1+1/n)^2}{(2+1/n)(2+2/n)} \rightarrow \frac{1}{4} < 1 \end{aligned}$$

So the series converges.

(b) $\sum_{n=1}^{\infty} (1 - \sqrt[n]{n})^n$

Solution Use the root test:

$$\sqrt[n]{|a_n|} = (1 - \sqrt[n]{n}) \rightarrow 0 < 1$$

because $\sqrt[n]{n} \rightarrow 1$. So the series converges.

(c) $1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \frac{1}{8} + \frac{1}{9} - \frac{1}{10} - \frac{1}{11} + \frac{1}{12} + \dots$

Solution This can be regrouped as follows:

$$\begin{aligned} &\left(1 - \frac{1}{2}\right) - \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) - \left(\frac{1}{7} - \frac{1}{8}\right) + \left(\frac{1}{9} - \frac{1}{10}\right) - \left(\frac{1}{11} - \frac{1}{12}\right) + \dots \\ &= \frac{1}{1 \cdot 2} - \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} - \frac{1}{7 \cdot 8} + \frac{1}{9 \cdot 10} - \dots \end{aligned}$$

The general term is

$$\frac{(-1)^{m+1}}{(2m-1)(2m)}; m = 1, 2, \dots$$

and so the regrouped series converges - in fact converges absolutely. The original series must also converge, since clearly

the partial sums of the original series can be approximated as closely as one pleases by the partial sums of the regrouped series because $\pm \frac{1}{n} \rightarrow 0$. More precisely, if the regrouped series has partial sums $(t_m)_{m \in \mathbb{N}}$ and converges to T and the original series has partial sums $(s_n)_{n \in \mathbb{N}}$ then for both m, n large enough we can make $|t_m - T|$ small, because $t_m \rightarrow T$, and we can also make $|s_n - t_m|$ small because if n is even, $s_n = t_{n/2}$ and if n is odd then $n = 2m + 1$ and

$$s_{2m+1} = t_m \pm \frac{1}{2m+1} \text{ or}$$

$$|s_n - t_m| = \frac{1}{n}$$

Since

$$|s_n - T| \leq |s_n - t_m| + |t_m - T|$$

it follows that $s_n \rightarrow T$ too.

- 18** **5. (a)** Show that if $\sum_{n=1}^{\infty} a_n$ converges and each $a_n \geq 0$, then $\sum_{n=1}^{\infty} a_n^2$ also converges. If the condition that each $a_n \geq 0$ is removed, is the result still true? Explain or give a counterexample.

Solution Since $\sum_{n=1}^{\infty} a_n$ converges we know $a_n \rightarrow 0$, so for some n_0 we know if $n \geq n_0$ then $|a_n| \leq 1$ and so $a_n^2 \leq |a_n|$. By the Comparison test $\sum_{n=n_0}^{\infty} a_n^2$ converges and hence $\sum_{n=0}^{\infty} a_n^2 = \sum_{n=0}^{n_0-1} a_n^2 + \sum_{n=n_0}^{\infty} a_n^2$ converges. If the condition that each $a_n \geq 0$ is removed, the result is no longer true in general. Consider $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$. This converges (a.s. test) but $\sum_{n=0}^{\infty} a_n^2 = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

- (b)** State the definition of uniform convergence of a sequence of functions $(f_n(x))_{n \in \mathbb{N}}$ to a function $F(x)$.

Solution $(f_n(x))_{n \in \mathbb{N}} \rightarrow F(x)$ on a domain D uniformly means: given $\varepsilon > 0 \exists n_0 \in \mathbb{N}$ such that $n \geq n_0 \implies |f_n(x) - F(x)| < \varepsilon$ for every $x \in D$. Here the number n_0 depends only on ε ; i.e. $n_0 = n_0(\varepsilon)$

- (c)** Suppose $f_n(x) \rightarrow F(x)$ uniformly on (a, b) and also that $f_n(a) \rightarrow F(a)$ and $f_n(b) \rightarrow F(b)$. Show that $f_n(x) \rightarrow F(x)$ uniformly on $[a, b]$.

Solution Given $\varepsilon > 0$ let n_1, n_2, n_3 be the numbers in \mathbb{N} satisfying the convergence definitions on (a, b) , at a and at b respectively - that is,

$$n \geq n_1 \implies |f_n(x) - F(x)| < \varepsilon \text{ for every } x \in (a, b)$$

$$n \geq n_2 \implies |f_n(a) - F(a)| < \varepsilon$$

$$n \geq n_3 \implies |f_n(b) - F(b)| < \varepsilon$$

If $n_0 = \max\{n_1, n_2, n_3\}$ then

$$n \geq n_0 \implies |f_n(x) - F(x)| < \varepsilon \text{ for every } x \in (a, b) \cup \{a\} \cup \{b\} = [a, b].$$

20 6. (a) Let $f_n(x) = \frac{x^n}{1+x^n}$ on the interval $[0, 1]$.

(i) Find $F(x) = \lim_{n \rightarrow \infty} f_n(x)$.

Solution If $0 \leq x < 1$ then $\lim_{n \rightarrow \infty} x^n = 0 \implies \lim_{n \rightarrow \infty} \frac{x^n}{1+x^n} = 0 = F(x)$. If $x = 1$ then $f_n(x) = \frac{1}{2} \forall n \in \mathbb{N}$ and so $\lim_{n \rightarrow \infty} f_n(1) = \frac{1}{2} = F(1)$.

