Practical Aspects of derivatives of $f^{ns}: \mathbb{R}^n \to \mathbb{R}^m$

- How can we decide if $f'(a)$ exists?

Step 1: Reduction to case $f: U \to \mathbb{R}$:

Theorem Suppose $U \subseteq \mathbb{R}^n$ (open) and $f: U \to \mathbb{R}^p$.

If we write $f = \begin{bmatrix} f_1 \\ \vdots \\ f_p \end{bmatrix}$ where $f_i: U \to \mathbb{R}$ $\forall i$, $1 \le i \le p$,

then for $a \in U$, $f'(a)$ exists $\Leftrightarrow$ $f_i'(a)$ exist, $\forall i$ $1 \le i \le p$,

and $f'(a)w = \begin{bmatrix} f_1'(a)w \\ \vdots \\ f_n'(a)w \end{bmatrix}$, $\forall w \in \mathbb{R}^n$

Pf First note that $L: \mathbb{R}^n \to \mathbb{R}^p$ is linear $\Leftrightarrow$ $L_i: \mathbb{R}^n \to \mathbb{R}$ is linear, $\forall i$, $1 \le i \le p$,

$\begin{bmatrix} L_1 \\ \vdots \\ L_p \end{bmatrix}$ and $L(w) = \begin{pmatrix} L_1(w) \\ \vdots \\ L_n(w) \end{pmatrix}$, $\forall w \in \mathbb{R}^n$.

and now that if $\begin{bmatrix} r_1 \\ \vdots \\ r_p \end{bmatrix} = r: U \to \mathbb{R}^p$, when $r_i: U \to \mathbb{R}$, $1 \le i \le p$

- $$\lim_{v \to v_0} r(v) = r(a) = 0 \Leftrightarrow \lim_{v \to v_0} r_i(v) - r_i(a) = 0, \quad \forall i, \ 1 \le i \le p.$$

Hence,
$$\lim_{v \to v_0} \frac{f(v) - f(a) - L(v - v_0)}{\|v - a\|} = 0$$

$$\Leftrightarrow \lim_{v \to v_0} \frac{f_i(v) - f_i(a) - L_i(v - v_0)}{\|v - a\|} = 0, \quad \forall i \ 1 \le i \le p.$$

Step 2: Partial derivatives. Suppose $f: U \to \mathbb{R}$, $U \subseteq \mathbb{R}^n$, $a \in U$.

- Def^n If $1 \le j \le n$, the partial derivative of f w.r.t x_j (jth coordinate) is $\frac{\partial f}{\partial x_j}(a) := \lim_{t \to 0} \frac{f(a + te_j) - f(a)}{t}$, when it exists

Theorem (cf Remark 3.4)

If $U \subseteq \mathbb{R}^n$ and $f : U \to \mathbb{R}$ is differentiable at $a \in U$, then $\forall j$, $1 \le j \le n$, $\frac{\partial f}{\partial x_j}(a)$ exists and $\forall w \in \mathbb{R}^n$

$$f'(a)(w) = df(a)(w) = \left[\frac{\partial f}{\partial x_1}(a) \;\ldots\; \frac{\partial f}{\partial x_n}(a)\right] . w .$$

i.e. $f'(a)e_j = \frac{\partial f}{\partial x_j}(a)$, $1 \le j \le n$.

i.e. (after identifying $f'(a)$ with its standard matrix)

$$df(a) = \left[\frac{\partial f}{\partial x_1}(a) \;\ldots\; \frac{\partial f}{\partial x_n}(a)\right].$$

Pf. If $f'(a)$ exist, $\displaystyle\lim_{v \to a} \frac{f(v) - f(a) - f'(a)(v-a)}{\|v-a\|} = 0$. $(*)$ Fix j, $1 \le j \le n$ & set $v = a + te_j$. Then $(*) \Rightarrow \displaystyle\lim_{t\to 0} \frac{f(a+te_j) - f(a) - f'(a)(te_j)}{|t|} = 0$

$$\Rightarrow \lim_{t\to 0} \frac{f(a+te_j) - f(a)}{|t|} = \lim_{t \to 0} \frac{t}{|t|} \cdot f'(a)e_j \qquad \left(\frac{|t|}{t}\right)$$

$$\Rightarrow \lim_{t\to 0} \frac{f(a+te_j) - f(a)}{t} = \lim_{t\to 0} f'(a)e_j = f'(a)e_j$$

i.e. $\frac{\partial f}{\partial x_j}(a)$ exist, and $\frac{\partial f}{\partial x_j}(a) = f'(a)e_j$.

Def^n If $U \subseteq \mathbb{R}^n$, $f : U \to \mathbb{R}^p$, $f = \begin{bmatrix} f_1 \\ \vdots \\ f_p \end{bmatrix}$, $f_i : U \to \mathbb{R}$,

the Jacobian matrix of f at $a \in U$ is $\begin{bmatrix} df_1(a) \\ \vdots \\ df_p(a) \end{bmatrix}$

$$= \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(a) & \frac{\partial f_1}{\partial x_2}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \vdots & & & \vdots \\ \frac{\partial f_p}{\partial x_1}(a) & \cdots & \cdots & \frac{\partial f_p}{\partial x_n}(a) \end{bmatrix} = \left[\frac{\partial f_i}{\partial x_j}(a)\right] \quad \begin{matrix} 1 \le i \le p \\ 1 \le j \le n. \end{matrix}$$

e.g. Consider $p_1, p_2 : \mathbb{R}^2 \to \mathbb{R}$ $\quad p_1(x_1,x_2) = x_1$, $\quad p_2(x_1,x_2) = x_2$ $\quad \begin{pmatrix} p_1(v) = [1\ 0]\,v \\ p_2(v) = [0\ 1]\,v \end{pmatrix}$

$$\frac{\partial p_i}{\partial x_j} = \delta_{ij} \quad ; \quad dp_1(a) = [1\ 0] = e_1^t, \quad dp_2(a) = [0\ 1] = e_2^t$$

