Then, if $v = (h, k)$, $f(v) - f(0) = \underbrace{\left[\frac{\partial f}{\partial x}(0) \ \ \frac{\partial f}{\partial y}(0)\right]}_{L} v + S(v)v$,

and so $\lim_{v \to 0} \frac{f(v) - f(0) - L(v)}{\|v\|} = \lim_{v \to 0} S(v)\frac{v}{\|v\|} = 0$!

Corollary $U \subseteq \mathbb{R}^n$, $f : U \to \mathbb{R}^p$, $f = \begin{bmatrix} f_1 \\ \vdots \\ f_p \end{bmatrix}$;

If $\frac{\partial f_i}{\partial x_j}$ exist in a ball around $a \in U$ and are cts at a, $f'(a)$ exists, and $f'(a) = \left[\frac{\partial f_i}{\partial x_j}(a)\right]$.

See on-line problems for more examples

e.g. $f(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

DGD next week

$(x,y) \neq (0,0)$ $\frac{\partial f}{\partial x}(x,y) = \frac{2xy^4}{x^2+y^2}$, $\frac{\partial f}{\partial y} = \frac{2x^4y}{x^2+y^2}$

$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = 0 = \frac{\partial f}{\partial y}(0,0)$

If $(x,y) \neq (0,0)$ $\left| \frac{\partial f}{\partial x}(x,y) - \underbrace{\frac{\partial f}{\partial x}(0,0)}_{0} \right| = \frac{2|xy^4|}{x^2+y^2} \leq \frac{2(x^2+y^2)^{5/2}}{x^2+y^2}$

$= 2\sqrt{x^2+y^2}$

$\therefore \frac{\partial f}{\partial x}(x,y) \to 0 = \frac{\partial f}{\partial x}(0,0)$ as $(x,y) \to (0,0)$ ("$\delta = \frac{\varepsilon}{2}$").

Similarly, $\frac{\partial f}{\partial y}$ is cts at $(0,0)$. Hence, f is diff'ble at $(0,0)$
$f'(0,0) = [0 \ \ 0] = 0$!

Level Curves and the Gradient
(Surfaces)

- Suppose $f : U \to \mathbb{R}$, $U \subseteq \mathbb{R}^n$ is open, f' exists on U.

Def^n The gradient of f at $a \in U$ is

$$\nabla f(a) = df(a)^t = \begin{bmatrix} \frac{\partial f}{\partial x_1}(a) \\ \vdots \\ \frac{\partial f}{\partial x_n}(a) \end{bmatrix}$$

Note: $\underbrace{\nabla f(a) \cdot w}_{\text{dot product}} = \underbrace{df(a) . w}_{\text{matrix product}}$, $\forall w \in \mathbb{R}^n$.

- $\|w\|=1$, $\left|\frac{\partial f}{\partial w}(a)\right| = |\nabla f(a) \cdot w| \leq \|\nabla f(a)\|$, equality obtains when $w \parallel \nabla f(a)$
parallel.

Prop^n: If $f'(a)$ exists, if it isn't zero the direction in which f increases the fastest at a is $\nabla f(a)$; the f^n decreases the fastest in the direction $-\nabla f(a)$. $\left(\nabla f(a) \cdot \frac{\nabla f(a)}{\|\nabla f(a)\|} = \|\nabla f(a)\|\right.$
$-$ "

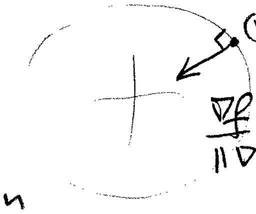
Pf. Let $g : (-1,1) \to U \subseteq \mathbb{R}^n$ be a differentiable curve s.t. $g(0) = a$, and $g'(0) = w$ satisfies $\|w\| = 1$.
($g'(0)$: tangent vector to curve)

Then the rate of change of f along g at $t=0$ is

- $$(f \circ g)'(0) = \frac{d}{dt} f(g(t))\Big|_{t=0} = df(g(0))(g'(0)) = \nabla f(a) \cdot g'(0). = \frac{\partial f}{\partial w}(a).$$

So $|(f \circ g)'(0)| \leq \|\nabla f(a)\|$, and equality obtains when $g'(0) = \frac{\nabla f(a)}{\|\nabla f(a)\|}$

e.g $f(x,y) = 1 - x^2 - y^2$; $\nabla f\left(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}\right) = -2\,[x_0 \cdot y_0]$;



Level curves, Level sets

- Suppose $g : (-1,1) \to U$ is a curve in $\mathbb{R}^n$.

Def^n The curve g is a level curve of $f : U \to \mathbb{R}$ if $f(g(t))$ is constant for $t \in (-1,1)$.