(ii) Show that the convergence is uniform on every interval $[0, a]$, $0 < a < 1$.

Solution Since $0 \leq \frac{x^n}{1+x^n} \leq x^n \leq a^n$ on $[0, a]$ and a^n can be made arbitrarily small ($a^n < \varepsilon$ if $n \geq n_0 > \frac{\ln(1/\varepsilon)}{\ln(1/a)}$), it follows that $\frac{x^n}{1+x^n}$ can be made arbitrarily small independently of x . So the convergence is uniform.

(iii) Show that the convergence is not uniform on $[0, 1]$.

Solution The easiest way to show it is to note that the limit function $F(x)$ is not continuous, even though each $f_n(x)$ is. So the convergence cannot be uniform.

(b) Use the Weierstrass M-test to prove uniform convergence of $\sum_{n=1}^{\infty} e^{-nx} x^n$ on $[0, \infty)$.

Solution $e^{-nx} x^n = (xe^{-x})^n$. Let's find the maximum value of xe^{-x} (for $x \geq 0$): straightforward calculus gives us $(x-1)e^{-x} = 0 \implies x = 1$ and the max. value is e^{-1} . So take $M_n = e^{-n}$. Now we know that

$$e^{-nx} x^n = (xe^{-x})^n \leq e^{-n} = M_n$$

and since $\sum_{n=1}^{\infty} e^{-n}$ is a convergent geometric series, we see that $\sum_{n=1}^{\infty} e^{-nx} x^n$ converges uniformly on $[0, \infty)$.

- 10 7. (a) Find the radius and interval of convergence of the power series $\sum_{n=2}^{\infty} \frac{x^n}{n \ln n}$

Solution Using the ratio test gives

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \frac{|x|^{n+1}}{(n+1) \ln(n+1)} \cdot \frac{n \ln n}{|x|^n} \\ &= |x| \frac{n \ln n}{(n+1) \ln(n+1)} \rightarrow |x| \end{aligned}$$

So the ratio is less than 1 $\iff |x| < 1 \iff -1 < x < 1$. The radius of convergence is therefore 1. To find the interval of convergence we must test the endpoints. First, let $x = 1$. Then we get $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$. This is easy to test with the integral test: Let $f(x) = \frac{1}{x \ln x}$ $x \geq 2$. Clearly the conditions are satisfied and also $\int_2^{\infty} \frac{1}{x \ln x} dx = \int_{\ln 2}^{\infty} \frac{1}{u} du$ (let $u = \ln x$) diverges. So the series diverges. For $x = -1$ we get $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ - an alternating series that passes the a.s. test and converges. So the interval of convergence is $[-1, 1)$.

- (b) Find the Taylor series that represents $x^2 e^{x-2}$.

Solution We start with the Taylor series for $g(x) = e^x$ centered at 0 (so it is actually the MacLaurin series, but there is no problem calling it a Taylor series).

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \end{aligned}$$

since $g^{(n)}(x) = e^x$ and so $g^{(n)}(0) = 1 \forall n \in \mathbb{N}$. Next, we multiply by e^{-2}

$$e^{x-2} = \sum_{n=0}^{\infty} e^{-2} \frac{x^n}{n!}$$

and finally we multiply by x^2 to get

$$x^2 e^{x-2} = \sum_{n=0}^{\infty} e^{-2} \frac{x^{n+2}}{n!}$$

Since this is a power series and since power series representations of functions are unique, this has to be the Taylor series of $f(x) = x^2 e^{x-2}$

Note that since we didn't develop the theory of when the Taylor series is **equal** to the original function (Taylor's Theorem; Theorem 5.6) this is all that is expected in answering the question. If we **had** covered the extra material, it would also be necessary to show that the remainder term

$$R_n(x) = \frac{f^{(n+1)}(t)}{(n+1)!} x^{n+1} \rightarrow 0 \text{ as } n \rightarrow \infty$$

where t is between 0 and x . Now $f^{(n+1)}(t) = e^{t-2} (t^2 + 2(n+1)t + n(n+1))$, as can be seen by induction, and clearly

$$\lim_{n \rightarrow \infty} \frac{e^{t-2} (t^2 + 2(n+1)t + n(n+1))}{(n+1)!} x^{n+1} = 0$$

for every fixed x and t between 0 and x [note that $x^{n+k}/n!$ is the general term of the MacLaurin series for $x^k e^x$ (which converges everywhere), and hence $\rightarrow 0$].

Bonus Question (5 marks)

Suppose $a_n \geq 0$ for each n and $\sum_{n=1}^{\infty} a_n$ converges. Let $s_n = \sum_{k=1}^n a_k$. Prove that $\sum_{n=1}^{\infty} a_n s_n$ also converges.

Solution The easiest way to show convergence is to let $S = \sum_{k=1}^{\infty} a_k = \lim_{n \rightarrow \infty} s_n$. Then

$$0 \leq a_n s_n \leq a_n S$$

and since $\sum_{n=1}^{\infty} a_n S = S \sum_{n=1}^{\infty} a_n$ converges, by the Comparison test $\sum_{n=1}^{\infty} a_n s_n$ also converges.