Remark If $T : \mathbb{R}^n \to \mathbb{R}^p$ is linear, $T'(a)$ exists $\forall a \in \mathbb{R}^n$, and $T'(a)(w) = T(w)$, since $T(v) - T(a) - T(v-a) = 0$, $\forall v, \forall a \in \mathbb{R}^n$. N.B. This does not say $\underbrace{T'(a)}_{\text{linear map } \mathbb{R}^n \to \mathbb{R}^p} = \underbrace{T(a)}_{\text{vector in } \mathbb{R}^p!}$!

e.g. $p_i : \mathbb{R}^n \to \mathbb{R}$ $\quad p_i\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_i$; $\quad dp_i(a) = e_i^t$ (transpose

e.g. If $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ is diff'ble at $a \in U$,

$$df(a) = \left[\frac{\partial f}{\partial x_1}(a) \ \cdots \ \frac{\partial f}{\partial x_j}(a) \right]$$
$$= \sum_{j=1}^n \frac{\partial f}{\partial x_j} e_j^t = \sum_{j=1}^n \frac{\partial f}{\partial x_j} dp_j \,. \qquad (*)$$

We often use "x_j" to denote the jth coordinate function $x_j \equiv p_j : \mathbb{R}^n \to \mathbb{R}$ $\quad x_j\left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\right) = a_j$. Then $(*)$ can be written

$$df(a) = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(a)\, dx_j \quad ; \quad dx_j(a) = x_j'(a) = e_j^t.$$

df is called a differential (one-) form on $\mathbb{R}^n$, since

$df(a)$ is a linear form on $\mathbb{R}^n$.

Directional Derivatives Let $w \in \mathbb{R}^n$ with $\|w\| = 1$, $f : U \to \mathbb{R}$, $U \subseteq \mathbb{R}^n$

The directional derivative of f at a in the direction w is

$$\frac{\partial f}{\partial w}(a) := \lim_{t \to 0} \frac{f(a+tw) - f(a)}{t}, \text{ when it exists}$$

Thm $f'(a)$ exists $\Rightarrow$ $\frac{\partial f}{\partial w}(a)$ exists $\forall w$, $\|w\|=1$,

and $\frac{\partial f}{\partial w}(a) = f'(a)(w)$

Pf As $f(a+tw) - f(a) = f'(a)(tw) + r(a+tw)\,|t|$,

$\lim_{t\to 0} \frac{f(a+tw)-f(a)}{t} = f'(a)(w)$, since $\lim_{t\to 0} r(a+tw)\frac{|t|}{t} = 0$.

Rmk: An example where all directional derivates exist but f is not diff'ble is given in the exercises!

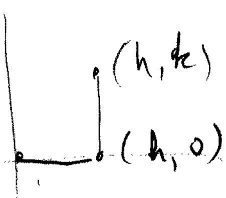
Is all lost?

Let $f: U \to \mathbb{R}$, $U \subseteq \mathbb{R}^n$.

Thm (3.6, P302) Suppose $a \in U$ and that all partial derivatives $\frac{\partial f}{\partial x_j}$, $1 \le j \le n$ exist in an open ball B around a, and that each $\frac{\partial f}{\partial x_j} : U \to \mathbb{R}$ is cts at a. Then $f'(a)$ exists. ($f' = [\frac{\partial f}{\partial x_1} \cdots \frac{\partial f}{\partial x_n}]$)

Pf. We do this for $n=2$. No new ideas are involved in the pf for $n>2$. W.l.o.g suppose $a=(0,0)$. Note that for (h,k) in B,

$$f(h,k) - f(0,0) = f(h,k) - f(h,0) + f(h,0) - f(0,0)$$



(MVT -Lagrange) $$= \frac{\partial f}{\partial y}(h, \xi(h,k)) \cdot k + \frac{\partial f}{\partial x}(\eta(h), 0)$$

with $\xi \in (0,k)$ and $\eta \in (0,h)$. Now define

$$\Delta(h,k) = \left[\frac{\partial f}{\partial x}(\eta, 0) - \frac{\partial f}{\partial x}(0,0) \quad \frac{\partial f}{\partial y}(h,\xi) - \frac{\partial f}{\partial y}(0,0)\right]$$ and

note that $\lim_{(h,k)\to(0,0)} \Delta(h,k) = (0,0)$ because the partials are cts at $(0,0)$ by hypothesis.

Then, if $v = (h, k)$, $f(v) - f(0) = \underbrace{\left[\frac{\partial f}{\partial x}(0) \;\; \frac{\partial f}{\partial y}(0)\right]}_{L} v + S(v)\, v$,

and so $\lim_{v \to 0} \frac{f(v) - f(0) - L(v)}{\|v\|} = \lim_{v \to 0} S(v) \frac{v}{\|v\|} = 0$!

$\square$

Corollary $U \subseteq \mathbb{R}^n$, $f : U \to \mathbb{R}^p$, $f = \begin{bmatrix} f_1 \\ \vdots \\ f_p \end{bmatrix}$;

If $\frac{\partial f_i}{\partial x_j}$ exist in a ball around $a \in U$ and are cts at a, $f'(a)$ exists, and $f'(a) = \left[\frac{\partial f_i}{\partial x_j}(a)\right]$.