Remark If $g'(t_0)$ exists, and $f'(g(t_0))$ exists, the chainrule shows

$$0 = \frac{d}{dt}\left(f(g(t))\right)\Big|_{t=t_0} = df(g(t_0))(g'(t_0))$$

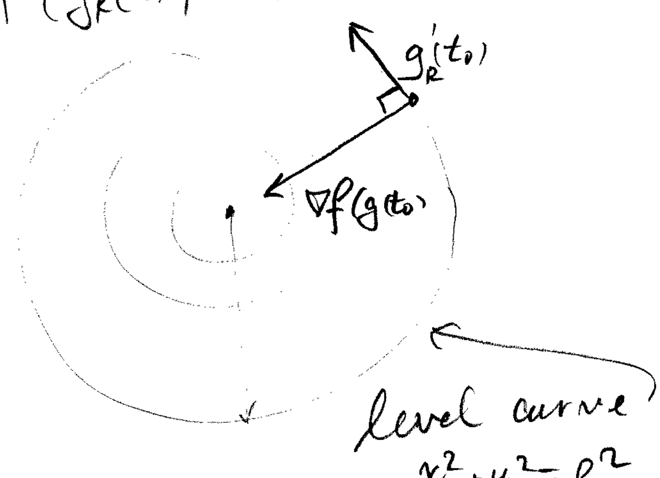
$$= \nabla f\, g(t_0) \cdot \underbrace{g'(t_0)}_{\text{tangent to level curve}}, \quad \forall t_0 \in (-1$$

- i.e. The gradient of f at $g(t_0)$ is perpendicular to the level curve of f at $g(t_0)$.

e.g. $f(x,y) = 1 - x^2 - y^2$; $g_R(t) = R(\cos t, \sin t)$; $\|g_R(t)\| = R$ defines a level curve of f, $\forall R \in \mathbb{R}$.

$$f(g_R(t)) = 1 - R\cos^2 t - R\sin^2 t = 1 - R^2 \quad (\text{constant.})$$

$$g_R'(t_0) = R(-\sin t, +\cos t)$$



If $f : U \to \mathbb{R}$, $U \subseteq \mathbb{R}^n$, $c \in \mathbb{R}$
$A_c = \{ v \in U \mid f(v) = c \}$ is called level set of f.

e.g. $f(x,y,z) = x^2 + y^2 + z^2$.
A_{c^2} = sphere of radius $|c|$, centre $0 \in \mathbb{R}^3$

Local Extrema of f^{ns} of several variables $f: U \to \mathbb{R}$

● **Theorem** Suppose f has a local max or a local min at $a \in U$, and that $\frac{\partial f}{\partial w}(a)$ exists for some dir^n w. Then, $\frac{\partial f}{\partial w}(a) = 0$.

Moreover, if $f'(a)$ exist, $f'(a) = df(a) = 0$. $(\nabla f(a) = 0)$

Pf Define $h(t) = f(a + tw)$, and suppose f has a local max. at a. Then $f(a) \geq f(a + tw)$, $\forall t \in (-\delta, \delta)$ for some $\delta > 0$. Then, $h(0) \geq h(t)$ $\forall t \in (-\delta, \delta)$ and so $h'(0) = 0$. But $0 = h'(0) = \frac{\partial f}{\partial w}(a)$.

● If $df(a)$ exists, all dir^{nl} derivatives exist and are zero (by the first part). But $\nabla f(a) \cdot w = \frac{\partial f}{\partial w}(a)$, $\forall w$ and so $\nabla f(a) = 0$ $(\Leftrightarrow df(a) = 0)$. ∎

Remark: As for f^{ns} of one variable, $df(a) = 0$ does not imply f has a local extremum there

eg

$f(x,y) = x^2 - y^2$ at $(0,0)$:

$\nabla f(0,0) = 2\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$ but

● f has neither a local max nor a local min at $(0,0)$

"Saddle point"

Global Extrema; Constraints & Lagrange's method

- e.g. Let $A = \{(x,y) \mid x^2+4y^2 \le 4\} \subset \mathbb{R}^2$

$$= \{(x,y) \mid x^2+4y^2 < 4\} \cup \{(x,y) \mid x^2+4y^2 = 4\} = V \cup C$$

The first set here is open, the second is closed and is the level set (curve) $g=0$ of $g(x,y) = x^2+4y^2-4$.

- We know how to find local extrema of $f: C \to \mathbb{R}$: we solve $df(a) = 0$ for $a \in V$.
- To find local extrema of f on C, we can use

Theorem (Lagrange) Suppose f and g are diff'ble on some open set containing $S = \{v \in \mathbb{R}^n \mid g(v) = 0\}$, and that f has a local extremum at $a \in S$ (e.g. local max: $\exists \delta > 0$ st $\forall v \in S \cap B(a,\delta)$, $f(a) \ge f(v)$.)

If $\nabla g(a) \ne 0$, then $\exists \lambda \in \mathbb{R}$ s.t. $\nabla f(a) = \lambda \nabla g(a)$

"Lagrange multiplier"
